Journal of Combinatorial Theory, Series B 103 (2013) 61–74

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Complete graph minors and the graph minor structure theorem ☆

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ARTICLE INFO

Article history:
Received 19 May 2011
Available online 11 September 2012

Keywords:
Graph theory
Graph minor
Structure theorem

ABSTRACT

The graph minor structure theorem by Robertson and Seymour shows that every graph that excludes a fixed minor can be constructed by a combination of four ingredients: graphs embedded in a surface of bounded genus, a bounded number of vortices of bounded width, a bounded number of apex vertices, and the clique-sum operation. This paper studies the converse question: What is the maximum order of a complete graph minor in a graph constructed using these four ingredients? Our main result answers this question up to a constant factor.

1. Introduction

Robertson and Seymour [8] proved a rough structural characterization of graphs that exclude a fixed minor. It says that such a graph can be constructed by a combination of four ingredients: graphs embedded in a surface of bounded genus, a bounded number of vortices of bounded width, a bounded number of apex vertices, and the clique-sum operation. Moreover, each of these ingredients is essential.

In this paper, we consider the converse question: What is the maximum order of a complete graph minor in a graph constructed using these four ingredients? Our main result answers this question up to a constant factor.

To state this theorem, we now introduce some notation; see Section 2 for precise definitions. For a graph G, let $\eta(G)$ denote the maximum integer n such that the complete graph K_n is a minor

☆ This work was supported in part by the Actions de Recherche Concertées (ARC) fund of the Communauté française de Belgique. Gwenaël Joret is a Postdoctoral Researcher of the Fonds National de la Recherche Scientifique (F.R.S.–FNRS). The research of David Wood is supported by a QEII Research Fellowship and a Discovery Project from the Australian Research Council.

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0095-8956/$ – see front matter
http://dx.doi.org/10.1016/j.jctb.2012.09.001

of G, sometimes called the *Hadwiger number* of G. For integers $g, p, k \geqslant 0$, let $\mathcal{G}(g, p, k)$ be the set of graphs obtained by adding at most p vortices, each with width at most k, to a graph embedded in a surface of Euler genus at most g. For an integer $a \geqslant 0$, let $\mathcal{G}(g, p, k, a)$ be the set of graphs G such that $G \setminus A \in \mathcal{G}(g, p, k)$ for some set $A \subseteq V(G)$ with $|A| \leqslant a$. The vertices in A are called *apex* vertices. Let $\mathcal{G}(g, p, k, a)^+$ be the set of graphs obtained from clique-sums of graphs in $\mathcal{G}(g, p, k, a)$.

The graph minor structure theorem of Robertson and Seymour [8] says that for every integer $t \geqslant 1$, there exist integers $g, p, k, a \geqslant 0$, such that every graph G with $\eta(G) \leqslant t$ is in $\mathcal{G}(g, p, k, a)^+$. We prove the following converse result.

Theorem 1.1. *For some constant $c > 0$, for all integers $g, p, k, a \geqslant 0$, for every graph G in $\mathcal{G}(g, p, k, a)^+$,*

$$\eta(G) \leqslant a + c(k+1)\sqrt{g+p} + c.$$

Moreover, for some constant $c' > 0$, for all integers $g, a \geqslant 0$ and $p \geqslant 1$ and $k \geqslant 2$, there is a graph G in $\mathcal{G}(g, p, k, a)$ such that

$$\eta(G) \geqslant a + c'k\sqrt{g+p}.$$

Let $\mathrm{RS}(G)$ be the minimum integer k such that G is a subgraph of a graph in $\mathcal{G}(k, k, k, k)^+$. The graph minor structure theorem [8] says that $\mathrm{RS}(G) \leqslant f(\eta(G))$ for some function f independent of G. Conversely, Theorem 1.1 implies that $\eta(G) \leqslant f'(\mathrm{RS}(G))$ for some (much smaller) function f'. In this sense, η and RS are “tied”. Note that such a function f' is widely understood to exist (see for instance Diestel [2, p. 340] and Lovász [5]). However, the authors are not aware of any proof. In addition to proving the existence of f', this paper determines the best possible function f' (up to a constant factor).

Following the presentation of definitions and other preliminary results in Section 2, the proof of the upper and lower bounds in Theorem 1.1 are respectively presented in Sections 3 and 4.

2. Definitions and preliminaries

All graphs in this paper are finite and simple, unless otherwise stated. Let $V(G)$ and $E(G)$ denote the vertex and edge sets of a graph G. For background graph theory see [2].

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. (Note that, since we only consider simple graphs, loops and parallel edges created during an edge contraction are deleted.) An *H-model* in G is a collection $\{S_x\colon x \in V(H)\}$ of pairwise vertex-disjoint connected subgraphs of G (called *branch sets*) such that, for every edge $xy \in E(H)$, some edge in G joins a vertex in S_x to a vertex in S_y. Clearly, H is a minor of G if and only if G contains an H-model. For a recent survey on graph minors see [4].

Let $G[k]$ denote the *lexicographic product* of G with K_k, namely the graph obtained by replacing each vertex v of G with a clique C_v of size k, where for each edge $vw \in E(G)$, each vertex in C_v is adjacent to each vertex in C_w. Let $\mathrm{tw}(G)$ be the treewidth of a graph G; see [2] for background on treewidth.

Lemma 2.1. *For every graph G and integer $k \geqslant 1$, every minor of $G[k]$ has minimum degree at most $k \cdot \mathrm{tw}(G) + k - 1$.*

Proof. A tree decomposition of G can be turned into a tree decomposition of $G[k]$ in the obvious way: in each bag, replace each vertex by its k copies in $G[k]$. The size of each bag is multiplied by k; hence the new tree decomposition has width at most $k(w+1) - 1 = kw + k - 1$, where w denotes the width of the original decomposition. Let H be a minor of $G[k]$. Since treewidth is minor-monotone,

$$\mathrm{tw}(H) \leqslant \mathrm{tw}\big(G[k]\big) \leqslant k \cdot \mathrm{tw}(G) + k - 1.$$

The claim follows since the minimum degree of a graph is at most its treewidth. $\square$

Note that Lemma 2.1 can be written in terms of contraction degeneracy; see [1,3].

Let G be a graph and let $\Omega = (v_1, v_2, \dots, v_t)$ be a circular ordering of a subset of the vertices of G. We write $V(\Omega)$ for the set $\{v_1, v_2, \dots, v_t\}$. A *circular decomposition of G with perimeter Ω* is

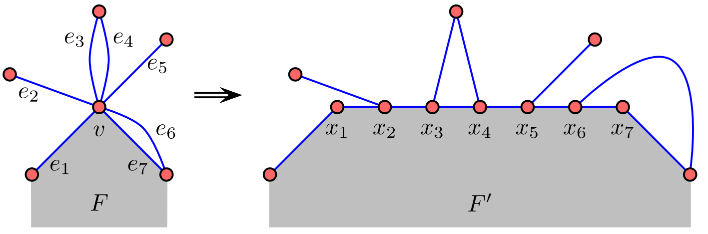

Fig. 1. Splitting a vertex v at a face F.

a multiset $\{C\langle w\rangle \subseteq V(G): w \in V(\Omega)\}$ of subsets of vertices of G, called *bags*, that satisfy the following properties:

- every vertex $w \in V(\Omega)$ is contained in its corresponding bag $C\langle w\rangle$;
- for every vertex $u \in V(G) \setminus V(\Omega)$, there exists $w \in V(\Omega)$ such that u is in $C\langle w\rangle$;
- for every edge $e \in E(G)$, there exists $w \in V(\Omega)$ such that both endpoints of e are in $C\langle w\rangle$; and
- for each vertex $u \in V(G)$, if $u \in C\langle v_i\rangle, C\langle v_j\rangle$ with $i < j$ then $u \in C\langle v_{i+1}\rangle, \dots, C\langle v_{j-1}\rangle$ or $u \in C\langle v_{j+1}\rangle, \dots, C\langle v_t\rangle, C\langle v_1\rangle, \dots, C\langle v_{i-1}\rangle$.

