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Belief linear programming

Hatem Masri [a,*], Fouad Ben Abdelaziz [b]

[a] Faculty of Economics, Management and Information Systems, University of Nizwa, P.O. Box 33, Nizwa, Oman
[b] Engineering System Management Graduate Program, College of Engineering, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates

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ABSTRACT

This paper proposes solution approaches to the belief linear programming (BLP). The BLP problem is an uncertain linear program where uncertainty is expressed by belief functions. The theory of belief function provides an uncertainty measure that takes into account the ignorance about the occurrence of single states of nature. This is the case of many decision situations as in medical diagnosis, mechanical design optimization and investigation problems. We extend stochastic programming approaches, namely the chance constrained approach and the recourse approach to obtain a certainty equivalent program. A generic solution strategy for the resulting certainty equivalent is presented.

1. Introduction

In stochastic programming, uncertainty on parameters is characterized by a known probability distribution. In many cases, the complete knowledge of the probability is not possible. Incomplete knowledge about probability distribution was considered in many cases. Dupacova [10], for example, studied stochastic programs where the probability distribution is expressed by some of its moments. Recently, Ben Abdelaziz and Masri [3] addressed the problem of stochastic programming with fuzzy probability distribution. In the literature, two main approaches were used to solve stochastic program and stochastic program with incomplete knowledge on probability distributions, namely, the recourse approach and the chance constrained approach. Each approach, under predefined hypotheses, leads to a certainty equivalent mathematical program.

In the case where the decision maker (DM) assigns a probability mass to subsets of the set of all states of nature Θ and not to each individual state of nature $w \in \Theta$, the uncertainty can be modeled as a belief function [21]. A belief function can also be considered when the DM handles incomplete observations, imprecise judgments and/or missing data. Despite the development of many researches using belief function to model uncertainty, attempts to provide decision models under a belief function framework are rather scarce.

The first belief decision models are due to Strat [24] and Jaffray [16]. Inspired from the Hurwicz principle, they proposed a weighted average of the upper and lower expectations related to all probability distributions that have the given belief function as a lower envelope. Yager [26] adapted the ordered weighted averaging (OWA) operators to provide an unifying framework for belief decision making. The Yager's model seems to be difficult to implement as we need to subjectively define the decision maker's (DM) coefficient of optimism and then solve a nonlinear program to define the OWA weights. Denoeux [9] examined some decision strategies for pattern classification in the context of Dempster–Shafer theory. Recently, Boujelben et al. [5] proposed a multiple criteria decision model, inspired by ELECTRE I, where the weights of criteria are expressed by a belief function.

* Corresponding author.
E-mail addresses: hatem.masri@unizwa.edu.om (H. Masri), fabdelaziz@aus.edu (F.B. Abdelaziz).

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As probability theory is a special case of evidence theory, few methods were proposed to map a belief function into a probability distribution with some predefined assumptions [14]. Among these methods, Smets [22,23] proposed the transferable belief model (TBM) that allows the DM to transform any belief decision model into a probabilistic decision problem by converting belief functions into probabilities using the pignistic transformation.

In the literature, the first attempt to incorporate belief functions within optimization problems was related to the reliability based design optimization (RBDO) problem [1]. The resulting program is called the evidence-based design optimization (EBDO) problem where the plausibility of the performance constraint violation has to be small [17]. Mourelatos and Zhou [18] proposed a hybrid solution algorithm for the EBDO problem where they apply first an RBDO algorithm to move to the vicinity of the optimum and then they use a derivative free optimizer that considers only the obtained active constraints to find the evidence-based optimum solution. Recently, Hermann [12] proposed an unified solution approach for both the EBDO and the imprecise probability design optimization (IPDO) problem.

In this paper, we introduce a general uncertain linear programming where uncertainty is characterized by a belief function. The resulting optimization program is called the belief linear program (BLP). As probability distributions are belief functions, we propose to generalize main solution approaches in stochastic programming, namely the chance constrained approach and the recourse approach, to state for a generic solution approaches.

In the next section, we recall some basic concepts of the belief function theory and then in Section 3, we introduce the BLP problem. In Section 4, we illustrate with an example the way we generate certainty equivalents to stochastic programs under the hypotheses of the chance constrained approach and the recourse approach. In Section 5, we extend the chance constrained approach, that we call belief constrained approach, to get a certainty equivalent program to the BLP problem. We discuss the convexity of the obtained certainty equivalent and we propose a solution strategy that can be used to solve it. In Section 6, we present the recourse approach for the BLP problem and a solution algorithm to solve the resulting certainty equivalent. All concepts introduced through the paper are illustrated with a simple example.

2. Basic concepts of the belief function theory

The belief function theory, also called the Dempster–Shafer theory of evidence, was initiated by Dempster [8] and then extended by Shafer [21]. Compared to other uncertainty measures such us fuzzy sets theory, the belief function theory is an extension to the probabilistic reasoning. The belief function is obtained using a probability distribution over the power set of the set of possible events. This allows assigning a mass of evidence (or probability of occurrence) to subsets and not only to singletons. In many decision situations, we can only measure the occurrence of a set of events, for example, in the medical diagnosis problems; we might have evidences about the presence of a set of bacteria and no evidence about the presence of particular bacteria [11].

