Kernels by monochromatic paths in digraphs with covering number 2

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We call the digraph D an k-colored digraph if the arcs of D are colored with k colors. A subdigraph H of D is called monochromatic if all of its arcs are colored alike. A set N ⊆ V(D) is said to be a kernel by monochromatic paths if it satisfies the following two conditions: (i) for every pair of different vertices u, v ∈ N, there is no monochromatic directed path between them, and (ii) for every vertex x ∈ (V(D) \ N), there is a vertex y ∈ N such that there is an xy-monochromatic directed path. In this paper, we prove that if D is an k-colored digraph that can be partitioned into two vertex-disjoint transitive tournaments such that every directed cycle of length 3, 4 or 5 is monochromatic, then D has a kernel by monochromatic paths. This result gives a positive answer for this family of digraphs of the following question, which has motivated many results in monochromatic kernel theory: Is there a natural number l such that if a digraph D is k-colored so that every directed cycle of length at most l is monochromatic, then D has a kernel by monochromatic paths? © 2010 Elsevier B.V. All rights reserved.

1. Introduction

Let D be a digraph. We denote by V(D) and A(D) the sets of vertices and the set of arcs of D, respectively. Let v ∈ V(D). We denote by N+(v) and N−(v) the out- and in-neighborhood of v in D, respectively. We define δ+(v) = |N+(v)| and δ−(v) = |N−(v)|. For S ⊆ V(D), we denote by D[S] the subdigraph of D induced by the vertex set S. For two disjoint subsets U, V of V(D), we denote by (U, V) = {uv ∈ A(D) : u ∈ U, v ∈ V} and [U, V] = (U, V) ∪ (V, U). An UV-arc is an arc from (U, V) if U = {u} (resp. V = {v}), we denote the UV-arc by uV-arc (resp. uV-arc). We call the digraph D an k-colored digraph if the arcs of D are colored with k colors. The digraph D will be an k-colored digraph and all the paths, cycles and walks considered in this paper will be directed paths, cycles or walks. If W = (x0, x1, . . . , xn) is a walk, the length of W is n. The length of a walk W is denoted by l(W). The path (u0, u1, . . . , un) will be called an UV-path whenever u0 ∈ U and un ∈ V. A tournament is a digraph T such that there is exactly one arc between any two vertices of T. An acyclic tournament is called a transitive tournament. A vertex v ∈ V(T) is called a sink if N−(v) = V(D) \ v. A subdigraph H of a k-colored digraph D is called monochromatic if all of its arcs are colored alike. Let N ⊆ V(D). Then, N is said to be m-independent if there is no monochromatic directed path between any pair of vertices of the set N, N is a m-absorvent (or m-dominant) if for every vertex x ∈ (V(D) \ N) there is a vertex y ∈ N such that there is an xy-monochromatic directed path and finally, N is a m-kernel (kernel by monochromatic paths) if it satisfies the following two conditions: (i) N is m-independent and (ii) N is m-absorvent. For general concepts, we refer the reader to [1,2,7].