(The last condition says that the bags in which u appears correspond to consecutive vertices of Ω.) The *width* of the decomposition is the maximum cardinality of a bag minus 1. The ordered pair (G, Ω) is called a *vortex*; its width is the minimum width of a circular decomposition of G with perimeter Ω.

A *surface* is a non-null compact connected 2-manifold without boundary. Recall that the *Euler genus* of a surface Σ is $2 - \chi(\Sigma)$, where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. Thus the orientable surface with h handles has Euler genus $2h$, and the non-orientable surface with c cross-caps has Euler genus c. The boundary of an open disc $D \subset \Sigma$ is denoted by $\mathrm{bd}(D)$.

See [6] for basic terminology and results about graphs embedded in surfaces. When considering a graph G embedded in a surface Σ, we use G both for the corresponding abstract graph and for the subset of Σ corresponding to the drawing of G. An embedding of G in Σ is 2-*cell* if every face is homeomorphic to an open disc.

Recall Euler's formula: if an n-vertex m-edge graph is 2-cell embedded with f faces in a surface of Euler genus g, then $n - m + f = 2 - g$. Since $2m \geqslant 3f$,

$$m \leqslant 3n + 3g - 6, \tag{1}$$

which in turn implies the following well-known upper bound on the Hadwiger number.

Lemma 2.2. *If a graph G has an embedding in a surface Σ with Euler genus g, then*

$$\eta(G) \leqslant \sqrt{6g} + 4.$$

Proof. Let $t := \eta(G)$. Then K_t has an embedding in Σ. It is well known that this implies that K_t has a 2-cell embedding in a surface of Euler genus at most g (see [6]). Hence $\binom{t}{2} \leqslant 3t + 3g - 6$ by (1). In particular, $t \leqslant \sqrt{6g} + 4$. $\square$

Let G be an embedded multigraph, and let F be a facial walk of G. Let v be a vertex of F with degree more than 3. Let $e_1, \dots, e_d$ be the edges incident to v in clockwise order around v, such that e_1 and e_d are in F. Let G' be the embedded multigraph obtained from G as follows. First, introduce a path $x_1, \dots, x_d$ of new vertices. Then for each $i \in [1, d]$, replace v as the endpoint of e_i by x_i. The clockwise ordering around x_i is as described in Fig. 1. Finally delete v. We say that G' is obtained from G by *splitting* v at F. Each vertex x_i is said to *belong* to v. By construction, x_i has degree at most 3. Observe that there is a one-to-one correspondence between facial walks of G and G'. This process can be repeated at each vertex of F. The embedded graph that is obtained is called the *splitting* of G at F. And more generally, if $F_1, \dots, F_p$ are pairwise vertex-disjoint facial walks of G, then the embedded graph that is obtained by splitting each F_i is called the *splitting* of G at $F_1, \dots, F_p$. (Clearly, the splitting of G at $F_1, \dots, F_p$ is unique.)

For $g, p, k \geqslant 0$, a graph G is (g, p, k)-*almost embeddable* if there exists a graph G_0 embedded in a surface Σ of Euler genus at most g, and there exist $q \leqslant p$ vortices $(G_1, \Omega_1), \dots, (G_q, \Omega_q)$, each of width at most k, such that

- $G = G_0 \cup G_1 \cup \dots \cup G_q$;
- the graphs $G_1, \dots, G_q$ are pairwise vertex-disjoint;
- $V(G_i) \cap V(G_0) = V(\Omega_i)$ for all $i \in [1, q]$; and
- there exist q disjoint closed discs in Σ whose interiors $D_1, \dots, D_q$ are disjoint from G_0, whose boundaries meet G_0 only in vertices, and such that $\mathrm{bd}(D_i) \cap V(G_0) = V(\Omega_i)$ and the cyclic ordering Ω_i is compatible with the natural cyclic ordering of $V(\Omega_i)$ induced by $\mathrm{bd}(D_i)$, for all $i \in [1, q]$.

Let $\mathcal{G}(g, p, k)$ be the set of (g, p, k)-almost embeddable graphs. Note that $\mathcal{G}(g, 0, 0)$ is exactly the class of graphs with Euler genus at most g. Also note that the literature defines a graph to be *h-almost embeddable* if it is (h, h, h)-almost embeddable. To enable more accurate results we distinguish the three parameters.

Let G_1 and G_2 be disjoint graphs. Let $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_k\}$ be cliques of the same cardinality in G_1 and G_2 respectively. A *clique-sum* of G_1 and G_2 is any graph obtained from $G_1 \cup G_2$ by identifying v_i with w_i for each $i \in [1, k]$, and possibly deleting some of the edges $v_i v_j$.

The above definitions make precise the definition of $\mathcal{G}(g, p, k, a)^+$ given in the introduction. We conclude this section with an easy lemma on clique-sums.

Lemma 2.3. *If a graph G is a clique-sum of graphs G_1 and G_2, then*

$$\eta(G) \leqslant \max\{\eta(G_1), \eta(G_2)\}.$$

Proof. Let $t := \eta(G)$ and let $S_1, \dots, S_t$ be the branch sets of a K_t-model in G. If some branch set S_i were contained in $G_1 \setminus V(G_2)$, and some branch set S_j were contained in $G_2 \setminus V(G_1)$, then there would be no edge between S_i and S_j in G, which is a contradiction. Thus every branch set intersects $V(G_1)$, or every branch set intersects $V(G_2)$. Suppose that every branch set intersects $V(G_1)$. For each branch set S_i that intersects $G_1 \cap G_2$ remove from S_i all vertices in $V(G_2) \setminus V(G_1)$. Since $V(G_1) \cap V(G_2)$ is a clique in G_1, the modified branch sets yield a K_t-model in G_1. Hence $t \leqslant \eta(G_1)$. By symmetry, $t \leqslant \eta(G_2)$ in the case that every branch set intersects G_2. Therefore $\eta(G) \leqslant \max\{\eta(G_1), \eta(G_2)\}$. □

3. Proof of upper bound

The aim of this section is to prove the following theorem.

Theorem 3.1. *For all integers $g, p, k \geqslant 0$, every graph G in $\mathcal{G}(g, p, k)$ satisfies*

$$\eta(G) \leqslant 48(k+1)\sqrt{g+p} + \sqrt{6g} + 5.$$

Combining this theorem with Lemma 2.3 gives the following quantitative version of the first part of Theorem 1.1.

Corollary 3.2. *For every graph $G \in \mathcal{G}(g, p, k, a)^+$,*

$$\eta(G) \leqslant a + 48(k+1)\sqrt{g+p} + \sqrt{6g} + 5.$$

Proof. Let $G \in \mathcal{G}(g, p, k, a)^+$. Lemma 2.3 implies that $\eta(G) \leqslant \eta(G')$ for some graph $G' \in \mathcal{G}(g, p, k, a)$. Clearly, $\eta(G') \leqslant \eta(G' \setminus A) + a$, where A denotes the (possibly empty) apex set of G'. Since $G' \setminus A \in \mathcal{G}(g, p, k)$, the claim follows from Theorem 3.1. □

The proof of Theorem 3.1 uses the following definitions. Two subgraphs A and B of a graph G *touch* if A and B have at least one vertex in common or if there is an edge in G between a vertex in A and another vertex in B. We generalize the notion of minors and models as follows. For an integer $k \geqslant 1$, a graph H is said to be an *(H,k)-minor* of a graph G if there exists a collection $\{S_x\colon x \in V(H)\}$ of connected subgraphs of G (called *branch sets*), such that S_x and S_y touch in G for every edge $xy \in E(H)$, and every vertex of G is included in at most k branch sets in the collection. The collection $\{S_x\colon x \in V(H)\}$ is called an *(H,k)-model* in G. Note that for $k=1$ this definition corresponds to the usual notions of H-minor and H-model. As shown in the next lemma, this generalization provides another way of considering H-minors in $G[k]$, the lexicographic product of G with K_k. (The easy proof is left to the reader.)