Let us denote by $\Theta = \{w_1, \ldots, w_N\}$ the set of mutually exclusive states of nature and 2^Θ the power set of Θ, called the frame of discernment. Over this frame, we define a probability distribution m

$$m : 2^\Theta \to [0,1],$$

such that

$$m(\phi) = 0 \quad \text{and} \quad \sum_{A \subseteq \Theta} m(A) = 1.$$

This function is called basic probability assignment (bpa). Obviously, m divides one unity over singletons and subsets of the frame of discernment. Each singleton or each subset with a nonzero mass is called focal element. The basic probability assignment function is a generalization of the probability mass function in probability theory where focal elements are not only singleton [21].

Associated with the bpa, we introduce two measures called belief measure (Bel) and plausibility measure (Pl) and are respectively defined by:

$$\begin{aligned} Bel : \quad & 2^\Theta \to [0,1], \\ & A \to Bel(A) = \sum_{B \subseteq A} m(B), \end{aligned}$$

and

$$\begin{aligned} Pl : \quad & 2^\Theta \to [0,1], \\ & A \to Pl(A) = \sum_{A \cap B \neq \phi} m(B). \end{aligned}$$

For all sets $A \subseteq \Theta$, $Bel(A)$ is the total mass of evidence attributed by m to the subsets of A and $Pl(A)$ is the maximum degree of evidence that can be assigned to A [21].

We note that for all $A \subseteq \Theta$

$$Pl(A^c) = 1 - Bel(A),$$
$$Bel(A) \leqslant Pl(A).$$

When all focal elements are singletons, the bpa become a probability distribution Pr and for all $A \subseteq \Theta$

$$Pl(A) = Bel(A) = Pr(A).$$

A probability distribution P dominates the belief function Bel if and only if for all $A \subseteq \Theta$, $Bel(A) \leqslant P(A)$. If a probability distribution dominates the belief function, then the probability is dominated by the plausibility function [25]

$$Bel(A) \leqslant P(A) \leqslant Pl(A) \quad \text{for all } A \subseteq \Theta.$$

Let us denote by π the set of all probability distribution dominating the belief function Bel

$$\pi = \left\{ \begin{array}{c} P := (p_1, \ldots, p_N)^t \in \Re^N : P(A) \geqslant Bel(A) \text{ for all } A \subseteq \Theta \\ \sum_{j=1}^{N} p_j = 1; \ p_j \geqslant 0, \quad j = 1, \ldots, N \end{array} \right\}.$$

$\pi$is a polyhedral bounded set of $\Re^N$ [15] and for all $A \subseteq \Theta$

$$Bel(A) = \operatorname*{Inf}_{P \in \pi} P(A).$$

Let us also denote by π_Σ the set of all extreme points of π. π_Σ is finite [7].

From all probability distributions in π a special interest was given to the pignistic probability distribution ($BetP$). It was used to replace the Bel when a decision is required. In the TBM, we derive the $BetP$ according to the following transformation

$$BetP(w) = \sum_{A \subseteq \Theta : w \in A} \frac{1}{|A|} \frac{m(A)}{(1 - m(\phi))},$$

where $|A|$ is the number of element of Θ in A [2]. We note that the above transformation is based only on the given bpa m. Therefore, the way in which the $BetP$ is obtained is not related to the structure of the decision problem but to the structure of the given bpa.

We conclude this section by the following proposition:

Proposition 1. *For any real valued mapping u and a bpa m over Θ, the minimum (resp. the maximum) over π of the mathematical expectation*

$$E_P(u) = \sum_{w \in \Theta} u(w) \ P(w)$$

is attained for some $P_S \in \pi_\Sigma$ [7].

3. Belief linear program

We define the belief linear program (BLP) as follows:

$$\begin{array}{ll} Min & c(w)x \\ s.t. & T(w)x - h(w) \geqslant 0, \\ & x \in X_0, \end{array} \tag{1}$$

where $X_0 = \{x \in IR^n : A_0 x = b_0, x \geqslant 0\}$ is the set of deterministic constraints with A_0 is $m_0 \times n$ matrix and b_0 is m_0 vector; $c(w)$, $T(w)$ and $h(w)$ are matrices of respective dimension $(1 \times n)$, $(m \times n)$ and $(m \times 1)$ depending on the discrete random vector $w \in \Theta$.

We note that in this paper, we deal with case where the set of mutually exclusive states of nature $\Theta = \{w_1, \ldots, w_N\}$ is discrete and finite, and where the dependence is defined by linear relations as follows:

$$T(w) = T + \sum_{i=1}^{N} T_i w_i,$$

$$h(w) = h + \sum_{i=1}^{N} h_i w_i,$$

where T, T_i are $(m \times n)$ deterministic matrices and h, h_i are $(m \times 1)$ deterministic matrices, $i = 1, \ldots, N$. The known information on the possible state of nature is described by a belief function Bel.

