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A structure theorem for posets admitting a "strong" chain partition: A generalization of a conjecture of Daykin and Daykin (with connections to probability correlation inequalities)

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Abstract

Suppose a finite poset P is partitioned into three non-empty chains so that, whenever $p, q \in P$ lie in distinct chains and $p < q$, then every other element of P is either above p or below q.

In 1985, the following conjecture was made by David Daykin and Jacqueline Daykin: such a poset may be decomposed into an ordinal sum of posets $\bigoplus_{i=1}^{n} R_i$ such that, for $1 \leqslant i \leqslant n$, one of the following occurs:

(1) R_i is disjoint from one of the chains of the partition; or
(2) if $p, q \in R_i$ are in distinct chains, then they are incomparable.

The conjecture is related to a question of R. L. Graham's concerning probability correlation inequalities for linear extensions of finite posets.

In 1996, a proof of the Daykin–Daykin conjecture was announced (by two other mathematicians), but their proof needs to be rectified.

In this note, a generalization of the conjecture is proven that applies to finite or infinite posets partitioned into a (possibly infinite) number of chains with the same property. In particular, it is shown that a poset admits such a partition if and only if it is an ordinal sum of posets, each of which is either a width 2 poset or else a disjoint sum of chains. A forbidden subposet characterization of these partial orders is also obtained.

MSC: 06A07; 06A06; 05A18

Keywords: (Partially) ordered set; Correlation inequality; Linear extension; Dilworth's Theorem; Ordinal sum; Partition; Chain

1. The Conjecture of Daykin and Daykin

This paper began as a dare.[1]

If one wants to understand a large ordered set, it helps if one can decompose it into smaller parts that have a simpler structure. For example, one can build series–parallel posets from one-element posets using the operations of disjoint

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[1] For details, consult Gerhard Paseman.

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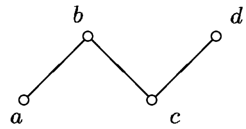

Fig. 1. The four-element fence F_4.

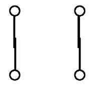

Fig. 2. The poset $\mathbf{2} + \mathbf{2}$.

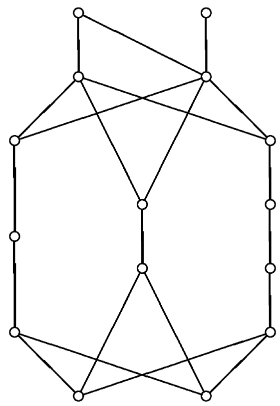

Fig. 3. A poset satisfying the hypotheses of the Daykin–Daykin conjecture.

sum (parallel composition) and ordinal sum (series composition). One of the most famous decomposition results for ordered sets is Dilworth's Theorem [5, Theorem 1.1]: it asserts that every partially ordered set that has no $(k+1)$-element *antichain* (a subset of pairwise incomparable elements) admits a partition into k or fewer *chains* (totally ordered subsets), where $k \in \mathbb{N}$. Both Dilworth's Theorem, series–parallel posets, and the so-called *interval orders* figure prominently in scheduling applications [1, Section 4; 7,10, pp. 268–269].

In many cases, the existence of the desired decomposition follows from the *non-existence* of certain subposets. For instance, for series–parallel posets, the forbidden subposet is the four-element fence F_4, in which $a < b > c < d$ and no other comparabilities hold (Fig. 1).

In the case of Dilworth's Theorem, the forbidden subposet is a $(k + 1)$-element antichain. In the case of interval orders, the forbidden subposet is the disjoint sum of two copies of the two-element chain $\mathbf{2}$ (Fig. 2).

In 1985, David Daykin and Jacqueline Daykin made a conjecture concerning posets that could be partitioned into chains in a particularly "strong" fashion—so that, whenever p and q belong to distinct chains, and $p < q$, then every other element of the poset is either greater than p or less than q.