Lemma 3.3. *Let $k \geqslant 1$. A graph H is an (H,k)-minor of a graph G if and only if H is a minor of $G[k]$.*

For a surface Σ, let Σ_c be Σ with c cuffs added; that is, Σ_c is obtained from Σ by removing the interior of c pairwise disjoint closed discs. (It is well known that the locations of the discs are irrelevant.) When considering graphs embedded in Σ_c we require the embedding to be 2-cell. We emphasize that this is a non-standard and relatively strong requirement; in particular, it implies that the graph is connected, and the boundary of each cuff intersects the graph in a cycle. Such cycles are called *cuff-cycles*.

For $g \geqslant 0$ and $c \geqslant 1$, a graph G is *(g,c)-embedded* if G has maximum degree $\Delta(G) \leqslant 3$ and G is embedded in a surface of Euler genus at most g with at most c cuffs added, such that *every* vertex of G lies on the boundary of the surface. (Thus the cuff-cycles induce a partition of the whole vertex set.) The graph G is *(g,c)-embeddable* if there exists such an embedding. Note that if C is a contractible cycle in a (g,c)-embedded graph, then the closed disc bounded by C is uniquely determined even if the underlying surface is the sphere (because there is at least one cuff).

Lemma 3.4. *For every graph $G \in \mathcal{G}(g,p,k)$ there exists a (g,p)-embeddable graph H with $\eta(G) \leqslant \eta(H[k+1]) + \sqrt{6g} + 4$.*

Proof. Let $t := \eta(G)$. Let $S_1,\dots,S_t$ be the branch sets of a K_t-model in G. Since $\eta(G)$ equals the Hadwiger number of some connected component of G, we may assume that G is connected. Thus we may 'grow' the branch sets until $V(S_1) \cup \cdots \cup V(S_t) = V(G)$.

Write $G = G_0 \cup G_1 \cup \cdots \cup G_q$ as in the definition of (g,p,k)-almost embeddable graphs. Thus G_0 is embedded in a surface Σ of Euler genus at most g, and $(G_1,\Omega_1),\dots,(G_q,\Omega_q)$ are pairwise vertex-disjoint vortices of width at most k, for some $q \leqslant p$. Let $D_1,\dots,D_q$ be the proper interiors of the closed discs of Σ appearing in the definition.

Define r and reorder the branch sets, so that each S_i contains a vertex of some vortex if and only if $i \leqslant r$. If $t > r$, then $S_{r+1},\dots,S_t$ is a K_{t-r}-model in the embedded graph G_0, and hence $t - r \leqslant \sqrt{6g} + 4$ by Lemma 2.2. Therefore, it suffices to show that $r \leqslant \eta(H[k+1])$ for some (g,p)-embeddable graph H.

Modify G, G_0, and the branch sets $S_1,\dots,S_r$ as follows. First, remove from G and G_0 every vertex of S_i for all $i \in [r+1,t]$. Next, while some branch set S_i $(i \in [1,r])$ contains an edge uv in G_0 where u is in some vortex, but v is in no vortex, contract the edge uv into u (this operation is done in S_i, G, and G_0). The above operations on G_0 are carried out in its embedding in the natural way. Now apply a final operation on G and G_0: for each $j \in [1,q]$ and each pair of consecutive vertices a and b in Ω_j, remove the edge ab if it exists, and embed the edge ab as a curve on the boundary of D_j.

When the above procedure is finished, every vertex of the modified G_0 belongs to some vortex. It should be clear that the modified branch sets $S_1,\dots,S_r$ still provide a model of K_r in G. Also observe that G_0 is connected; this is because $V(\Omega_j)$ induces a connected subgraph for each $j \in [1,q]$, and each vortex intersects at least one branch set S_i with $i \in [1,r]$. By the final operation, the boundary of the disc D_j of Σ intersects G_0 in a cycle C_j of G_0 with $V(C_j) = V(\Omega_j)$ and such that C_j (with the right orientation) defines the same cyclic ordering as Ω_j for every $j \in [1,q]$.

We claim that G_0 can be 2-cell embedded in a surface Σ' with Euler genus at most that of Σ, such that each C_j $(j \in [1,q])$ is a facial cycle of the embedding. This follows by considering the combinatorial embedding (that is, circular ordering of edges incident to each vertex, and edge signatures)

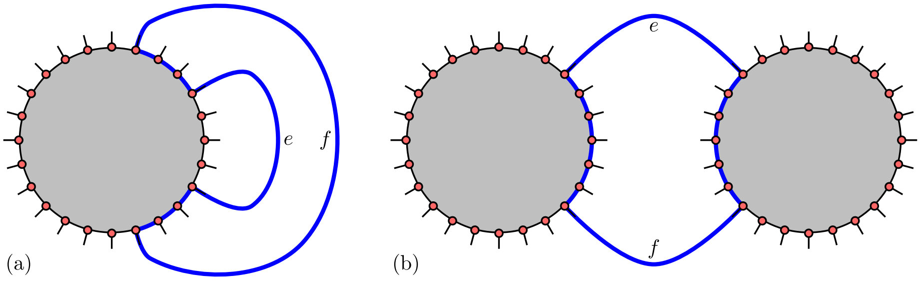

Fig. 2. Homotopic edges: (a) one cuff, (b) two cuffs.

determined by the embedding in Σ (see [6]), and observing that under the above operations, the Euler genus of the combinatorial embedding does not increase, and facial walks remain facial walks (so that each C_j is a facial cycle). Now, removing the q open discs corresponding to these facial cycles gives a 2-cell embedding of G_0 in Σ'_q.

We now prove that $\eta(G_0[k+1]) \geqslant r$. For every $i \in [1,q]$, let $\{C\langle w\rangle \subseteq V(G_i)\colon\ w \in V(\Omega_i)\}$ denote a circular decomposition of width at most k of the i-th vortex. For each $i \in [1,r]$, mark the vertices w of G_0 for which S_i contains at least one vertex in the bag $C\langle w\rangle$ (recall that every vertex of G_0 is in the perimeter of some vortex), and define S'_i as the subgraph of G_0 induced by the marked vertices. It is easily checked that S'_i is a connected subgraph of G_0. Also, S'_j and S'_i touch in G_0 for all $i \neq j$. Finally, a vertex of G_0 will be marked at most $k+1$ times, since each bag has size at most $k+1$. It follows that $\{S'_1,\dots,S'_r\}$ is a $(K_r,k+1)$-model in G_0, which implies by Lemma 3.3 that K_r is minor of $G_0[k+1]$, as claimed.

Finally, let H be obtained from G_0 by splitting each vertex v of degree more than 3 along the cuff boundary that contains v. (Clearly the notion of splitting along a face extends to splitting along a cuff.) By construction, $\Delta(H) \leqslant 3$ and H is (g,q)-embedded. The $(K_r,k+1)$-model of G_0 constructed above can be turned into a $(K_r,k+1)$-model of H by replacing each branch set S'_i by the union, taken over the vertices $v \in V(S'_i)$, of the set of vertices in H that belong to v. Hence $r \leqslant \eta(G_0[k+1]) \leqslant \eta(H[k+1])$. □

We need to introduce a few definitions. Consider a (g,c)-embedded graph G. An edge e of G is said to be a *cuff* or a *non-cuff* edge, depending on whether e is included in a cuff-cycle. Every non-cuff edge has its two endpoints in either the same cuff-cycle or in two distinct cuff-cycles. Since $\Delta(G) \leqslant 3$, the set of non-cuff edges is a matching.