As in stochastic programming, we suppose that our decision has no influence on the belief function of the uncertain parameters in the BLP problem (1).

Example. An olive oil company ROSBINA produces two kinds of olive oil, virgin oil and normal oil, from two varieties of olive, "Chemlali" and "Chetoui". The following table illustrates the production rates (barrel per ton) of the two kinds of olive oil from the two varieties of olive:

	Virgin oil	Normal oil
Chemlali variety	2	4
Chetoui variety	6	3

ROSBINA has a production capacity of 100 tons of olives. It needs to satisfy an uncertain demand on virgin and normal olive oil. Production costs are also uncertain and are functions of the manpower cost, storage cost and olive production levels. For simplicity, we suppose that demand and production costs depend on three possible states of the market w_1, w_2, w_3:

State of the market, w	w_1	w_2	w_3
Demand on virgin oil (per barrel), $h_1(w)$	160	200	150
Demand on normal oil (per barrel), $h_2(w)$	195	135	120
Production cost of one ton of the Chemlali variety, $c_1(w)$	2	2.2	1.4
Production cost of ton of the Chetoui variety, $c_2(w)$	3	2.9	3.3

For the last 24 years, ROSBINA manager's has a partial historical data about the state of the market w. He knows that the state of the market was w_1 for six years, w_2 for ten years, w_3 for two years and either w_1 or w_2 for three years. He ignores the state of the market during three of the 24 years. We characterize the olive oil producer knowledge by the following bpa m:

w	w_1	w_2	w_3	$\{w_1, w_2\}$	Θ
m	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{8}$

where $m(\{w_1, w_2\}) = \frac{1}{8}$ and $m(\Theta) = \frac{1}{8}$ because of the missing data during six years.

The derived belief function is given as follows:

w	w_1	w_2	w_3	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_2, w_3\}$	Θ
Bel	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{19}{24}$	$\frac{1}{3}$	$\frac{1}{2}$	1

The set π of all probability distribution dominating the belief function *Bel* is defined by:

$$\pi = \left\{ P := (p_1, p_2, p_3) : \sum_{j=1}^{3} p_j = 1;\ p_1 \geqslant \frac{1}{4}, p_2 \geqslant \frac{5}{12}, p_3 \geqslant \frac{1}{12}, p_1 + p_2 \geqslant \frac{19}{24} \right\}.$$

The ROSBINA olive oil production problem can be written as a BLP problem:

$$\begin{aligned} Min \quad & c_1(w)x_1 + c_2(w)x_2 \\ s.t \quad & 2x_1 + 6x_2 \geqslant h_1(w), \\ & 4x_1 + 3x_2 \geqslant h_2(w), \\ & x_1 + x_2 \leqslant 100, \\ & x_1 \geqslant 0,\ x_2 \geqslant 0. \end{aligned} \tag{2}$$

4. Stochastic programming

A first attempt to solve the BLP problem is to derive the pignistic probability distribution from *Bel* and then transform the BLP problem into a stochastic program.

This section will present main concepts of stochastic programming as we will be using them in solving the BLP.

We note that a stochastic program can be viewed as a BLP problem where the belief function is a probability distribution.

In the literature, to solve a stochastic program, we need to propose an equivalent mathematical optimization model to the stochastic program under predefined approaches. This equivalent optimization model is called certainty equivalent. The main approaches in stochastic programming are the chance constrained approach and the recourse approach.

4.1. Chance constrained approach

In the chance constrained approach, the certainty equivalent is obtained by optimizing the expected value of the random objective function; subject to all feasible decisions for deterministic constraints respecting the uncertain constraints with a given probability level [6].

Example. The *BetP* of the TBM model is obtained by dividing uniformly the mass of ignorance $\frac{1}{8}$ over all the elementary events and the mass:

w	w_1	w_2	w_3
BetP	$\frac{17}{48}$	$\frac{25}{48}$	$\frac{3}{24}$

As noticed in the previous section, the pignistic distribution is obtained using only the information structure of *Bel* and is not related to the BLP problem itself.

Under a chance constrained approach, the resulting certainty equivalent is as follows

$$\begin{array}{ll} Min & E_{BetP}[c_1(w)x_1 + c_2(w)x_2] \\ s.t & BetP\begin{pmatrix} 2x_1 + 6x_2 \geqslant h_1(w) \\ 4x_1 + 3x_2 \geqslant h_2(w) \end{pmatrix} \geqslant \alpha, \\ & x_1 + x_2 \leqslant 100, \\ & x_1 \geqslant 0,\ x_2 \geqslant 0, \end{array} \tag{3}$$

where $\alpha \in [0,1]$ is the probability (or reliability) level and represents (at least) the satisfaction degree on the realization of the uncertain constraints, and $E_{BetP}[c_1(w)x_1 + c_2(w)x_2]$ is the expected value of the random objective function. The problem (3) is called the chance constrained program.