Precisely, Daykin and Daykin conjecture in [3, Section 9] that, if a finite poset P admits a strong partition into three non-empty chains, then it admits a partition into posets R_i $(i = 1, \ldots, n)$ where every element of R_i is less than every element of R_j when $1 \leqslant i < j \leqslant n$, and where, for $i = 1, \ldots, n$, either

(1) R_i is disjoint from one of the original three chains; or else
(2) p and q are incomparable if $p, q \in R_i$ lie in distinct chains.

(See Fig. 3 for an example; let one chain be all the elements on the left; let another be all the elements on the right; and let the third chain be the remainder.)

The motivation for this conjecture is a question of Graham's [8, pp. 232–233]: it deals with probability correlation inequalities for linear extensions of finite posets that admit strong partitions into *two* subsets (not necessarily chains). We explain this question in Section 3. (Much of this material is repeated from [6].)

According to [4, p. 84], Robson proved the Daykin–Daykin conjecture using an inductive argument; but this fact was not publicized. Consequently, Tseng and Horng announced a proof in [12]. Their argument, however, rests on an invalid assertion, which we explain in Section 4. Moreover, their proof seems to rely on the fact that the poset is finite and that there are three (and no more than three) chains.

In [6], we proved a generalization of the conjecture for finite posets admitting a strong partition into n chains, where $n \geqslant 3$. (This paper also contains an interesting theorem of Bill Sands: a finite distributive lattice with an even number of elements can be partitioned into two-element chains.)

In this note, we present a generalization of the conjecture that applies to any poset (finite or infinite) admitting a strong partition into any number (finite or infinite) of chains. We show that any such poset is an ordinal sum, each summand of which is either a width 2 poset or else a disjoint sum of chains. Moreover, any such poset admits a strong chain partition. Finally, we show that such posets have a forbidden subposet characterization (Theorem 5.1 and Corollary 5.2).

2. Basic definitions and notation

For standard terminology and notation, see [2].

Let $^{\#}S$ denote the cardinality of a set S.

Let P be a poset. Elements $p, q \in P$ are *comparable* if $p \leqslant q$ or $q \leqslant p$; otherwise they are *incomparable*, denoted $p \| q$. A *chain* is a subset of pairwise comparable elements; an *antichain* is a subset of pairwise incomparable elements. The supremum of the cardinalities of antichains in P is the *width* of P.

For $p \in P$, let $\downarrow p = \{q \in P \mid q \leqslant p\}$ and let $\uparrow p = \{q \in P \mid q \geqslant p\}$. For $Q \subseteq P$, let $\downarrow Q = \bigcup_{q \in Q} \downarrow q$ and let $\uparrow Q = \bigcup_{q \in Q} \uparrow q$.

A partition of a poset P into chains is *strong* if, whenever $p, q \in P$ belong to distinct chains and $p < q$, then $P = \uparrow p \cup \downarrow q$. (Unless specifically stated otherwise, we will allow the chains in our strong partition to be empty.)

Let R and S be posets with disjoint underlying sets. The *disjoint sum* $R + S$ of R and S is the poset on underlying set $R \cup S$ such that $r \| s$ for all $r \in R$ and $s \in S$. The *ordinal sum* $R \oplus S$ of R and S is the poset on underlying set $R \cup S$ such that $r < s$ for all $r \in R$ and $s \in S$. (See Figs. 4 and 5.) One can similarly define disjoint and ordinal sums for a finite or infinite number of posets.

A poset P is $\oplus$*-indecomposable* if whenever $P = R \oplus S$, then either $R = \emptyset$ or $S = \emptyset$.

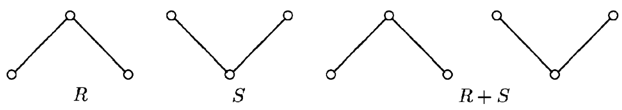

Fig. 4. The disjoint sum.