A cycle C of G is an *F-cycle* where F is the set of non-cuff edges in C. A non-cuff edge e is *contractible* if there exists a contractible $\{e\}$-cycle, and is *noncontractible* otherwise. Two non-cuff edges e and f are *homotopic* if G contains a contractible $\{e,f\}$-cycle. Observe that if e and f are homotopic, then they have their endpoints in the same cuff-cycle(s), as illustrated in Fig. 2. We now prove that homotopy defines an equivalence relation on the set of noncontractible non-cuff edges of G.

Lemma 3.5. *Let G be a (g,c)-embedded graph, and let e_1, e_2, e_3 be distinct noncontractible non-cuff edges of G, such that e_1 is homotopic to e_2 and to e_3. Then e_2 and e_3 are also homotopic. Moreover, given a contractible $\{e_1,e_2\}$-cycle C_{12} bounding a closed disc D_{12}, for some distinct $i,j \in \{1,2,3\}$, there is a contractible $\{e_i,e_j\}$-cycle bounding a closed disc containing e_1, e_2, e_3 and all noncontractible non-cuff edges of G contained in D_{12}.*

Proof. Let C_{13} be a contractible $\{e_1,e_3\}$-cycle. Let P_{12}, Q_{12} be the two paths in the graph $C_{12} \setminus \{e_1,e_2\}$. Let P_{13}, Q_{13} be the two paths in the graph $C_{13} \setminus \{e_1,e_3\}$. Exchanging P_{13} and Q_{13} if necessary, we may denote the endpoints of e_i $(i=1,2,3)$ by u_i, v_i so that the endpoints of P_{12} and P_{13} are u_1, u_2 and u_1, u_3, respectively, and similarly, the endpoints of Q_{12} and Q_{13} are v_1, v_2 and v_1, v_3, respectively.

Let D_{13} be the closed disc bounded by C_{13}. Each edge of P_{1i} and Q_{1i} $(i=2,3)$ is on the boundaries of both D_{1i} and a cuff; it follows that every non-cuff edge of G incident to an internal vertex of P_{1i}

or Q_{1i} is entirely contained in the disc D_{1i}. The paths P_{12} and P_{13} are subgraphs of a common cuff-cycle C_P, and Q_{12} and Q_{13} are subgraphs of a common cuff-cycle C_Q. Note that these two cuff-cycles could be the same.

Recall that non-cuff edges of G are independent (that is, have no endpoint in common). This will be used in the arguments below. We claim that

$$\text{every noncontractible non-cuff edge } f \text{ contained in } D_{1i} \text{ has one endpoint in } P_{1i} \text{ and the other in } Q_{1i}, \text{ for each } i \in \{2,3\}. \tag{2}$$

The claim is immediate if $f \in \{e_1, e_i\}$. Now assume that $f \notin \{e_1, e_i\}$. The edge f is incident to at least one of P_{1i} and Q_{1i} since there is no vertex in the proper interior of D_{1i}. Without loss of generality, f is incident to P_{1i}. The edge f can only be incident to internal vertices of P_{1i}, since f is independent of e_1 and e_i. Say $f = xy$. If $x, y \in V(P_{1i})$ then the $\{f\}$-cycle obtained by combining the x–y subpath of P_{1i} with the edge f is contained in D_{1i} and thus is contractible. Hence f is a contractible non-cuff edge, a contradiction. This proves (2).

First we prove the lemma in the case where e_3 is incident to P_{12}. Since e_3 is incident to an internal vertex of P_{12}, it follows that e_3 is contained in D_{12}. This shows the second part of the lemma. To show that e_2 and e_3 are homotopic, consider the endpoint v_3 of e_3. Since e_3 is in D_{12} and $u_3 \in V(P_{12})$, we have $v_3 \in V(Q_{12})$ by (2). Now, combining the u_2–u_3 subpath of P_{12} and the v_2–v_3 subpath of Q_{12} with e_2 and e_3, we obtain an $\{e_2, e_3\}$-cycle contained in D_{12}, which is thus contractible. This shows that e_2 and e_3 are homotopic.

By symmetry, the above argument also handles the case where e_3 is incident to Q_{12}. Thus we may assume that e_3 is incident to neither P_{12} nor Q_{12}.

Suppose $P_{12} \subseteq P_{13}$. Then, by (2), all noncontractible non-cuff edges contained in D_{12} are incident to P_{12}, and thus also to P_{13}. Hence they are all contained in the disc D_{13}. Moreover, a contractible $\{e_2, e_3\}$-cycle can be found in the obvious way. Therefore the lemma holds in this case. Using symmetry, the same argument can be used if $P_{12} \subseteq Q_{13}$, $Q_{12} \subseteq P_{13}$, or $Q_{12} \subseteq Q_{13}$. Thus we may assume

$$P_{12} \nsubseteq P_{13}; \qquad P_{12} \nsubseteq Q_{13}; \qquad Q_{12} \nsubseteq P_{13}; \qquad Q_{12} \nsubseteq Q_{13}. \tag{3}$$

Next consider P_{12} and P_{13}. If we orient these paths starting at u_1, then they either go in the same direction around C_P, or in opposite directions. Suppose the former. Then one path is a subpath of the other. Since by our assumption u_3 is not in P_{12}, we have $P_{12} \subseteq P_{13}$, which contradicts (3). Hence the paths P_{12} and P_{13} go in opposite directions around C_P. If $V(P_{12}) \cap V(P_{13}) \neq \{u_1\}$, then u_3 is an internal vertex of P_{12}, which contradicts our assumption on e_3. Hence

$$V(P_{12}) \cap V(P_{13}) = \{u_1\}. \tag{4}$$

By symmetry, the above argument shows that Q_{12} and Q_{13} go in opposite directions around C_Q (starting from v_1), which similarly implies

$$V(Q_{12}) \cap V(Q_{13}) = \{v_1\}. \tag{5}$$

Now consider P_{12} and Q_{13}. These two paths do not share any endpoint. If $C_P \neq C_Q$ then obviously the two paths are vertex-disjoint. If $C_P = C_Q$ and $V(P_{12}) \cap V(Q_{13}) \neq \emptyset$, then at least one of v_1 and v_3 is an internal vertex of P_{12}, because otherwise $P_{12} \subseteq Q_{13}$, which contradicts (3). However $v_1 \notin V(P_{12})$ since $v_1 \in V(Q_{12})$, and $v_3 \notin V(P_{12})$ by our assumption that e_3 is not incident to P_{12}. Hence, in all cases,

$$V(P_{12}) \cap V(Q_{13}) = \emptyset. \tag{6}$$

By symmetry,

$$V(Q_{12}) \cap V(P_{13}) = \emptyset. \tag{7}$$

It follows from (4)–(7) that C_{12} and C_{13} only have e_1 in common. This implies in turn that D_{12} and D_{13} have disjoint proper interiors. Thus the cycle $C_{23} := (C_{12} \cup C_{13}) - e_1$ bounds the disc obtained by gluing D_{12} and D_{13} along e_1. Hence C_{23} is an $\{e_2, e_3\}$-cycle of G bounding a disc containing e_3 and all edges contained in D_{12}. This concludes the proof. □

The next lemma is a direct consequence of Lemma 3.5. An equivalence class $\mathcal{Q}$ for the homotopy relation on the noncontractible non-cuff edges of G is *trivial* if $|\mathcal{Q}| = 1$, and *non-trivial* otherwise.