The topic of domination in graphs has been widely studied by many authors. A very complete study of this topic is presented in [19,20]. A special class of domination is the domination in digraphs, and it is defined as follows. In a digraph D,
a set of vertices $S \subseteq V(D)$ dominates whenever for every $w \in (V(D) \setminus S)$ there exists a $wS$-arc in $D$. Dominating independent sets in digraphs (kernels in digraphs) have found many applications in different topics of mathematics (see for instance [21, 22,8,9,26]) and they have been studied by several authors; interesting surveys of kernels in digraphs can be found in [6,9]. The concepts of $m$-domination, $m$-independence and $m$-kernel in edge-colored digraphs are generalization of those of domination, independence and kernel in digraphs. The study of the existence of $m$-kernels in edge-colored digraphs starts with the theorem of Sands, Sauer and Woodrow, proved in [25], which asserts that every two-colored digraph possesses an $m$-kernel. In the same work, the authors proposed the following question: let $D$ be an $k$-colored tournament such that every directed cycle of length 3 is quasi-monochromatic (a subdigraph $H$ of an $k$-colored digraph $D$ is said to be quasi-monochromatic if, with at most one exception, all of its arcs are colored alike) must $D$ have a $m$-kernel? Minggan [24] proved that if $D$ is an $k$-colored tournament such that every directed cycle of length 3 and every transitive tournament of order 3 is quasi-monochromatic, then $D$ has a $m$-kernel. He also proved that this result is best possible for $m \geq 4$. In [15], it was proved that the result is best possible for $m \geq 4$. The question for $m = 3$ is still open: Does every 3-colored tournament such that every directed cycle of length 3 is quasi-monochromatic have a $m$-kernel? Sufficient conditions for the existence of $m$-kernels in edge-colored digraphs have been obtained mainly in tournaments and generalized tournaments, and ask for the monochromaticity or quasi-monochromaticity of small digraphs (due to the difficulty of the problem) in several papers (see [10,11,15,16,18,24]). Other interesting results can be found in [27,28]. Another question which has motivated many results in $m$-kernel theory is the following (proposed in the abstract): Given a digraph $D$ is there an integer $k$ such that if every directed cycle of length at most $k$ is monochromatic (resp. quasi-monochromatic), then $D$ has a $m$-kernel? In [11] (resp. in [16]) it was proved that if $D$ is an $k$-colored tournament (resp. bipartite tournament) such that every directed cycle of length 3 (resp. every directed cycle of length 4) is monochromatic, then $D$ has a $m$-kernel. Later the following generalization of both results was proved in [17]; if $D$ is an $k$-colored $k$-partite tournament, such that every directed cycle of length 3 and every directed cycle of length 4 is monochromatic, then $D$ has a $m$-kernel. In [18] were considered quasi-monochromatic cycles, the authors proved that if $D$ is an $k$-colored tournament such that for some $k$ every directed cycle of length $k$ is quasi-monochromatic and every directed cycle of length less than $k$ is not polymonochromatic (a subdigraph $H$ of $D$ is called polymonochromatic whenever it is colored with at least three colors), then $D$ has a $m$-kernel. In [13] this result was extended for nearly complete digraphs. The covering number of a digraph $D$ is the minimum number of transitive tournaments of $D$ that partition $V(D)$. Digraphs with a small covering number are a nice class of nearly tournament digraphs. The existence of kernels in digraphs with a covering number at most 3 has been studied by several authors, in particular by Berge [3], Maffray [23] and others [4,5,12,14].

In this paper, we study the existence of $m$-kernels in edge-colored digraphs with covering number 2, asking for the monochromaticity of small directed cycles. We prove that if $D$ is an $k$-colored digraph with covering number 2 such that every directed cycle of length 3, 4 or 5 is monochromatic, then $D$ has a $m$-kernel.

2. Structural properties

We consider the family $\mathcal{D}$ of digraphs $D$ with covering number 2. Since $D$ has covering number 2, there exists a non-trivial partition of $V(D)$ into two sets $U$, $V$ such that $D[U]$, $D[V]$ are transitive tournaments. Throughout this paper, the non-trivial partition of the vertex set into $U$, $V$ is such that $D[U]$, $D[V]$ are transitive tournaments. Let $T$ be a transitive tournament of order $n$. Throughout this paper, $(v_0$, $v_1$, $\ldots$, $v_n)$ will denote the Hamiltonian path in $T$. Thus for any $1 \leq i \leq n$, the vertex $v_i$ is the sink of $T \setminus \{v_{i-1}, v_i, \ldots, v_n\}$, in particular, $v_1$ is the sink of $T$. When $P = (u_0, u_1, \ldots, u_k)$ is a path, we will denote by $(u_i, P, u_j)$, for $0 \leq i < j \leq k$, the $u_i$-$u_j$-path contained in $P$. Let $u_iu_{i+1}$ and $u_ju_{j+1}$ be two distinct arcs on $P$. We say that the arc $u_iu_{i+1}$ precedes (resp. follows) the arc $u_ju_{j+1}$ on the path $P$, if $i < j$ (resp. $j < i$).

Throughout this paper, the vertex $z$ will be fixed and arbitrary.

First, we prove some structural properties of the $wz$-paths of minimum length with $w \in \{u_1, v_1\}$ in digraphs of the family $\mathcal{D}$. Next, we extend these properties for $wz$-paths of minimum length with $w \in \{u_1, v_1\}$ in $k$-colored digraphs of the family $\mathcal{D}$ with every directed cycle of length 3, 4 or 5 monochromatic.

Let $u_nv_n$, $v_iku_i$ $\in [U, V]$. We say that $u_nv_n$, $v_iku_i$ are crossing arcs if $u_nv_n$, $v_iku_i$ $\in V(D)$, $v_n$, $v_k$ $\in V(D)$, and $m \leq l$, $k \leq n$, except when $n = k$ and $m = l$ (see Fig. 1). Let $u_nv_n$ $\in [U, V]$. If $xy$, $u_nv_n$ $\in [U, V]$ are crossing arcs, then clearly $xy$ $\in (V, U)$.