4.2. Recourse approach

In the recourse approach, any shortage in the uncertain constraints generates a penalty cost in the objective function via the recourse function. The optimal decision provides the minimum long run average of the objective function in the stochastic program augmented by the recourse function, among all feasible solutions of the deterministic constraints [13].

Example. To illustrate the recourse approach, let us suppose that when oil demand is not satisfied, ROSBINA should buy the quantity of shortage from the market with a price higher than the production cost. Let us denote by $q_1(w)$ and $q_2(w)$ the price of one barrel from virgin oil and normal oil, respectively. These prices depend on the state of the market and are given by the following table:

w	w_1	w_2	w_3
$q_1(w)$	24	23	25
$q_2(w)$	18	19	17

ROSBINA seeks to minimize the expected cost of production augmented by the expected cost of the recourse decision.

Under a recourse approach, the certainty equivalent program to the problem (2) is:

$$\begin{array}{ll} Min & E_{BetP}[c_1(w)x_1 + c_2(w)x_2 + Q(x,w)] \\ s.t & x_1 + x_2 \leqslant 100, \\ & x_1 \geqslant 0,\ x_2 \geqslant 0, \end{array} \tag{4}$$

where

$$\begin{array}{lll} Q(x,w_i) = Min & q_1(w_i)y_{1i} + q_2(w_i)y_{2i} \\ & s.t & y_{1i} - y_{3i} = h_1(w_i) - 2x_1 - 6x_2, \\ & & y_{2i} - y_{4i} = h_2(w_i) - 4x_1 - 3x_2, \\ & & y_{1i} \geqslant 0,\ y_{2i} \geqslant 0,\ y_{3i} \geqslant 0,\ y_{4i} \geqslant 0 \end{array}$$

with y_{1i}, y_{2i} are the quantity of shortage in demands and y_{3i}, y_{4i} are the levels of overproduction when scenario w_i occurs. In this case, whatever the first-stage decision (x_1,x_2) and the state of the market w_i the recourse problem $Q(x_1,x_2,w_i)$ will be feasible [13]. The problem (4) is called the recourse program.

5. Belief linear programming

Another strategy to solve the problem (1) is to extend stochastic programming solution strategy and then provide a certainty equivalent to the BLP problem (1). In this following, we propose to generalize the chance constrained approach and the recourse approach to deal with the BLP problem.

5.1. Belief constrained approach

5.1.1. Introduction

In this section, we propose to adapt the chance constrained approach to the BLP problem. We call the resulting approach the belief constrained approach.

Under a belief constrained approach, we suppose that:

(1) The DM may accept solutions which are feasible within a given reliability level $\alpha \in [0,1]$. Therefore, a decision $x \in X_0$ is feasible, if and only if for all probability distribution P dominating the belief function *Bel*, $P[T(w)x - h(w) \geqslant 0] \geqslant \alpha$. This restriction is satisfied if $Bel[T(w)x - h(w) \geqslant 0] \geqslant \alpha$.

(2) The DM is pessimistic as he aims to optimize an expected value of the uncertain objective function while he is aware of the worst value of the objective function. The DM tends to minimize the worst expected value of the objective function regarding all probability distributions dominating the belief function *Bel*:

$$Z(x) = \underset{P \in \pi}{Max}\ E_P[c^t(w)x].$$

The equivalent program for the BLP problem (1), under a belief constrained approach, can be written as follows:

$$\begin{aligned} &Min\ \underset{P\in\pi}{Max}\ E_P[c^t(w)x] \\ &s.t\quad Bel[T(w)x - h(w) \geqslant 0] \geqslant \alpha, \\ &\qquad\ x \in X_0. \end{aligned} \tag{5}$$

We call the problem (5) the (joint) belief constrained program. The word joint comes from the fact that we consider one reliability level to all uncertain constraints. In the case, where we set different reliability levels $\alpha^i \in [0,1]$ for each constraint $T_i(w)x - h_i(w) \geqslant 0$, $i = 1,\dots,m$, with $T_i(w)$ is the i^{th} row of the matrix $T(w)$ and $h_i(w)$ is the i^{th} component of the vector $h(w)$, the equivalent program to the BLP problem (1) can be written as:

$$\begin{aligned} &Min\ \underset{P\in\pi}{Max}\ E_P[c^t(w)x] \\ &s.t\quad Bel[T_i(w)x - h_i(w) \geqslant 0] \geqslant \alpha^i, \quad i = 1,\dots,m, \\ &\qquad\ x \in X_0. \end{aligned} \tag{6}$$

We call the problem (6) the separate belief constrained program.