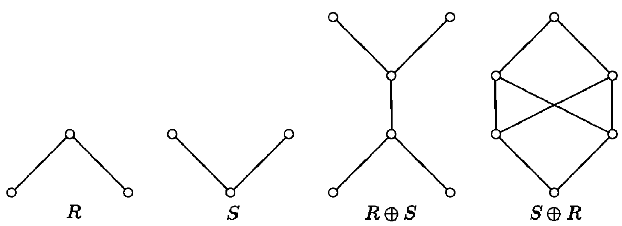

Fig. 5. The ordinal sum.

3. Probability correlation inequalities and a question of Graham: the parable of the tennis players

Our main theorem is a generalization of a conjecture of Daykin and Daykin that deals with special sorts of chain partitions. The motivation for the conjecture comes from a parable, due to Graham et al. [9, Section 1]:

Imagine that there are two teams of tennis players, A and B. The players of Team A are linearly ranked from best to worst, as are the players from Team B; but we know only in a few cases how individual players from Team A compare with individuals from Team B.

We may describe this set-up using a poset: the players are the elements, and a relation $p<q$ means that player p is worse than player q. Hence the subset corresponding to Team A is a chain, as is the subset corresponding to Team B (so that we have a poset of width at most 2).

In Fig. 6, we do not know whether Asmodeus (in Team A) is better or worse than Beezelbub (in Team B). We might ask for the *probability* that Asmodeus is worse than Beezelbub, Pr (Asmodeus<Beezelbub). We calculate this probability by looking at all the possible linear rankings of the players in both teams that are consistent with what we already know about which players are better than which.

Formally, we are looking at all the possible *linear extensions* of the poset, the bijective order-preserving maps from P into a chain of cardinality $^{\#}P$. By counting the number of these in which (the image of) Asmodeus is below (the image of) Beezelbub, and dividing by the total number of linear extensions (assuming all are equally likely), we have Pr (Asmodeus<Beezelbub).

Now suppose we are given additional information. Namely, suppose we learn that certain players from Team A are worse than certain players from Team B. (Perhaps the two teams have just finished playing a tournament.) This information supports the idea that the players from Team A are worse than the players from Team B, making it more likely that Asmodeus is, in fact, worse than Beezelbub.

Formally, we would expect:

$$\Pr(\text{Asmodeus} < \text{Beezelbub}) \leqslant \Pr(\text{Asmodeus} < \text{Beezelbub} | a < b \ \& \cdots \& \ a' < b'),$$

the conditional probability that Asmodeus is worse than Beezelbub given that a is worse than b, a' worse than b', etc. (See Fig. 7.)

We can formalize a more general situation. Let P be a poset partitioned into subsets A and B (not necessarily chains). Suppose that, whenever x and y are disjunctions of statements of the form

$$a < b \ \& \cdots \& \ a' < b'$$

we know that

$$\Pr(x)\Pr(y) \leqslant \Pr(x \& y).$$

Then we say the partition has the *positive correlation property*.

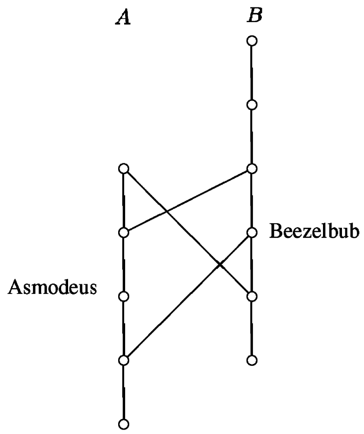

Fig. 6. A tale of two tennis teams.

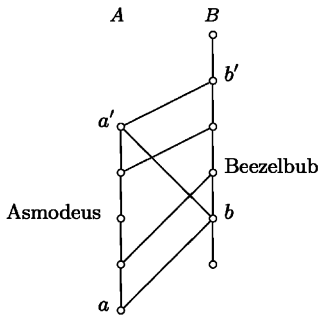

Fig. 7. The poset of tennis players after a tournament.