![Fig. 1. $u_nv_n$ and $v_iku_i$ are crossing arcs.](image-url)
**Lemma 1.** Let $D$ be a digraph of the family $\mathcal{D}$ such that the sinks $u_1, v_1$ of $D[U]$ and $D[V]$ respectively has a nonempty out-neighborhood. Let $P$ be a $wz$-path of minimum length, with $w \in \{u_1, v_1\}$. Then, any $a \in [U, V] \cap A(P)$ has at most one preceding crossing arc on $P$ and at most one following crossing arc on $P$.

**Proof.** Let $D$ be a digraph that satisfies the hypothesis of this lemma and let $P$ be a $wz$-path of minimum length starting at the vertex $u_1$ or $v_1$ and $|[U, V] \cap A(P)| \geq 2$.

Suppose, for a contradiction, that there is an arc of $[U, V] \cap A(P)$ with at least two following (preceding) crossing arcs on $P$. By symmetry, we may assume that $u_1v_2 \in (U, V) \cap A(P)$ is such that $u_1v_2$ has at least two following (preceding) crossing arcs in $(V, U)$. Let $v_2u_1 \in (V, U)$ be the first following crossing arc on $P$ and let $v_2u_t \in (V, U)$ be the last following crossing arc on $P$ (resp. let $v_2u_t \in (V, U)$ be the first preceding crossing arc on $P$). Since $u_1v_j$ and $v_ju_k$ are crossing arcs, let $P = (v_2u_t)$ has length at most $4$ (see Fig. 2), and since $P$ is a path, $v_jv_2$ is not on $P$, then $P' = (w, P, v_j) \cup (v_j, v_s) \cup (v_s, P, z)$ (resp. $P' = (w, P, u_t) \cup (u_t, u_j) \cup (u_j, P, z)$) is a $wz$-directed path such that $l(P') < l(P)$, which contradicts the way we chose $P$. So $P$ has at most one following (preceding) crossing arc; by symmetry any arc of $[U, V]$ has at most one following (preceding) crossing arc and we are done. □

**Remark 1.** Let $D$ be a digraph of the family $\mathcal{D}$ such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let $P$ be a $wz$-path of minimum length with $w \in \{u_1, v_1\}$. Then, $xy$ is the first arc of $[U, V] \cap A(P)$ if and only if $x = w$.

**Lemma 2.** Let $D$ be a digraph of the family $\mathcal{D}$ such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let $P$ be a directed $wz$-path of minimum length, with $w \in \{u_1, v_1\}$. Then, any two consecutive $[U, V]$-arcs on $P$ are crossing arcs.

**Proof.** Let $D$ be a digraph that satisfies the hypothesis of this lemma and let $P$ be a path of minimum length, starting at the vertex $u_1$ or $v_1$, with $|[U, V] \cap A(P)| \geq 2$.

Suppose, for a contradiction, that there are two consecutive $[U, V]$-arcs on $P$ such that they are not crossing arcs. By symmetry, we may assume that $u_1v_j \in (U, V), v_ju_t \in (V, U)$ is the first pair of consecutive $[U, V]$-arcs that does not form a crossing pair, then $1 \leq l < t$ and by Remark 1, the arc $u_1v_j$ must have a preceding $vu_j$-arc, and this arc must be a crossing arc by the way we chose the arc $u_1v_j$. Let $v_ju_h \in (V, U)$ be the preceding crossing arc of $u_1v_j$. Since $D[U]$ is transitive, then $P' = (w, P, u_t) \cup (u_t, u_j) \cup (u_j, P, z)$ is a $wz$-path such that $l(P') < l(P)$ (see Fig. 3), which is a contradiction. □

**Proposition 1.** Let $D$ be a digraph of the family $\mathcal{D}$ such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let $P$ be a $wz$-path of minimum length with $|[U, V] \cap A(P)| \geq 2$. Then, the first and the last $[U, V]$-arc on $P$ have exactly one crossing arc and any $[U, V]$-arc except the first and the last one, has exactly one preceding and exactly one following crossing $[U, V]$-arc.