Compared to the chance constrained approach, in the belief constrained approach, we added the hypothesis that the DM is pessimistic. If it is not the case and the DM is risk taker or optimistic then the resulting certainty equivalent program is

$$\begin{aligned} &Min\ \underset{P\in\pi}{Min}\ E_P[c^t(w)x] \\ &s.t\quad Bel[T(w)x - h(w) \geqslant 0] \geqslant \alpha, \\ &\qquad\ x \in X_0. \end{aligned}$$

Also, we can follow Strat [24] and consider a moderate DM attitude and suggest optimizing a convex combination of the two bound $Max_{P\in\pi}E_P(\cdot)$ and $Min_{P\in\pi}E_P(\cdot)$:

$$\begin{aligned} &Min\ \lambda \underset{P\in\pi}{Max}\ E_P[c^t(w)x] + (1-\lambda)\underset{P\in\pi}{Min}\ E_P[c^t(w)x] \\ &s.t\quad Bel[T(w)x - h(w) \geqslant 0] \geqslant \alpha, \\ &\qquad\ x \in X_0, \end{aligned}$$

where $\lambda \in [0,1]$. In the following, we focus only on the case of a pessimistic DM.

The EBDO problem [17] is a belief constrained program with a deterministic objective function and where a plausibility function is used instead of a belief function. This is due to the fact that chance constraint characterizes the probability of failure and that this probability should be small. Therefore the chance constraint must be written using a plausibility function

$$Pl[T(w)x - h(w) \geqslant 0] \leqslant \alpha$$

to ensure that for all probability distribution P dominating the belief function *Bel*, $P[T(w)x - h(w) \geqslant 0] \leqslant \alpha$.

The solution strategy proposed by Mourelatos and Zhou [18] to the EBDO problem is specific and cannot be extended, in its actual form, to solve the belief constrained program. In the following, we will try to extend results from stochastic programming literature to solve certainty equivalent programs (5) and (6). But due to the nonlinear form of the belief constraint, we are going in the next subsection to verify the convexity of these certainty equivalents. We note that convexity is an important assumption in optimization without it optimality is not guaranteed [13].

5.1.2. Convexity of the belief constrained program

Let us denote by $X(\alpha)$ the set of feasible solutions of the belief constrained program (5):

$$X(\alpha) = \{x \in X_0 : Bel[T(w)x - h(w) \geqslant 0] \geqslant \alpha\}.$$

We note that if for all $i = 1,\dots,m$, the set

$$X_i(\alpha_i) := \left\{x \in X_0 : Bel[T(w)x - h(w) \geqslant 0] \geqslant \alpha^i\right\}$$

is convex, then the set of feasible solutions in the separate belief constrained program (6) is also convex. Therefore, we are interested in the remaining part of this section on the convexity of the set $X(\alpha)$.

Proposition 2.

$$X(\alpha) = \cap_{P\in\pi_\Sigma}\{x \in X_0 : P[T(w)x - h(w) \geqslant 0] \geqslant \alpha\}.$$

Proof. If π_Σ, the set of extreme probability distributions, is finite then any probability distribution $P \in \pi$ can be written as the convex combination of the elements from π_Σ, such that for all $A \subset \Theta$:

$$P(A) = \sum_{P_i\in\pi_\Sigma} \lambda_i\, P_i(A),$$

where $\lambda_i \geqslant 0$, $i = 1,\dots,|\pi_\Sigma|$ and $\sum_{P_i\in\pi_\Sigma}\lambda_i = 1$. □

If for all $P_i \in \pi_\Sigma$, $P_i[T(w)x - h(w) \geqslant 0] \geqslant \alpha$ then for all $\lambda_i \geqslant 0$, $i = 1,\dots,|\pi_\Sigma|$, $\sum_{P_i\in\pi_\Sigma}\lambda_i = 1$, we have $\sum_{P_i\in\pi_\Sigma}\lambda_i P_i[T(w)x - h(w) \geqslant 0] \geqslant \alpha$. Hence, for all $P \in \pi$, $P[T(w)x - h(w) \geqslant 0] \geqslant \alpha$. Therefore

$$Bel(T(w)x - h(w) \geqslant 0) = \operatorname*{Inf}_{P\in\pi} P(T(w)x - h(w) \geqslant 0) \geqslant \alpha.$$

In other words, we have

$$\cap_{P\in\pi_\Sigma}\{x \in X_0 : P[T(w)x - h(w) \geqslant 0] \geqslant \alpha\} \subset X(\alpha).$$

Reciprocally, if $Bel[T(w)x - h(w) \geqslant 0] \geqslant \alpha$ then for all $P \in \pi$, $P[T(w)x - h(w) \geqslant 0] \geqslant \alpha$. As $\pi_\Sigma \subset \pi$, then we have for all $P_i \in \pi_\Sigma$, $P_i[T(w)x - h(w) \geqslant 0] \geqslant \alpha$. In other words, we have

$$X(\alpha) \subset \cap_{P\in\pi_\Sigma}\{x \in X_0 : P[T(w)x - h(w) \geqslant 0] \geqslant \alpha\}.$$

Then, we conclude that for a given reliability level $\alpha \in [0,1]$, we have

$$X(\alpha) = \cap_{P\in\pi_\Sigma}\{x \in X_0 : P[T(w)x - h(w) \geqslant 0] \geqslant \alpha\}.$$

Based on the previous proposition and on the fact that the intersection of convex sets yields to a convex set, then if for all $P \in \pi_\Sigma$ the set $\{x \in X_0: P[T(w)x - h(w) \geqslant 0] \geqslant \alpha\}$ is convex then $X(\alpha)$ is also convex.

