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NOTE

END VERTICES IN INTERVAL GRAPHS

John GIMBEL

Department of Mathematical Sciences, University of Alaska, Fairbanks, AK 99775-1110, USA

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Given an interval representation of an interval graph G, an interval is an end interval if its right (or left) endpoint is further to the right (left) of all other intervals in the collection. We characterize those vertices v in interval graphs for which there is some representation of G where the interval corresponding to v is an end interval. We also present a short proof of a characterization of homogeneously representable interval graphs.

We say that G is an *interval graph* if there is a collection of intervals $\{I_v\}_{v\in G}$, where uv is an edge of G if and only if $I_u \cap I_v \neq \emptyset$. By $v\in G$ we mean v is in the vertex set of G. We will say that I_v is the interval representing v. The collection $\{I_v\}_{v\in G}$ will be called a representation of G.

When the context allows, we shall not mention the indexing set. We may assume that all intervals in any representation are closed and have positive length. Further, no intervals agree on any endpoint. Given I, an interval, let $R(I)$ and $L(I)$ respectively be the right and left endpoints of I. Given a family of intervals $\{I_v\}$ we say that I_v is an *end* interval if $L(I_u)=\text{Min}\bigcup I_v$ or $R(I_u)=\text{Max}\bigcup I_v$. A given vertex v in an interval graph G is an *end* vertex if there is some representation of G where v is represented by an end interval.

There are many practical applications of interval graph theory, such as archaeology and behavioral psychology, where one is interested in constructing time lines from interval graphs. Each interval corresponds to a period of time when a par-

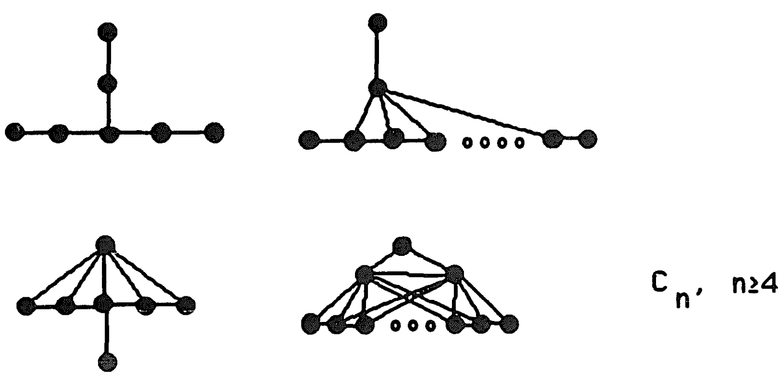

Fig. 1.

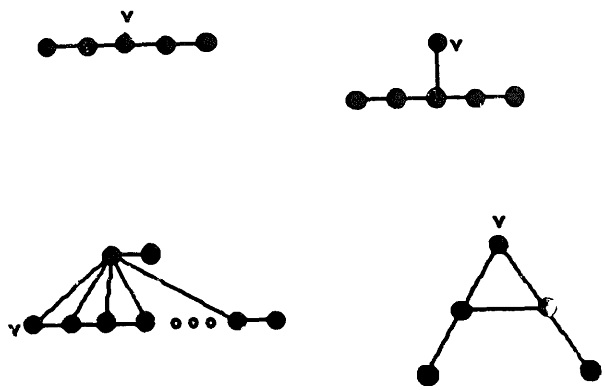

Fig. 2

ticular phenomenon occurred (see [1,2]). In such applications one might be interested in knowing which phenomenon could possibly come first (or last). Here, the subject of end intervals becomes important.

In our characterization of end vertices, we will need the following remark on interval graphs. Our lemma refers to Fig. 1 where C_n denotes a chordless cycle on n vertices.

Lemma. *A graph G is an interval graph if and only if it contains none of the graphs in Fig. 1 as induced subgraphs.*

The original proof, due to Lekkerkerker and Boland can be found in [3].

Theorem 1. *A vertex v in an interval graph G is an end interval if and only if G contains as induced subgraphs, none of the graphs in Fig. 2, where v is the designated vertex.*

Proof. Clearly, if v is the designated vertex in any of the graphs in Fig. 2 it cannot be an end vertex.

Conversely, suppose none of the configurations in Fig. 2 occur in G. Without loss of generality, we may assume G is connected. Form H from G by attaching a single vertex u to v. We first notice that H is an interval graph. For if it were not, H would have to contain one of the graphs in Fig. 1 as an induced subgraph where u is one of the vertices of degree 1. But when we remove any of the vertices of degree 1 from the graphs in Fig. 1 we always form one of the graphs in Fig. 2, where v is the designated vertex. This is a contradiction of the hypothesis.

So, let $\{I_w\}_{w\in H}$ be a representation of H. If I_v is an end interval in this representation, then we may remove I_u from the family and see that v is an end vertex in G. Hence, we shall assume I_v is not and end interval of $\{I_w\}_{w\in H}$. If there are no intervals I_t where $R(I_v)<L(I_t)$, then we can extend I_v as far as we wish to the right without forming any new intersections. In which case v would be an end vertex. Hence, we may assume that there is some interval entirely to the right of I_v. Since

G is connected, we may assume that there are intervals I_s and I_t where I_v is entirely to the left of I_t and I_s intersects both I_t and I_v.

If we cannot extend I_v on the left indefinitely we consider this same argument. We can find intervals I_q and I_r where I_q is entirely to the left of I_v and I_r intersects both I_v and I_q. Since I_u intersects I_v and no other interval, we see it is strictly between I_r and I_s. Hence these two intervals do not intersect. We must now conclude that the intervals I_q, I_r, I_v, I_s, I_t corresponds in G to an induced P_5 with v as the center vertex as represented by the first graph in Fig. 2.

Since this configuration is forbidden, the proof is complete. □

We note that the last graph in Fig. 2, the so called A-graph, is a special case of the third one. As we shall momentarily see, it is useful to make such a distinction.

An interval graph is *homogeneously representable* if each vertex is an end vertex. Such graphs were originally characterized in [4]. We are now prepared to present a short proof of the result.

Theorem 2. *An interval graph G is homogeneously representable if and only if it contains no 5-path or A-graph as an induced subgraph.*

Proof. It is clear that no homogeneously representable interval graph contains either the 5-path or the A-graph as induced subgraphs.

To see the converse, simply note that each of the graphs in Fig. 2 contains either (or both) of these two graphs. □

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