**Proof.** A consequence of the Lemmas 1 and 2. □

We will now extend the results Lemmas 1, 2 and Proposition 1 to $k$-colored digraphs of the family $\mathcal{D}$, with every directed cycle of length 3, 4 or 5 monochromatic.
Remark 2. Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. Then any two crossing arcs have the same color, and so if $m < l$ and $k < n$, then $(u_l, u_m, v_n, v_j, v_k)$ is a 4-cycle, if $m < l$ and $n = k$, then $(u_m, v_n, u_l, u_k)$ is a 3-cycle and if $m = l$ and $k < n$, then $(u_m, v_n, v_k, u_m)$ is a 3-cycle (see Fig. 1).

Moreover for any integers $i$ or $j$, $m < i < l$ and $n < j < k$ (if they exist), the arcs $u_l u_i, u_i u_m, v_n v_j, v_j v_k$ have the same color as the crossing arcs $u_m v_n, v_k u_l$ (see Fig. 1).

Lemma 3. Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. If $\delta^+(u_1)$, $\delta^+(v_1) > 0$, then $u_1 v_j$ and $v_1 u_i$ has the same color for any $u_i \in N^+(v_1)$ and $v_j \in N^+(u_1)$.

Proof. Let $i, j$ be two integers such that $u_i \in N^+(v_1)$ and $v_j \in N^+(u_1)$. Then, the arcs $u_1 v_j, v_1 u_i$ are crossing arcs and by Remark 2, they have the same color. $\square$

Corollary 1. Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. If $\delta^+(u_1)$, $\delta^+(v_1) > 0$, then all the monochromatic paths from $u_1$ or from $v_1$ have the same color.

By Lemma 3, we may assume that every arc $u_1 v_j, v_1 u_i$ has color 1, and by Corollary 1, any monochromatic $u_1 x$-path, $v_1 x$-path has color 1, for all $x \in V(D)$.

Lemma 4. Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3 and 4 are monochromatic and such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let $P$ be a monochromatic $wz$-path of minimum length, with $w \in \{u_1, v_1\}$, and let $a \in [U, V] \cap A(P)$. Then the arc $a$ has at most one preceding crossing $[U, V]$-arc on $P$ and at most one following crossing $[U, V]$-arc on $P$.

Proof. Let $D$ be a digraph that satisfies the hypothesis of the Lemma 4 and let $P$ be a monochromatic $wz$-path of minimum length starting at the vertex $u_1$ or $v_1$ and $|[U, V] \cap A(P)| \geq 2$.

Suppose, for a contradiction, that $a \in [U, V] \cap A(P)$ such that $a$ has at least two following crossing arcs on $P$. By symmetry, we may assume that $u_i v_j \in (U, V)$ is the first $[U, V]$-arc of $P$ such that $u_i v_j$ has at least two following crossing arcs on $P$.

Proceed as in the proof of Lemma 1. Since the cycle $(u_1, v_j, v_1, u_i, u_j, v_k)$ has length at most 4, it is monochromatic. Moreover, the arc $u_i v_j$ has color 1, so the cycle and the path $P'$ are both monochromatic of color 1 (see Fig. 2) and $P'$ is a monochromatic $wz$-path such that $l(P') < l(P)$. So any $[U, V]$-arc on $P$ has at most one following crossing $[U, V]$-arc on $P$. Analogously, any $[U, V]$-arc on $P$ has at most one preceding crossing $[U, V]$-arc on $P$. $\square$

Remark 3. Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. Let $P$ be a monochromatic $wz$-path of minimum length with $w \in \{u_1, v_1\}$. Then $xy$ is the first $[U, V]$-arc on $P$ if and only if $x = w$.

Lemma 5. Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3, 4 and 5 are monochromatic and such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let $P$ be a monochromatic $wz$-path of minimum length, with $w \in \{u_1, v_1\}$. Then any two consecutive $[U, V]$-arcs on $P$ are crossing arcs.

Proof. Let $D$ be a digraph that satisfies the hypothesis of this lemma and let $P$ be a monochromatic path of minimum length starting at the vertex $u_1$ or $v_1$, with $|[U, V] \cap A(P)| \geq 2$.

Suppose, for a contradiction, that there are two consecutive $[U, V]$-arcs on $P$ that are not crossing arcs. By symmetry, we may assume that $u_i v_j \in (U, V)$ is the first $[U, V]$-arc such that the following $[U, V]$-arc $v_k u_l$ is not a crossing arc of $u_i v_j$.

Claim 1. If $v_k u_l \in A(P)$ is the first $[U, V]$-arc on $P$, then $x < l$.