Usually, it is not easy to verify the convexity of the set $\{x \in X_0: P[T(w)x - h(w) \geqslant 0] \geqslant \alpha\}$ for all $P \in \pi_\Sigma$. The following result provides conditions on m to ensure the convexity of $X(\alpha)$.

Proposition 3. *If $m(w_j) > 0$, $j = 1,\dots,N$ and the reliability level verifies*

$$\alpha > 1 - \operatorname*{Min}_{j=1,\dots,N} m(w_j),$$

then the set of feasible solution $X(\alpha)$ is convex.

Proof. We can notice that $X(1)$ is a convex subset of $\Re^n$

$$X(1) = \{x \in X_0 : Bel[T(w)x - h(w) \geqslant 0] \geqslant 1\} = \{x \in X_0 : T(w)x - h(w) \geqslant 0\}.$$

To prove the above theorem, we have to establish that $X(\alpha) = X(1)$.

We can easily see that $X(1) \subset X(\alpha)$ for all $\alpha \in [0,1]$.

Let $\alpha > 1 - Min_{j=1,\dots,N} m(w_j)$. We know that

$$Bel(\Theta \setminus \{w_i\}) + Bel(w_i) = \sum_{B\subseteq\Theta\setminus\{w_i\}} m(B) + m(w_i) \leqslant 1,$$

then

$$Bel(\Theta \setminus \{w_i\}) \leqslant 1 - \underset{i=1,\dots,N}{Min}\ Bel(w_i) < \alpha.$$

For all $A \subset \Theta (A \neq \Theta)$, there exist $w_i \in \Theta$, such that $A \subset \Theta \setminus \{w_i\}$. As Bel is a monotonic function over the power set of Θ, then for all $A \subset \Theta$, $Bel(A) < \alpha$.

Let $x \in X(\alpha)$.

If $Bel[T(w)x - h(w) \geqslant 0] \geqslant \alpha$, then $\{w: T(w)x - h(w) \geqslant 0\} = \Theta$. Hence, $X(\alpha) \subset X(1)$.

We conclude that $X(\alpha) = X(1)$.

Note that in the case of $Min_{j=1,\dots,N} m(w_j) = 0$ then the convexity of the set of feasible solution is guaranteed only for $\alpha = 1$. Therefore, the Proposition 3 may be useful to assert the convexity of the set of feasible solutions for the case where focal elements are singleton ($m(w_j) > 0$). □

Following research progress in stochastic programming where for some type of probability distributions (for example quasi-concave distribution [13]) the convexity of the chance constraint program is proved, further studies are needed to characterize under which assumptions the convexity of the belief constrained program is obtained.

In the next section, we discuss a solution strategy for the belief constrained program (5) under the hypothesis that the set of feasible solution of program (5) is convex.

5.1.3. Solution strategy for the belief constrained program

In the following, we consider only the case where $T(w) = T$ and $h(w) = w$.

To solve the problem (5), we propose first to rewrite the set of feasible solutions $X(\alpha) = \{x \in X_0\text{: } Bel[w \leqslant Tx] \geqslant \alpha\}$ using linear constraints.

Let us define, for each elementary event $w_j \in \Theta$, $j = 1,\dots,N$, the (lower) belief cumulative distribution [26]

$$F(w_j) = Bel(w \leqslant w_j).$$

To express the set $X(\alpha)$, by linear constraints, we extend the notion of p-level efficient points (pLEP), proposed by Prékopa [19] for the probability cumulative distribution, to the case of the belief cumulative distribution:

Definition. An event $z \in \Theta$ is called pLEP event for the belief cumulative distribution F, if $F(z) \geqslant p$ and there is no $y \in \Theta$ satisfying $y \leqslant z$, $y \neq z$ and $F(y) \geqslant p$.

Prékopa et al. [20] presented an algorithm to determine pLEP events for any cumulative measures. We propose to use this algorithm to determine the pLEP events z^i, $i = 1,\dots,s$, for the belief cumulative distribution F, where s is the number of pLEP events. Therefore, the set of feasible solutions $X(\alpha)$ may be written as follows

$$X(\alpha) = \{x \in X_0 : Tx \geqslant z^i \text{ for at least one } z^i, \quad i = 1,\dots,s\}. \tag{7}$$

To represent the set of feasible solutions $X(\alpha)$ by linear constraints, we propose the following relaxation of the set (7):

$$X(\alpha) = \left\{ x \in X_0 : Tx \geqslant \sum_{i=1}^{s} \mu_i\, z^i, \sum_{i=1}^{s} \mu_i = 1, \mu_i \geqslant 0, \quad i = 1,\dots,s \right\}.$$