Lemma 3.6. *Let G be a (g, c)-embedded graph and let $\mathcal{Q}$ be a non-trivial equivalence class of the noncontractible non-cuff edges of G. Then there are distinct edges $e, f \in \mathcal{Q}$ and a contractible $\{e, f\}$-cycle C of G, such that the closed disc bounded by C contains every edge in $\mathcal{Q}$.*

Our main tool in proving Theorem 3.1 is the following lemma, whose inductive proof is enabled by the following definition. Let G be a (g, c)-embedded graph and let $k \geqslant 1$. A graph H is a *k-minor* of G if there exists an $(H, 4k)$-model $\{S_x\colon x \in V(H)\}$ in G such that, for every vertex $u \in V(G)$ incident to a noncontractible non-cuff edge in a non-trivial equivalence class, the number of subgraphs in the model including u is at most k. Such a collection $\{S_x\colon x \in V(H)\}$ is said to be a *k-model* of H in G. This provides a relaxation of the notion of (H, k)-minor since some vertices of G could appear in up to $4k$ branch sets (instead of k). We emphasize that this definition depends heavily on the embedding of G.

Lemma 3.7. *Let G be a (g, c)-embedded graph and let $k \geqslant 1$. Then every k-minor H of G has minimum degree at most $48k\sqrt{c+g}$.*

Proof. Let $q(G)$ be the number of non-trivial equivalence classes of noncontractible non-cuff edges in G. We proceed by induction, firstly on $g+c$, then on $q(G)$, and then on $|V(G)|$. Now G is embedded in a surface of Euler genus $g' \leqslant g$ with $c' \leqslant c$ cuffs added. If $g' < g$ or $c' < c$ then we are done by induction. Now assume that $g' = g$ and $c' = c$.

We repeatedly use the following observation: If C is a contractible cycle of G, then the subgraph of G consisting of the vertices and edges contained in the closed disc D bounded by C is outerplanar, and thus has treewidth at most 2. This is because the proper interior of D contains no vertex of G (since all the vertices in G are on the cuff boundaries).

Let $\{S_x\colon x \in V(H)\}$ be a k-model of H in G. Let d be the minimum degree of H. We may assume that $d \geqslant 20k$, as otherwise $d \leqslant 48k\sqrt{c+g}$ (since $c \geqslant 1$) and we are done. Also, it is enough to prove the lemma when H is connected, so assume this is the case.

Case 1: Some non-cuff edge *e* of *G* is contractible. Let C be a contractible $\{e\}$-cycle. Let u, v be the endpoints of e. Remove from G every vertex in $V(C) \setminus \{u, v\}$ and modify the embedding of G by redrawing the edge e where the path $C - e$ was. Thus e becomes a cuff-edge in the resulting graph G', and u and v both have degree 2. Also observe that G' is connected and remains simple (that is, this operation does not create loops or parallel edges). Since the embedding of G' is 2-cell, G' is (g, c)-embedded also.

If e_1 and e_2 are noncontractible non-cuff edges of G' that are homotopic in G', then e_1 and e_2 were also noncontractible and homotopic in G. Hence, $q(G') \leqslant q(G)$. Also, $|V(G')| < |V(G)|$ since we removed at least one vertex from G. Thus, by induction, every k-minor of G' has minimum degree at most $48k\sqrt{c+g}$. Therefore, it is enough to show that H is also a k-minor of G'.

Let G_1 be the subgraph of G lying in the closed disc bounded by C; observe that G_1 is outerplanar. Moreover, (G_1, G') is a separation of G with $V(G_1) \cap V(G') = \{u, v\}$. (That is, $G_1 \cup G' = G$ and $V(G_1) \setminus V(G') \neq \emptyset$ and $V(G') \setminus V(G_1) \neq \emptyset$.)

First suppose that $S_x \subseteq G_1 \setminus \{u, v\}$ for some vertex $x \in V(H)$. Let H' be the subgraph of H induced by the set of such vertices x. In H, the only neighbors of a vertex $x \in V(H')$ that are not in H' are vertices y such that S_y includes at least one of u, v. There are at most $2 \cdot 4k = 8k$ such branch sets S_y. Hence, H' has minimum degree at least $d - 8k \geqslant 12k$. However, H' is a minor of $G_1[4k]$ and hence has minimum degree at most $4k \cdot \mathrm{tw}(G_1) + 4k - 1 \leqslant 12k - 1$ by Lemma 2.1, a contradiction.

It follows that every branch set S_x $(x \in V(H))$ contains at least one vertex in $V(G')$. Let $S'_i := S_i \cap G'$. Using the fact that $uv \in E(G')$, it is easily seen that the collection $\{S'_x\colon x \in V(H)\}$ is a k-model of H in G'.

Case 2: Some equivalence class $\mathcal{Q}$ is non-trivial. By Lemma 3.6, there are two edges $e, f \in \mathcal{Q}$ and a contractible $\{e, f\}$-cycle C such that every edge in $\mathcal{Q}$ is contained in the disc bounded by C. Let P_1, P_2 be the two components of $C \setminus \{e, f\}$. These two paths either belong to the same cuff-cycle or to two distinct cuff-cycles of G.

Our aim is to eventually contract each of P_1, P_2 into a single vertex. However, before doing so we slightly modify G as follows. For each cuff-cycle C^* intersecting C, select an arbitrary edge in $E(C^*) \setminus E(C)$ and subdivide it *twice*. Let G' be the resulting (g, c)-embedded graph. Clearly $q(G') = q(G)$, and there is an obvious k-model $\{S'_x\colon x \in V(H)\}$ of H in G': simply apply the same subdivision operation on the branch sets S_x.

Let G'_1 be the subgraph of G' lying in the closed disc D bounded by C. Observe that G'_1 is outerplanar with outercycle C. Suppose that some edge xy in $E(G'_1) \setminus E(C)$ has both its endpoints in the same path P_i, for some $i \in \{1, 2\}$. Then the cycle obtained by combining xy and the x–y path in P_i is a contractible cycle of G', and its only non-cuff edge is xy. The edge xy is thus a contractible edge of G', and hence also of G, a contradiction.

It follows that every non-cuff edge included in G'_1 has one endpoint in P_1 and the other in P_2. Hence, every such edge is homotopic to e and therefore belongs to $\mathcal{Q}$.

Consider the k-model $\{S'_x\colon x \in V(H)\}$ of H in G' mentioned above. Let $e = uv$ and $f = u'v'$, with $u, u' \in V(P_1)$ and $v, v' \in V(P_2)$. Let $X := \{u, u', v, v'\}$. For each $w \in X$, the number of branch sets S'_x that include w is at most k, since e and f are homotopic noncontractible non-cuff edges.

Let $J := G'_1 \setminus X$. Note that $\operatorname{tw}(J) \leqslant 2$ since G'_1 is outerplanar. Let $Z := \{x \in V(H)\colon S'_x \subseteq J\}$. First, suppose that $Z \neq \emptyset$. Every vertex of J is in at most $4k$ branch sets S'_x $(x \in Z)$. It follows that the induced subgraph $H[Z]$ is a minor of $J[4k]$. Thus, by Lemma 2.1, $H[Z]$ has a vertex y with degree at most $4k \cdot \operatorname{tw}(J) + 4k - 1 \leqslant 4k \cdot 2 + 4k - 1 = 12k - 1$. Consider the neighbors of y in H. Since X is a cutset of G' separating $V(J)$ from $G' \setminus V(G'_1)$, the only neighbors of y in H that are not in $H[Z]$ are vertices x such that $V(S'_x) \cap X \neq \emptyset$. As mentioned before, there are at most $4k$ such vertices; hence, y has degree at most $12k - 1 + 4k = 16k - 1$. However this contradicts the assumption that H has minimum degree $d \geqslant 20k$. Therefore, we may assume that $Z = \emptyset$; that is, every branch set S'_x $(x \in V(H))$ intersecting $V(G'_1)$ contains some vertex in X.