Let $x \geq l$. Suppose, for a contradiction, that $v_k u_l$ is the first $[U, V]$-arc on $P$, then $(u_1, u_k, u_l, v_1, v_j, v_k)$ is a monochromatic cycle of color 1 and of length at most 5 (where $v_j$ is any vertex in $N^+(u_1)$). Thus $P' = (w = v_1, u_1, u_k, u_l, v_1, P, z)$ is a directed monochromatic $wz$-path of color 1.

Since $|[U, V] \cap A(P)| \geq 2$, $P'$ is such that $l(P') < l(P)$, and the Claim 1 is valid. $\square$

Claim 2. $P$ has no crossing arcs $u_a v_b, v_c u_d$ preceding the arc $a u_i v_j$ with $a < l < d$.

Let $a < l < d$. For a contradiction, suppose that $u_a v_b, v_c u_d$ is a pair of crossing arcs, both preceding the arc $u_i v_j$ on the path $P$. The length of the cycle $C = (u_a, v_b, v_c, u_d, u_i, u_k)$ is at most 5. The path $P$ is monochromatic of color 1, and so is the cycle $C$. If $u_a v_b, v_c u_d$ are consecutive crossing arcs, then $P' = (w, P, u_d) \cup (u_b, u_i) \cup (u_i, P, z)$ is a directed monochromatic $wz$-path. The arcs $u_a v_b, v_c u_d$ are preceding the arc $u_i v_j$; thus, $P'$ is such that $l(P') < l(P)$. If $v_b u_c, u_d v_a$ are consecutive crossing arcs, then $P' = (w, P, u_a) \cup (u_b, u_i) \cup (u_i, P, z)$ is a directed monochromatic $wz$-path. The arcs $u_a v_b, v_c u_d$ are preceding the arc $u_i v_j$; thus, $P'$ is such that $l(P') < l(P)$. So Claim 2 is valid. $\square$

The arcs $u_i v_j$ and $v_k u_l$ are not crossing arcs, then $i > l > 0$ and by Remark 3, the arc $u_i v_j$ is not the first $[U, V]$-arc on $P$. Let $v_k u_l \in (V, U)$ be the preceding crossing arc of $u_i v_j$. Then, $h > i > l$ and $g < j$.

Note that $g > j > l$. By Claim 1 and Remark 3, the arc $v_k u_l$ is not the first $[U, V]$-arc on $P$. Let $v_k u_l \in (V, U)$ be the first arc on $P$ such that $f > l$, such arc does exist (for instance $v_k u_l$). By Claim 1 and Remark 3, the arc $v_k u_l$ is not the first $[U, V]$-arc on $P$. Then, the arc $v_k u_l$ must have a preceding $[U, V]$-arc, and this arc must be a crossing arc by the way we chose the arc $u_i v_j$. Let $u_a v_b \in (U, V)$ be the preceding crossing arc of $v_k u_l$. By Claim 2 and the fact that $f > l$, we have that $c > l$; moreover, since $P$ is a path $c > l$, then Claim 1 and Remark 3 imply that the arc $u_a v_b$ is not the first $[U, V]$-arc on $P$. So the arc $v_k u_l$
Lemma 4

Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3, 4 and 5 are monochromatic and such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let $P$ be a monochromatic $wz$-path of minimum length with $\| [U, V] \cap A(P) \| \geq 2$. Then, the first and the last $[U, V]$-arc on $P$ have exactly one crossing arc and any $[U, V]$-arc, except the first and the last one has exactly one preceding and exactly one following crossing $[U, V]$-arc. Moreover, two crossing arcs on $P$ must be consecutive $[U, V]$-arcs on $P$.

**Proof.** A consequence of the Lemmas 4 and 5. \hfill $\Box$

The following theorem collects the results of Lemmas 4, 5 and Corollary 2. Moreover, it describes the structure of a monochromatic path of minimum length, as shown in Fig. 5.

Corollary 2. Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3, 4 and 5 are monochromatic and such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let $P$ be a monochromatic $wz$-path of minimum length with $\| [U, V] \cap A(P) \| \geq 2$. Then, the first and the last $[U, V]$-arc on $P$ have exactly one crossing arc and any $[U, V]$-arc, except the first and the last one has exactly one preceding and exactly one following crossing $[U, V]$-arc. Moreover, two crossing arcs on $P$ must be consecutive $[U, V]$-arcs on $P$. 

**Proof.** A consequence of the Lemmas 4 and 5. \hfill $\Box$

The following theorem collects the results of Lemmas 4, 5 and Corollary 2. Moreover, it describes the structure of a monochromatic path of minimum length, as shown in Fig. 5.