The belief constrained program can be rewritten as follows:

$$\underset{x \in X(\alpha)}{Min}\ \underset{P \in \pi}{Max}\, E_P[c^t(w) \cdot x]. \tag{8}$$

Let us denote by $Z(x) = Max_{P \in \pi} E_P[c^t(w) \cdot x]$ the objective function of the problem (8). According to Proposition 1, we have $Z(x) = \underset{P \in \pi_S}{Max}\ E_P[c^t(w) \cdot x]$. Therefore $Z(x)$ is the maximum of a finite number of linear expressions and then $Z(\cdot)$ is piecewise linear convex. Based on this property, we propose the following cutting plane algorithm to solve the program (8):

Step 0: Set $k = s = 0$
Step 1: Set $k = k + 1$. Solve the following linear program

$$\begin{aligned} & Min\ \theta \\ & \qquad E_l\, x + \theta \geqslant 0,\ l = 1,\dots,s, \\ & \qquad x \in X(\alpha),\ \theta \in IR. \end{aligned} \tag{9}$$

Let (x^k, θ^k) be an optimal solution. For the first step, no constraint $E_l x + \theta \geqslant 0$ is present, then θ^k is set equal to $-\infty$ and x^k is chosen arbitrary from the set X. Go to step 2.
Step 2: Solve the following linear program

$$Z(x^k) = \underset{P \in \pi}{Max} \sum_{i=1}^{N} p_i \left[c^t(w_i).x^k\right]. \tag{10}$$

Let $P^k = (p_1^k, \dots, p_N^k)^t$ be an optimal solution of the problem (10). If $\theta^k \geqslant Z(x^k)$, stop; x^k is an optimal solution. Otherwise, let $E_{s+1} = -\sum_{i=1}^{N} p_i^k [c^t(w_i)]$, add the following cut $E_{s+1}x + \theta \geqslant 0$ to the program (9) constraints, set $s = s + 1$, and return to step 1.

Theorem. *The above algorithm finitely converges to an optimal solution when it exists.*

Proof. The problem (9) is equivalent to the following program:

$$\begin{aligned} &Min\ \theta\\ &s.t.\quad Z(x) \leqslant \theta,\\ &\qquad\ \ x \in X(\alpha),\ \ \theta \in \Re. \qquad \square \end{aligned}$$

The convergence of the above algorithm can be concluded based on the following points:

- The set π_Σ is finite. The number of cuts added to the problem (9) is finite. These cuts are supporting hyperplanes of $Z(x)$.
- If (x^k,θ^k) is the optimal solution of the problem (9) then $Z(x^k) = \theta^k$. Therefore, if (x^k,θ^k) is the optimal solution then $\theta^k = \sum_{i=1}^N p_i^k [c^t(w_i)x^k]$ corresponds to the optimal stopping criteria of the algorithm.

5.1.4. Example

Let us return to the hypothesis that ROSBINA must satisfy the demand with a reliability level $\alpha = 0.95$ and let us suppose that the company is aware of the worst value of the objective function. Under a belief constrained approach, the certainty equivalent program to the BLP problem (2) is

$$\begin{aligned} &Min\ \underset{P\in\pi}{Max}\ E_P[c_1(w)x_1 + c_2(w)x_2]\\ &s.t\quad Bel\begin{bmatrix} 2x_1 + 6x_2 \geqslant h_1(w)\\ 4x_1 + 3x_2 \geqslant h_2(w)\end{bmatrix} \geqslant 0.95,\\ &\qquad x_1 + x_2 \leqslant 100,\\ &\qquad x_1 \geqslant 0 \quad x_2 \geqslant 0. \end{aligned} \tag{11}$$

As $0.95 > 1 - Min_{j=1,\dots,N} Bel(\{w_j\})$ then based on the Proposition 3, the belief constrained program (11) is convex.

We can easily verify that we have an unique pLEP event $\begin{pmatrix}200\\195\end{pmatrix}$ for the belief cumulative distribution of *Bel*. Therefore, problem Eq. (11) can be rewritten as follows:

$$\begin{aligned} &Min\ \underset{P\in\pi}{Max}\ E_P[c_1(w)x_1 + c_2(w)x_2]\\ &s.t\quad 2x_1 + 6x_2 \geqslant 200,\\ &\qquad 4x_1 + 3x_2 \geqslant 195,\\ &\qquad x_1 + x_2 \leqslant 100,\\ &\qquad x_1 \geqslant 0 \quad x_2 \geqslant 0. \end{aligned} \tag{12}$$

The table below summarizes the different steps of the cutting plane algorithm for problem (12) with an initial solution equal to $(0,65)$:

k	θ^k	x_1^k	x_2^k	$Z(x^k)$	P^k	E_{s+1}
1	$-\infty$	0	65	133.34	$(\frac{1}{4},\frac{2}{3},\frac{1}{12})$	$(-2.083,-2.958)$
2	133.34	31.667	22.778	133.34	$(\frac{1}{4},\frac{2}{3},\frac{1}{12})$	

The obtained optimal solution is $(31.67,22.78)$ with an optimal value of 133.34. This optimal solution is achieved with a probability distribution $\overline{P} = (\frac{1}{4},\frac{2}{3},\frac{1}{12})$.