Using a result of Daykin and Daykin [3, Theorem 9.1], a question of Graham's [8, pp. 232–233] becomes:

Question (*Graham*). *Let $P = A \cup B$ be a partition of a finite poset such that, for all $a \in A$ and $b \in B$,*

$$a < b \text{ implies } P = \uparrow a \cup \downarrow b$$

and

$$a > b \text{ implies } P = \downarrow a \cup \uparrow b.$$

Does the partition have the positive correlation property?

4. The argument of Tseng and Horng

The article [12] attempts to prove the original conjecture of Daykin and Daykin. Both the fact that the poset is finite and the fact that the number of chains in the partition is 3 seem to be integrally used. (Hence, even if the argument were correct, it would resist generalization.)

Another integral component of the argument of [12] is a "lemma" that is used a number of times, but is false, as we now explain.

Let T_1, T_2, and T_3 be the chains in the strong partition of the finite poset P. For T_i ($1 \leqslant i \leqslant 3$), an equivalence relation is defined by

$$t \equiv t' \quad \text{if} \quad \uparrow t \backslash T_i = \uparrow t' \backslash T_i \quad \text{and} \quad \downarrow t \backslash T_i = \downarrow t' \backslash T_i \quad (t, t' \in T_i).$$

The blocks of the equivalence relation are denoted $T_i^1, \ldots, T_i^{m_i}$. A relation on the set of blocks for all three chains is defined by

$$T_i^j \equiv T_k^l \quad \text{if either} \quad \uparrow T_i^j \backslash T_i^j = \uparrow T_k^l \backslash T_k^l \quad \text{or} \quad \downarrow T_i^j \backslash T_i^j = \downarrow T_k^l \backslash T_k^l,$$

where $1 \leqslant i, k \leqslant 3$, $1 \leqslant j \leqslant m_i$, and $1 \leqslant l \leqslant m_k$. It is claimed in [12], Lemma 2.8 that this is an equivalence relation; let the unions of the "equivalence classes" be the subsets $M_1, \ldots, M_m$ of P.

In [12], the ordinal summands of P—$R_1, \ldots, R_n$—are supposed to be defined as follows. Initially let $R_1 = M_1$, and then iterate the following rule: If there exists some M_i, some $p \in M_i$, and some $r \in R_1$ such that $p \| r$, then extend R_1 to $R_1 \cup M_i$. Similarly, $R_2, \ldots, R_n$ are defined so that $P = \bigcup_{i=1}^n R_i$.

Example 4.1. Let P be the poset $\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c\}$ where $a_1 < a_2 < a_3 < a_4 < c$; $b_1 < b_2 < b_3 < b_4 < c$; $b_1 < a_2$; $b_2 < a_3$; $b_3 < a_4$; and no other comparabilities hold but the necessary ones (Fig. 8).

Let $T_1 = \{a_1, a_2, a_3, a_4\}$, $T_2 = \{b_1, b_2, b_3, b_4\}$, and $T_3 = \{c\}$.

Then $m_1 = 4$, $m_2 = 4$, $m_3 = 1$, and every T_i^j ($1 \leqslant i \leqslant 3$; $1 \leqslant j \leqslant m_i$) is a singleton (say, $T_1^j = \{a_j\}$, $T_2^j = \{b_j\}$, $T_3^1 = \{c\}$, $1 \leqslant j \leqslant 4$).

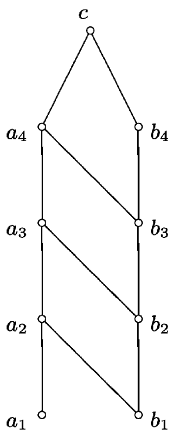

Fig. 8. The poset P.