**Theorem 1.** Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3, 4 and 5 are monochromatic and such that $\delta^+(u_1), \delta^+(v_1) > 0$. Let $P$ be a monochromatic $wz$-path of minimum length with $\| [U, V] \cap A(P) \| \geq 2$. Then, the first and the last $[U, V]$-arc on $P$ have exactly one crossing arc and any $[U, V]$-arc, except the first and the last one has exactly one preceding and exactly one following crossing $[U, V]$-arc. Moreover, two crossing arcs on $P$ must be consecutive $[U, V]$-arcs on $P$.

**Proof.** Let $D$ be a digraph that satisfies the hypothesis of this lemma and let $P$ be a monochromatic path of minimum length starting at the vertex $u_1$ or $v_1$.

(i) In order to prove the item (i) we take $u_4v_7$ and $u_7v_7$ the preceding and the following crossing arc respectively of $v_7u_7$ on the path $P$, by the definition of crossing arcs $a$, $e < d$ and $c < b < f$. Suppose, for a contradiction, that $a > e$, then $u_4v_7$ is not the first $[U, V]$-arc on $P$, by Remark 3. By Lemma 5, $u_4v_7$ has a preceding crossing arc, say $h$, then the arc $u_4v_7$ would have two preceding crossing arcs, namely $v_7u_7$ and $h$ and by Lemma 4, we have a contradiction, so $a < e$. Analogously $b < f$.

(ii) In order to prove that an induced path of $P$ has length at most 1, we take two consecutive crossing arcs on the path $P$, say $u_4v_7$, $v_7u_7$, and prove that $v_7u_7 \in A(P)$. The length of the cycle $C = (u_7, v_7, v_7, u_7)$ is at most 4 and the path $P$ is monochromatic of color 1, and so is the cycle $C$. Then $P' = (w, P, v_7) \cup (v_7, v_7) \cup (v_7, P, z)$ is a directed monochromatic $wz$-path. If $v_7u_7 \not\in A(P)$, then $P'$ would be a monochromatic $zw$-path such that $l(P') < l(P)$, so $v_7u_7 \in A(P)$. \hfill $\Box$

**3. m-kernel**

Let $D$ be an $k$-colored digraph. A subset $S$ of $V(D)$ is a $m$-semi-kernel of $D$ if it satisfies the following two conditions:

(a) $S$ is $m$-independent, and

(b) for every vertex $z \not\in S$ for which there exists a $Sz$-monochromatic directed path, there also exists a $zS$-monochromatic directed path.
A kernel of a digraph $D$ is also a semi-kernel of $D$, but the converse is not true.

We prove that an $k$-colored digraph $D$ of the family $\mathcal{D}$ with any cycle of length 3, 4 and 5 monochromatic has a $m$-semi-kernel of only one vertex. This fact will lead us to the main theorem.

The main idea in the proof of Proposition 2 is the following.

Let $x$ be any vertex, say $u$, on a monochromatic $uw$-path $P$ of minimum length, with $w \in \{u_1, v_1\}$. We prove that if $P$ has at least two $\{U, V\}$-arcs, then there is a pair of crossing arcs on $P$, say $uiuj$ and $viuk$, such that $i < l$ and $i \leq s \leq l$.

**Proposition 2.** Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. Then, $u_1$ (resp. $v_1$) is a $m$-semi-kernel of one vertex of $D$.

Moreover if there is a monochromatic $uw$-path, with $w \in \{u_1, v_1\}$, of color 1, then there is a monochromatic $zu$-path of color 1.

**Proof.** If $\delta^+(u_1) = 0$ (resp. $\delta^+(v_1) = 0$), then $u_1$ (resp. $v_1$) is a $m$-semi-kernel. Let $\delta^+(u_1), \delta^+(v_1) > 0$, and let $k, l$ be maximum integers such that $u_k \in N^+(u_1)$ and $v_l \in N^+(v_1)$. By Corollary 1, we may assume that any monochromatic path from $v_i$ or $u_1$ has color 1.

We prove that if there is a monochromatic $uw$-path, with $w \in \{u_1, v_1\}$, then there is a monochromatic $zu$-path of color 1. Since $u_1$ is the sink of $D[U]$ (resp. $v_1$ is the sink of $D[V]$), there is an $uw_1$ arc in $D$, for any $u \in U \setminus u_1$ (resp. there is a $vw_1$ arc in $D$ for any $v \in V \setminus v_1$), but this arc is not necessarily of color 1.