5.2. Recourse approach

5.2.1. Introduction

We propose to extend the stochastic programming recourse approach to the BLP problem as follows:

(1) the DM accepts that any violation in the uncertain constraints generates a penalty cost in the objective function;
(2) the DM aims to optimize an expected value of the uncertain objective augmented by the penalty cost;
(3) the DM is pessimistic as he aims to optimize an expected value of the uncertain objective function while he is aware of the worst value of the objective function. The DM tends to minimize the worst expected value of the objective function regarding all probability distributions dominating the belief function *Bel*.

Therefore, if a feasible solution $x \in X_0$ does not satisfy the uncertain constraints, then we penalize such a solution by introducing an additional cost function, called the recourse function, as follows:

$$\begin{aligned} Q(x,w) = Min \quad & q(w)^t y \\ s.t \quad & W(w)y = h(w) - T(w)x, \\ & y \geqslant 0, \end{aligned}$$

where $q(w)$ is the recourse cost, $W(w)$ is the recourse matrix and y is the recourse decision. As the DM is pessimistic, then he tends to minimize the worst expected cost, i.e.:

$$R(x) = \underset{P\in\pi}{Max}\ E_P[c^t(w)x + Q(x,w)].$$

The obtained certainty equivalent is called recourse program and can be written as:

$$\underset{x\in X_0}{Min}\ \underset{P\in\pi}{Max}\ E_P[c^t(w)x + Q(x,w)]. \tag{13}$$

5.2.2. Solution strategy for the recourse program

For a fixed w, $Q(x,w)$ is linear convex function of x [4]. Therefore, for a given probability distribution P of π, $E_P[c^t(w)x + Q(x,w)]$ is piecewise linear convex function of x. The function

$$R(x) = \underset{P\in\pi}{Max}\ E_P[c^t(w)x + Q(x,w)] = \underset{P\in\pi_\Sigma}{Max}\ E_P[c^t(w)x + Q(x,w)]$$

is the maximum of a finite number of piecewise linear convex functions, then $Z(\cdot)$ is piecewise linear convex. We conclude that the recourse problem (13) is a minimax problem with a piecewise linear convex objective function.

Ben Abdelaziz and Masri [3] proposed a modified L-shaped algorithm to solve problems with the same structure as the problem (13). The modified L-shaped algorithm is a cutting plane algorithm. It generates cuts to approximate the piecewise linear convex function $R(x)$. These cuts are of two kinds: feasibility cuts, that ensure the feasibility of the obtained solution, and optimality cuts, that outer linearize the function $R(x)$. A more detailed description of the modified L-shaped algorithm may be found in [3].

5.2.3. Example

Let us return to the hypothesis that when oil demand is not satisfied, ROSBINA should buy the quantity of shortage from the market with a price higher than the production cost.

We add the hypothesis that ROSBINA is pessimistic. Under a recourse approach, the certainty equivalent program to the BLP problem (2) is:

$$\begin{aligned} & Min\ \underset{P\in\pi}{Max}\ E_P[c_1(w)x_1 + c_2(w)x_2 + Q(x,w)] \\ & s.t \quad x_1 + x_2 \leqslant 100, \\ & \qquad x_1 \geqslant 0 \quad x_2 \geqslant 0. \end{aligned} \tag{14}$$

The table below summarizes the different steps of the modified L-shaped method for problem (14):

k	θ^k	x_1^k	x_2^k
1	$-\infty$	20	20
2	-4479.33	0	100
3	128.58	0	43.47
4	131.26	17.91	31.76
5	133.31	31.62	22.8

The optimal solution is (31.62,22.8) with an optimal value of 133.31. This optimal solution is achieved with a probability distribution $\overline{P} = (\frac{1}{2}, \frac{5}{12}, \frac{1}{12})$.

6. Conclusion

In this paper, we extended stochastic programming approaches to solve the belief linear program (BLP). These approaches are the belief constrained approach and the recourse approach. The belief constrained approach may be used in the case where partially feasible solutions may be considered. The recourse approach deals with situations where infeasible solutions for some scenarios are considered subject to a penalty cost added to the objective function. These approaches lead to certainty equivalent programs, namely the belief constrained program and the recourse program.

For the belief constrained program, we proved some convexity results. Further research is needed to find types of belief function under which the convexity is guaranteed. In the case of a convex belief constrained program, we presented a solution strategy based on the concept of pLEP and a cutting plane algorithm. For the recourse program, we proposed to solve this program using the modified L-shaped algorithm.

Future research may be conducted to enhance proposed solution strategies, to deal with other optimization problems as for example the multiple objective case.

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