We have

$$
\begin{aligned}
&\uparrow T_1^1 \backslash T_1^1 = \{a_2, a_3, a_4, c\},\\
&\uparrow T_1^2 \backslash T_1^2 = \{a_3, a_4, c\},\\
&\uparrow T_1^3 \backslash T_1^3 = \{a_4, c\},\\
&\uparrow T_1^4 \backslash T_1^4 = \{c\},\\
&\uparrow T_2^1 \backslash T_2^1 = \{a_2, a_3, a_4, b_2, b_3, b_4, c\},\\
&\uparrow T_2^2 \backslash T_2^2 = \{a_3, a_4, b_3, b_4, c\},\\
&\uparrow T_2^3 \backslash T_2^3 = \{a_4, b_4, c\},\\
&\uparrow T_2^4 \backslash T_2^4 = \{c\},\\
&\uparrow T_3^1 \backslash T_3^1 = \emptyset,\\
&\downarrow T_1^1 \backslash T_1^1 = \emptyset,\\
&\downarrow T_1^2 \backslash T_1^2 = \{a_1, b_1\},\\
&\downarrow T_1^3 \backslash T_1^3 = \{a_1, a_2, b_1, b_2\},\\
&\downarrow T_1^4 \backslash T_1^4 = \{a_1, a_2, a_3, b_1, b_2, b_3\},\\
&\downarrow T_2^1 \backslash T_2^1 = \emptyset,\\
&\downarrow T_2^2 \backslash T_2^2 = \{b_1\},\\
&\downarrow T_2^3 \backslash T_2^3 = \{b_1, b_2\},\\
&\downarrow T_2^4 \backslash T_2^4 = \{b_1, b_2, b_3\},\\
&\downarrow T_3^1 \backslash T_3^1 = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&M_1 = \{a_2\},\\
&M_2 = \{a_3\},\\
&M_3 = \{b_2\},\\
&M_4 = \{b_3\},\\
&M_5 = \{a_1, b_1\},\\
&M_6 = \{a_4, b_4\},\\
&M_7 = \{c\}.
\end{aligned}
$$

Thus $R_1 = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$ and $R_2 = \{c\}$.

The following statement is Lemma 3.1 in [12]; it is used a number of times in the "proof" of the Daykin–Daykin conjecture: *Suppose* $1 \leqslant i \leqslant n$ *and* $1 \leqslant j, k \leqslant m$. *Suppose that* $M_j, M_k \subseteq R_i$ *and* $a < b$ *for all* $a \in M_j$, $b \in M_k$.

Then for some $l \in \{1, \ldots, m\}$ *such that* $M_l \subseteq R_i$, *there exist* $a \in M_j$, $b \in M_k$, $c, d \in M_l$ *such that* $a \| c$ *and* $b \| d$.

Example 4.1 is a counterexample: let $i = 1$, $j = 3$, and $k = 2$.

5. Proof of a generalization of the Daykin–Daykin conjecture

Theorem 5.1. *Let P be a poset (finite or infinite). The following are equivalent:*

(1) *P admits a strong chain partition;*

($\sqrt{2}$) *P does not contain* $F_3 + \mathbf{1}$ *or its dual as a subposet (Fig. 9);*

(2) *P is an ordinal sum of posets, each of which either has width 2 or else is a disjoint sum of chains.*

Proof. First we prove that (1) implies ($\sqrt{2}$). For $p \in P$, let $T(p)$ denote the chain in the partition containing p. Assume for a contradiction that $F_3 + \mathbf{1} = \{a, b, c, x\}$ is a subposet of P, where $b < a > c$; $b \| c$; and $a, b, c \| x$ (Fig. 9).

Clearly $T(b) \neq T(c)$, so, without loss of generality, $T(a) \neq T(b)$. Hence $P = \downarrow a \cup \uparrow b$; but $x \| a, b$, a contradiction.

Now we show that ($\sqrt{2}$) implies (2). Represent P as an ordinal sum of $\oplus$-indecomposable posets. (Every ordinal sum decomposition of P corresponds to an equivalence relation; and the meet of all such equivalence relations still corresponds to an ordinal sum decomposition.)