Now, remove from G' every edge in $\mathcal{Q}$ except e, and contract each of P_1 and P_2 into a single vertex. Ensuring that the contractions are done along the boundary of the relevant cuffs in the embedding. This results in a graph G'' which is again (g, c)-embedded. Note that G'' is guaranteed to be simple, thanks to the edge subdivision operation that was applied to G when defining G'.

If a non-cuff edge is contractible in G'' then it is also contractible in G', implying all non-cuff edges in G'' are noncontractible. Two non-cuff edges of G'' are homotopic in G'' if and only if they are in G'. It follows $q(G'') = q(G') - 1 = q(G) - 1$, since e is not homotopic to another non-cuff edge in G''. By induction, every k-minor of G'' has minimum degree at most $48k\sqrt{c + g}$. Thus, it suffices to show that H is also a k-minor of G''.

For $x \in V(H)$, let S''_x be obtained from S'_x by performing the same contraction operation as when defining G'' from G': every edge in $\mathcal{Q} \setminus \{e\}$ is removed and every edge in $E(P_1) \cup E(P_2)$ is contracted. Using that every subgraph S'_x either is disjoint from $V(G'_1)$ or contains some vertex in X, it can be checked that S''_x is connected.

Consider an edge $xy \in E(H)$. We now show that the two subgraphs S''_x and S''_y touch in G''. Suppose S'_x and S'_y share a common vertex w. If $w \notin V(G'_1)$, then w is trivially included in both S''_x and S''_y. If $w \in V(G'_1)$, then each of S'_x and S'_y contains a vertex from X, and hence either u or v is included in both S''_x and S''_y, or u is included in one and v in the other. In the latter case uv is an edge of G'' joining S''_x and S''_y. Now assume S'_x and S'_y are vertex-disjoint. Thus there is an edge $ww' \in E(G')$ joining these two subgraphs in G'. Again, if neither w nor w' belongs to $V(G'_1)$, then obviously ww' joins S''_x and S''_y in G''. If $w, w' \in V(G'_1)$, then each of S'_x and S'_y contains a vertex from X, and we are done exactly as previously. If exactly one of w, w' belongs to $V(G'_1)$, say w, then $w \in X$ and w' is the unique neighbor of w in G' outside $V(G'_1)$. The contraction operation naturally maps w to a vertex $m(w) \in \{u, v\}$. The edge $w'm(w)$ is included in G'' and thus joins S''_x and S''_y.

In order to conclude that $\{S''_x\colon x \in V(H)\}$ is a k-model of H in G'', it remains to show that, for every vertex $w \in V(G'')$, the number of branch sets including w is at most $4k$, and is at most k if

w is incident to a non-cuff edge homotopic to another non-cuff edge. This condition is satisfied if $w \notin \{u, v\}$, because two non-cuff edges of G'' are homotopic in G'' if and only if they are in G'. Thus assume $w \in \{u, v\}$. By the definition of G'', the edge $e = uv$ is *not* homotopic to another non-cuff edge of G''. Moreover, for each $z \in X$, there are at most k branch sets S'_x ($x \in V(H)$) containing z. Since $|X| = 4$, it follows that there are at most $4k$ branch sets S''_x ($x \in V(H)$) containing w. Therefore, the condition holds also for w, and H is a k-minor of G''.

Case 3: There is at most one non-cuff edge. Because G is connected, this implies that G consists either of a unique cuff-cycle, or of two cuff-cycles joined by a non-cuff edge. In both cases, G has treewidth exactly 2. Since H is a minor of $G[4k]$, Lemma 2.1 implies that H has minimum degree at most $4k \cdot \operatorname{tw}(G) + 4k - 1 = 12k - 1 \leqslant 48k\sqrt{c+g}$, as desired.

Case 4: Some cuff-cycle C contains three consecutive degree-2 vertices. Let u, v, w be three such vertices (in order). Note that C has at least four vertices, as otherwise $G = C$ and the previous case would apply. It follows $uw \notin E(G)$. Let G' be obtained from G by contracting the edge uv into the vertex u. In the embedding of G', the edge uw is drawn where the path uvw was; thus uw is a cuff-edge, and G' is (g, c)-embedded. We have $q(G') = q(G)$ and $|V(G')| < |V(G)|$, hence by induction, G' satisfies the lemma, and it is enough to show that H is a k-minor of G'.

Consider the k-model $\{S_x\colon\ x \in V(H)\}$ of H in G. If $V(S_x) = \{v\}$ for some $x \in V(H)$, then x has degree at most $3 \cdot 4k - 1 = 12k - 1$ in H, because $xy \in E(H)$ implies that S_y contains at least one of u, v, w. However this contradicts the assumption that H has minimum degree $d \geqslant 20k$. Thus every branch set S_x that includes v also contains at least one of u, w (since S_x is connected).

For $x \in V(H)$, let S'_x be obtained from S_x as expected: contract the edge uv if $uv \in E(S_x)$. Clearly S'_x is connected. Consider an edge $xy \in E(H)$. If S_x and S_y had a common vertex then so do S'_x and S'_y. If S_x and S_y were joined by an edge e, then either e is still in G' and joins S'_x and S'_y, or $e = uv$ and $u \in V(S'_x), V(S'_y)$. Hence in each case S'_x and S'_y touch in G'. Finally, it is clear that $\{S'_x\colon\ x \in V(H)\}$ meets remaining requirements to be a k-model of H in G', since $V(S'_x) \subseteq V(S_x)$ for every $x \in V(H)$ and the homotopy properties of the non-cuff edges have not changed. Therefore, H is a k-minor of G'.

Case 5: None of the previous cases apply. Let t be the number of non-cuff edges in G (thus $t \geqslant 2$). Since there are no three consecutive degree-2 vertices, every cuff-edge is at distance at most 1 from a non-cuff edge. It follows that

$$\left|E(G)\right| \leqslant 9t. \tag{8}$$

(This inequality can be improved but is good enough for our purposes.)

For a facial walk F of the embedded graph G, let $\operatorname{nc}(F)$ denote the number of occurrences of non-cuff edges in F. (A non-cuff edge that appears twice in F is counted twice.) We claim that $\operatorname{nc}(F) \geqslant 3$. Suppose on the contrary that $\operatorname{nc}(F) \leqslant 2$.

First suppose that F has no repeated vertex. Thus F is a cycle. If $\operatorname{nc}(F) = 0$, then F is a cuff-cycle, every vertex of which is not incident to a non-cuff edge, contradicting the fact that G is connected with at least two non-cuff edges. If $\operatorname{nc}(F) = 1$ then F is a contractible cycle that contains exactly one non-cuff edge e. Thus e is contractible, and Case 1 applies. If $\operatorname{nc}(F) = 2$ then F is a contractible cycle containing exactly two non-cuff edges e and f. Thus e and f are homotopic. Hence there is a non-trivial equivalence class, and Case 2 applies.