Proceeding by contradiction, we take a monochromatic $uw$-path $P$ of minimum length (thus $P$ is colored 1) with $z$ as the first vertex on $P$ such that there is no monochromatic $zu$-path colored 1. By symmetry, we may assume that $z = u_i$ for some integer $1 \leq s \leq n$. Let $v_2u_1 \in (V, U)$ be the first arc on $P$ such that $l \geq s$. Such arc does exist because $D[U]$ is a transitive tournament and then for each $u_i \in U \cap V(P)$, the preceding vertex on $P$ is a vertex of the set $\{u_1, u_2, \ldots, u_i\} \cup U$. If $v_2u_1$ is the first $\{U, V\}$-arc on $P$, then $k = 1$ and for any $v \in N^+(u_1)$ the cycle $(v_1 = v_h, u_1, u_s, u_i, v, v_1)$ has length at most 5 and is monochromatic of color 1. Then $P' = (u_i, u_1)$ (resp. $P'' = (u_s, u_1, v_i, v_1)$) is a monochromatic $u_iu_1$-path (resp. $v_iu_1$-path) of color 1 and we are done.

Therefore, $v_2u_1$ is not the first $\{U, V\}$-arc on $P$. Let $u_1v_j \in (V, U)$ be the preceding crossing arc on $P$; this arc exists by Lemma 5. Since $P$ is a path, $i \neq l$. If $i > s$, then $u_1v_j$ is not the first $\{U, V\}$-arc on $P$ and by Lemma 5, $u_1v_j$ has a preceding crossing arc $v_2u_1 \in (V, U)$. By (ii) of Theorem 1, $l < h < l$, which contradicts the choice of the arc $v_2u_1$. Then, $i < s$. The cycle $(v_1, u_i, v_1, v_j, v_k)$ has length at most 5 and it is monochromatic of color 1 (see Fig. 6). By the choice of the vertex $u_i$, there is a monochromatic $u_iw$-path $P'$ colored 1. Then, $(u_i, u_1) \cup P'$ is a monochromatic $u_iw$-path colored 1, and we are done.

**Fig. 6.** The 5-cycle $(v_k, u_i, u_i, u_j, v_j, v_k)$ is monochromatic of color 1.

So, if there is a monochromatic $uw_1$-path, with $w \in \{u_1, v_1\}$, then there is a monochromatic $u_iw$-path of color 1. Analogously, if there is a monochromatic $uw_1$-path, with $w \in \{u_1, v_1\}$, then there is a monochromatic $v_iw$-path of color 1. Therefore, $\{u_1\}$ and $\{v_1\}$ are both semi-kernels of $D$. \qed

**Theorem 2.** Let $D$ be an $k$-colored digraph of the family $\mathcal{D}$ such that the cycles of length 3, 4 and 5 are monochromatic. Then, $D$ has a $m$-kernel.

**Proof.** If $\delta^+(u_1) = 0$ (resp. $\delta^+(v_1) = 0$), then $u_1$ (resp. $v_1$) is a $m$-semi-kernel; else Proposition 2 implies that $u_1$ or $v_1$ is a $m$-semi-kernel of $D$. We may assume that $v_1$ is a $m$-semi-kernel of $D$. Suppose that $v_1$ is not a $m$-kernel of $D$. Let $U'$ be the subset of the vertices of $V(D)$ such that there is no monochromatic $U'v_1$-directed path. As $v_1$ is a semi-kernel of $D$, we have that there are no monochromatic directed paths between $v_1$ and a vertex $x \in U'$. Since $D[V]$ is a transitive tournament, $v_jv_1 \in A(D)$ for every $1 < j \leq m$; therefore, $U' \subset U$. As $v_1$ is not a $m$-kernel of $D$, then $U' \neq \emptyset$ and $D[U]$ is a transitive tournament, then $D[U']$ has a sink. Let $u_p$ be the sink of $D[U']$. Then $\{v_1, u_p\}$ is a kernel by monochromatic paths of $D$. \qed

4. Final remarks

In this section, we show three digraphs from the family $\mathcal{D}$. The first one (Example 1) is a digraph colored with a large number of colors. Next we show two digraphs without $m$-kernel; the first one (Example 2) has 4- and 5-cycles that are not monochromatic and the second one (Example 3) has 3- and 5-cycles that are not monochromatic. The Example 2 shows that the condition of monochromatic 3-cycles is not sufficient, and Example 3 shows that monochromatic 4-cycles is not sufficient.
**Example 1.** We define the digraph $D$ as follows.
Let $U = \{u_1, u_2, \ldots, u_k\}$, $V = \{v_1, v_2, \ldots, v_l\}$ be a partition of the vertex set $V(D)$ such that $D[U]$, $D[V]$ are transitive tournaments. Let

$$A(D) = \{u_i u_j : j < i\} \cup \{v_i v_j : j < i\} \cup \{v_1 u_2, u_1 v_2\} \cup A'$$

$$A' \subset \{u_i u_j : j \leq i\} \cup \{v_i v_j : j < i\} \setminus \{v_2 u_1, u_2 v_1\}.$$