Let R be one of the ordinal summands. If R does not have width 1 or 2, then it contains a maximal antichain $A \subseteq R$ such that ${}^{\#}A \geqslant 3$. We claim that $R = \sum_{a \in A} (\uparrow a \cup \downarrow a)$.

Suppose not, for a contradiction. Without loss of generality (if necessary, by choosing a new A), there exists $r \in R$ such that ${}^{\#}(A \cap \downarrow r) \geqslant 2$. By ($\sqrt{2}$), for any such r we must have $A \subseteq \downarrow r$.

Now let $T = \{t \in R \mid A \subseteq \downarrow t\}$ and let $S = R \backslash T$. If $s \nleqslant t$ for some $s \in S$, $t \in T$, then $A \cap \downarrow s = \{a\}$ for some $a \in A$, so, for some $b, c \in A$, $\{s, t, b, c\}$ is isomorphic to $F_3 + \mathbf{1}$, a contradiction (Fig. 10). Hence $R = S \oplus T$, and (since $T \neq \emptyset$) we have $S = \emptyset$. Thus $R = T$ and $A \subseteq T$, which is absurd, as ${}^{\#}A \geqslant 2$. The claim follows.

Finally, since ${}^{\#}A \geqslant 2$ and ($\sqrt{2}$) holds, it is clear that $\uparrow a$ and $\downarrow a$ must be chains for all $a \in A$.

The fact that (2) implies (1) follows from Dilworth's Theorem (applied to the width 2 summands). □

As a consequence, we get a generalization of the conjecture of Daykin and Daykin for infinite posets.

Corollary 5.2. *Let P be a (finite or infinite) poset admitting a strong chain partition, with* $\mathscr{T}$ *being the set of chains. Assume that* ${}^{\#}\mathscr{T} \geqslant 3$. *(Possibly* $\mathscr{T}$ *is infinite.) For* $p \in P$, *let* $T(p)$ *be the chain in* $\mathscr{T}$ *containing p.*

Then P is an ordinal sum of posets such that, for each summand R, either

(1) *R is disjoint from some* $T \in \mathscr{T}$; *or*

(2) *for all* $p, q \in R$ *such that* $T(p) \neq T(q)$, *we have* $p \| q$.

Proof. By the proof of Theorem 5.1, we may assume that each ordinal summand R is $\oplus$-indecomposable and either width 2 or else a disjoint sum of chains. In the latter case, (2) clearly holds. So assume R has width 2, with a maximal antichain $\{r, r'\}$.

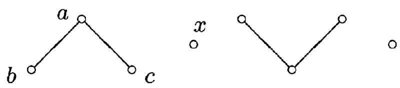

Fig. 9. Forbidden subposets.

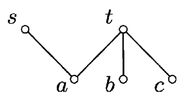

Fig. 10. An impossible scenario.

There is a unique chain $T \in \mathscr{T}$ containing an element incomparable to r. Let $U = T \cup T(r)$ and let

$$V = \{u \in U \mid \uparrow r \backslash U \subseteq \uparrow u \text{ and } \downarrow r \backslash U \subseteq \downarrow u\}.$$

Suppose $s \in P \backslash V$.

If $s \notin U$, then, without loss of generality, $r < s$, so $V \subseteq \downarrow s$, by definition of V.

If $s \in U$, then, without loss of generality, there exists $p \in P \backslash U$ such that $r < p$ but $s \nleqslant p$. Hence, for all $v \in V$, $v < p$, so that $v < s$. (The partition is strong.)

Therefore,

$$R = (R \cap \downarrow r \backslash V) \oplus (R \cap V) \oplus (R \cap \uparrow r \backslash V),$$

so that $R \subseteq V \subseteq U$. As ${}^{\#}\mathscr{T} \geqslant 3$, (1) holds. $\square$

Acknowledgment

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