Now assume that F contains a repeated vertex v. Let

$$F = (x_1, x_2, \dots, x_{i-1}, x_i = v, x_{i+1}, x_{i+2}, \dots, x_{j-1}, x_j = v).$$

All of x_1, x_{i-1}, x_{i+1}, x_{j-1} are adjacent to v. Since $x_1 \neq x_{j-1}$ and $x_{i-1} \neq x_{i+1}$ and $\deg(v) \leqslant 3$, we have $x_{i+1} = x_{j-1}$ or $x_1 = x_{i-1}$. Without loss of generality, $x_{i+1} = x_{j-1}$. Thus the path $x_{i-1}vx_1$ is in the boundary of the cuff-cycle C that contains v. Moreover, the edge $vx_{i+1} = vx_{j-1}$ counts twice in $\operatorname{nc}(F)$. Since $\operatorname{nc}(F) \leqslant 2$, every edge on F except vx_{i+1} and vx_{j-1} is a cuff-edge. Thus every edge in the walk $v, x_1, x_2, \dots, x_{i-1}, x_i = v$ is in C, and hence $v, x_1, x_2, \dots, x_{i-1}, x_i = v$ is the cycle C. Similarly, $x_{i+1}, x_{i+2}, \dots, x_{j-2}, x_{j-1} = x_{i+1}$ is a cycle C' bounding some other cuff. Hence vx_{i+1} is the only non-cuff edge incident to C, and the same for C'. Therefore G consists of two cuff-cycles joined by a non-cuff edge, and Case 3 applies.

Therefore, $\mathrm{nc}(F) \geqslant 3$, as claimed.

Let $n := |V(G)|$, $m := |E(G)|$, and f be the number of faces of G. It follows from Euler's formula that

$$n - m + f + c = 2 - g. \tag{9}$$

Every non-cuff edge appears exactly twice in faces of G (either twice in the same face, or once in two distinct faces). Thus

$$2t = \sum_{F \text{ face of } G} \mathrm{nc}(F) \geqslant 3f. \tag{10}$$

Since $n = m - t$, we deduce from (9) and (10) that

$$t = f + c + g - 2 \leqslant \frac{2}{3}t + c + g - 2.$$

Thus $t \leqslant 3(c + g)$, and $m \leqslant 9t \leqslant 27(c + g)$ by (8). This allows us, in turn, to bound the number of edges in $G[4k]$:

$$\left|E\big(G[4k]\big)\right| = \binom{4k}{2}n + (4k)^2 m \leqslant (4k)^2 \cdot 2m \leqslant 54(4k)^2(c + g) \leqslant 2(24k)^2(c + g).$$

Since H is a minor of $G[4k]$, we have $|E(H)| \leqslant |E(G[4k])|$. Thus the minimum degree d of H can be upper bounded as follows:

$$2\left|E(H)\right| \geqslant d\left|V(H)\right| \geqslant d^2,$$

and hence

$$d \leqslant \sqrt{2\left|E(H)\right|} \leqslant \sqrt{2\left|E\big(G[4k]\big)\right|} \leqslant \sqrt{2 \cdot 2(24k)^2(c + g)} = 48k\sqrt{c + g},$$

as desired. □

Now we put everything together and prove Theorem 3.1.

Proof of Theorem 3.1. Let $G \in \mathcal{G}(g, p, k)$. By Lemma 3.4, there exists a (g, p)-embedded graph G' with

$$\eta(G) \leqslant \eta\big(G'[k + 1]\big) + \sqrt{6g} + 4.$$

Let $t := \eta(G'[k + 1])$. Thus K_t is a $(k + 1)$-minor of G' by Lemma 3.3. Lemma 3.7 with $H = K_t$ implies that

$$\eta\big(G'[k + 1]\big) - 1 = t - 1 \leqslant 48(k + 1)\sqrt{g + p}.$$

Hence $\eta(G) \leqslant 48(k + 1)\sqrt{g + p} + \sqrt{6g} + 5$, as desired. □

4. Constructions

This section describes constructions of graphs in $\mathcal{G}(g, p, k, a)$ that contain large complete graph minors. The following lemma, which in some sense, is converse to Lemma 3.4 will be useful.

Lemma 4.1. *Let G be a graph embedded in a surface with Euler genus at most g. Let $F_1, \dots, F_p$ be pairwise vertex-disjoint facial cycles of G, such that $V(F_1) \cup \dots \cup V(F_p) = V(G)$. Then for all $k \geqslant 1$, some graph in $\mathcal{G}(g, p, k)$ contains $G[k]$ as a minor.*

Proof. Let G' be the embedded multigraph obtained from G by replacing each edge vw of G by k^2 edges between v and w bijectively labeled from $\{(i, j)\colon i, j \in [1, k]\}$. Embed these new edges 'parallel'

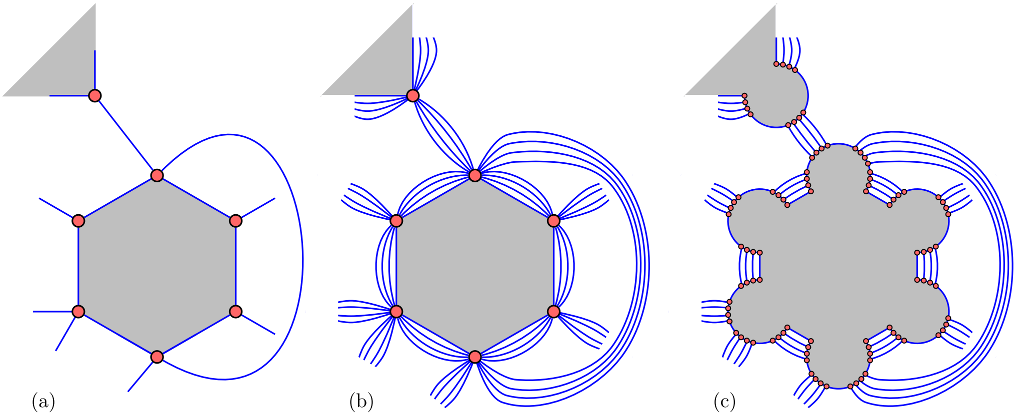

Fig. 3. Illustration for Lemma 4.1: (a) original graph G, (b) multigraph G', (c) splitting H_0 of G'.

to the original edge vw. Let H_0 be the splitting of G' at $F_1, \dots, F_p$. Edges in H_0 inherit their label in G'. For each $\ell \in [1, p]$, let J_ℓ be the face of H_0 that corresponds to F_ℓ (see Fig. 3).

Let H_ℓ be the graph with vertex set $V(J_\ell) \cup \{(v, i)\colon\ v \in V(F_\ell),\ i \in [1, k]\}$, where:

(a) each vertex x in J_ℓ that belongs to a vertex v in F_ℓ is adjacent to each vertex (v, i) in H_ℓ; and
(b) vertices (v, i) and (w, j) in H_ℓ are adjacent if and only if $v = w$ and $i \neq j$.

We now construct a circular decomposition $\{B\langle x\rangle\colon\ x \in V(J_\ell)\}$ of H_ℓ with perimeter J_ℓ. For each vertex x in J_ℓ that belongs to a vertex v in F_ℓ, let $B\langle x\rangle$ be the set $\{x\} \cup \{(v, i)\colon\ i \in [1, k]\}$ of vertices in H_ℓ. Thus $|B\langle x\rangle| \leqslant k + 1$. For each type-(a) edge between x and (v, i), the endpoints are both in bag $B\langle x\rangle$. For each type-(b) edge between (v, i) and (v, j) in H_ℓ, the endpoints are in every bag $B\langle x\rangle$ where x belongs to v. Thus the endpoints of every edge in H_ℓ are in some bag $B\langle x\rangle$. Thus $\{B\langle x\rangle\colon\ x \in V(J_\ell)\}$ is a circular decomposition of H_ℓ with perimeter J_ℓ and width at most k.

Let H be the graph $H_0 \cup H_1 \cup \dots \cup H_p$. Thus $V(H_0) \cap V(H_\ell) = V(J_\ell)$ for each $\ell \in [1, p]$. Since $J_1, \dots, J_p$ are pairwise vertex-disjoint facial cycles of H_0, the subgraphs $H_1, \dots, H_p$ are pairwise vertex-disjoint. Hence H is (g, p, k)-almost embeddable.