The only cycles of $D$ are the cycles in $D[u_1, u_2, v_1, v_2]$. Since there are no other cycles, we can color each arc outside $D[u_1, u_2, v_1, v_2]$ with a different color and still have an arc coloring of $D$, with all cycles of length 3, 4 and 5 monochromatic.

If $D$ is a tournament, then $D$ has $\binom{k+1}{2}$ arcs. There are 6-arcs in $D[u_1, u_2, v_1, v_2]$, so the maximum number of colors of $D$ is

$$\binom{k + I}{2} - 5.$$

So, we have $k$-colored digraphs in the family $\mathcal{D}$ with any cycle of length 3, 4 and 5 monochromatic, such that the $D$ is not a tournament nor a nearly complete digraphs, and $m = A(D) - 6$.

**Example 2.** Let $D$ be the digraph in Fig. 7(a). Note that the 4-cycle $(u_1, v_3, u_4, u_2, u_1)$ is not monochromatic. We show that $D$ has no $m$-kernel. First observe that the only vertex that absorbs the vertex $v_2$ is $v_1$. If $D$ has a kernel $K$, then $v_1$ or $v_2$ are vertices of $K$, but not both. Suppose that $v_1 \in K$. Since $(v_1, u_3, u_1)$ is a monochromatic path, $u_1 \notin K$. The only vertices that absorb the vertex $u_1$ are the vertices $u_4$ and $v_3$, but $v_3$ is not independent to $v_1$ and $u_4$ does not absorb the vertex $u_2$; then $v_1 \notin K$ and $v_2 \notin K$. In this case, $v_1$ is not absorbed by $v_2$. The vertices of $D$ that absorb the vertex $v_1$ are $u_1$ and $u_3$, but $u_1$ is not independent to $v_2$ and $u_3$ does not absorb the vertex $u_2$, so $v_2 \notin K$. Therefore, $D$ has no kernel.

**Example 3.** Let $D$ be the digraph in Fig. 7(b). Note that the 3-cycle $(u_2, v_3, u_3, u_2)$ and the 5-cycle $(u_2, v_3, u_3, u_4, u_4, u_2)$ are not monochromatic. We show that $D$ has no $m$-kernel. First observe that there is no vertex $x \in V(D)$ such that $x$ absorbs every vertex from $V(D) \setminus x$. Thus, if $D$ has a kernel $K$, then $|K| \geq 2$ and $K \cap \{u_1, u_2, u_3, u_4\} \neq \emptyset$ and $K \cap \{v_1, v_2, v_3, v_4\} \neq \emptyset$. Any vertex of the set $\{v_1, v_2, v_3\}$ absorbs all the vertices of $D$ except the vertices $u_3$, $u_4$. If $\{v_1, v_2, v_3\} \cap K \neq \emptyset$, then $u_3 \notin K$, because there is a monochromatic path from any of the vertices $v_1$, $v_2$, $v_3$ to the vertex $u_3$. Thus, $u_4 \in K$, but the vertex $u_4$ does not absorb the vertex $u_3$. Therefore, $\{v_1, v_2, v_3\} \cap K = \emptyset$ and $u_4 \in K$. The vertex $u_4$ absorbs all the vertices except the vertex $u_4$, but $u_4 u_4$ is a directed monochromatic path. So, the digraph $D$ has no $m$-kernel.

In Fig. 7 there are two digraphs, 3- and 4-colored respectively, with the colors.

In Fig. 7(a) the 3-cycles are all monochromatic, but there are 4-cycles and 5-cycles, which are not monochromatic (for instance $(u_1, v_3, u_4, u_2, u_1)$). In Fig. 7(b) the 4-cycles are all monochromatic, but there are 3-cycles and 5-cycles which are not monochromatic. In both cases, the digraph has no $m$-kernel. These digraphs shows that monochromatic 3-cycles are not sufficient and that monochromatic 4-cycles are not sufficient.

![Fig. 7. Digraphs without a m-kernel.](image)

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**References**