To complete the proof, we now construct a model $\{D_{v,i}\colon\ v^{(i)} \in V(G[k])\}$ of $G[k]$ in H, where $v^{(i)}$ is the i-th vertex in the k-clique of $G[k]$ corresponding to v. Fix an arbitrary total order $\preccurlyeq$ on $V(G)$. Consider a vertex $v^{(i)}$ of $G[k]$. Say v is in face F_ℓ. Add the vertex (v, i) of H_ℓ to $D_{v,i}$. For each edge $v^{(i)}w^{(j)}$ of $G[k]$ with $v \prec w$, by construction, there is an edge xy of H_0 labeled (i, j), such that x belongs to v and y belongs to w. Add the vertex x to $D_{v,i}$. Thus $D_{v,i}$ induces a connected star subgraph of H consisting of type-(a) edges in H_ℓ. Since every vertex in J_ℓ is incident to at most one labeled edge, $D_{v,i} \cap D_{w,j} = \emptyset$ for distinct vertices $v^{(i)}$ and $w^{(j)}$ of $G[k]$.

Consider an edge $v^{(i)}w^{(j)}$ of $G[k]$. If $v = w$ then $i \neq j$ and v is in some face F_ℓ, in which case a type-(b) edge in H_ℓ joins the vertex (v, i) in $D_{v,i}$ with the vertex (w, j) in $D_{w,j}$. Otherwise, without loss of generality, $v \prec w$ and by construction, there is an edge xy of H_0 labeled (i, j), such that x belongs to v and y belongs to w. By construction, x is in $D_{v,i}$ and y is in $D_{w,j}$. In both cases there is an edge of H between $D_{v,i}$ and $D_{w,j}$. Hence the $D_{v,i}$ are the branch sets of a $G[k]$-model in H. □

Our first construction employs just one vortex and is based on an embedding of a complete graph.

Lemma 4.2. *For all integers $g \geqslant 0$ and $k \geqslant 1$, there is an integer $n \geqslant k\sqrt{6g}$ such that K_n is a minor of some $(g, 1, k)$-almost embeddable graph.*

Proof. The claim is vacuous if $g = 0$. Assume that $g \geqslant 1$. The map color theorem [7] implies that K_m triangulates some surface if and only if $m \bmod 6 \in \{0, 1, 3, 4\}$, in which case the surface has Euler genus $\frac{1}{6}(m-3)(m-4)$. It follows that for every real number $m_0 \geqslant 2$ there is an integer m such that $m_0 \leqslant m \leqslant m_0+2$ and K_m triangulates some surface of Euler genus $\frac{1}{6}(m-3)(m-4)$. Apply this result with $m_0 = \sqrt{6g}+1$ for the given value of g. We obtain an integer m such that $\sqrt{6g}+1 \leqslant m \leqslant \sqrt{6g}+3$ and K_m triangulates a surface Σ of Euler genus $g' := \frac{1}{6}(m-3)(m-4)$. Since $m-4 < m-3 \leqslant \sqrt{6g}$, we have $g' \leqslant g$. Every triangulation has facewidth at least 3. Thus, deleting one vertex from the embedding of K_m in Σ gives an embedding of K_{m-1} in Σ, such that some facial cycle contains every vertex. Let $n := (m-1)k \geqslant k\sqrt{6g}$. Lemma 4.1 implies that $K_{m-1}[k] \cong K_n$ is a minor of some $(g', 1, k)$-almost embeddable graph. $\square$

Now we give a construction based on grids. Let L_n be the $n \times n$ planar grid graph. This graph has vertex set $[1,n] \times [1,n]$ and edge set $\{(x,y)(x',y')\colon |x-x'|+|y-y'|=1\}$. The following lemma is well known; see [9].

Lemma 4.3. *K_{nk} is a minor of $L_n[2k]$ for all $k \geqslant 1$.*

Proof. For $x, y \in [1,n]$ and $z \in [1,2k]$, let (x,y,z) be the z-th vertex in the $2k$-clique corresponding to the vertex (x,y) in $L_n[2k]$. For $x \in [1,n]$ and $z \in [1,k]$, let $B_{x,z}$ be the subgraph of $L_n[2k]$ induced by $\{(x,y,2z-1),(y,x,2z)\colon\ y \in [1,n]\}$. Clearly $B_{x,z}$ is connected. For all $x, x' \in [1,n]$ and $z, z' \in [1,k]$ with $(x,z) \neq (x',z')$, the subgraphs $B_{x,z}$ and $B_{x',z'}$ are disjoint, and the vertex $(x,x',2z-1)$ in $B_{x,z}$ is adjacent to the vertex $(x,x',2z')$ in $B_{x',z'}$. Thus the $B_{x,z}$ are the branch sets of a K_{nk}-minor in $L_n[2k]$. $\square$

Lemma 4.4. *For all integers $k \geqslant 2$ and $p \geqslant 1$, there is an integer $n \geqslant \frac{2}{3\sqrt{3}}k\sqrt{p}$, such that K_n is a minor of some $(0,p,k)$-almost embeddable graph.*

Proof. Let $m := \lfloor\sqrt{p}\rfloor$ and $\ell := \lfloor\frac{k}{2}\rfloor$. Let $n := 2m\ell \geqslant 2 \cdot \sqrt{\frac{p}{3}} \cdot \frac{k}{3} = \frac{2}{3\sqrt{3}}k\sqrt{p}$. For $x, y \in [1,m]$, let $F_{x,y}$ be the face of L_{2m} with vertex set $\{(2x-1,2y-1),(2x,2y-1),(2x,2y),(2x-1,2y)\}$. There are m^2 such faces, and every vertex of L_{2m} is in exactly one such face. By Lemma 4.3, K_n is a minor of $L_{2m}[2\ell]$. Since L_{2m} is planar, by Lemma 4.1, K_n is a minor of some $(0,m^2,2\ell)$-almost embeddable graph. The result follows since $p \geqslant m^2$ and $k \geqslant 2\ell$. $\square$

The following theorem summarizes our constructions of almost embeddable graphs containing large complete graph minors.

Theorem 4.5. *For all integers $g \geqslant 0$ and $p \geqslant 1$ and $k \geqslant 2$, there is an integer $n \geqslant \frac{1}{4}k\sqrt{p+g}$, such that K_n is a minor of some (g,p,k)-almost embeddable graph.*

Proof. First suppose that $g \geqslant p$. By Lemma 4.2, there is an integer $n \geqslant k\sqrt{6g}$, such that K_n is a minor of some $(g,1,k)$-almost embeddable graph, which is also (g,p,k)-embeddable (since $p \geqslant 1$). Since $n \geqslant k\sqrt{3p+3g} > \frac{1}{4}k\sqrt{p+g}$, we are done.

Now assume that $p > g$. By Lemma 4.4, there is an integer $n \geqslant \frac{2}{3\sqrt{3}}k\sqrt{p}$, such that K_n is a minor of some $(0,p,k)$-almost embeddable graph, which is also (g,p,k)-embeddable (since $g \geqslant 0$). Since $n \geqslant \frac{2}{3\sqrt{3}}k\sqrt{\frac{g}{2}+\frac{p}{2}} = \frac{\sqrt{2}}{3\sqrt{3}}k\sqrt{g+p} > \frac{1}{4}k\sqrt{g+p}$, we are done. $\square$

Adding a dominant vertices to a graph increases its Hadwiger number by a. Thus Theorem 4.5 implies:

Theorem 4.6. *For all integers $g, a \geqslant 0$ and $p \geqslant 1$ and $k \geqslant 2$, there is an integer $n \geqslant a + \frac{1}{4}k\sqrt{p+g}$, such that K_n is a minor of some graph in $\mathcal{G}(g,p,k,a)$.*

Corollary 3.2 and Theorem 4.6 together prove Theorem 1.1.

Acknowledgment

Thanks to Bojan Mohar for instructive conversations.

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