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L^p Inequalities

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1. INTRODUCTION

Riesz–Thorin interpolation, in one of its various forms, may be viewed as the key to establishing a number of classical L^p inequalities of which the following due to Clarkson and Schoenberg are representative [2, 9, 10]:

$$\text{(a)}\quad (\|x+y\|_p^{p'} + \|x-y\|_p^{p'})^{1/p'} \leqslant 2^{1/p'}(\|x\|_p^p + \|y\|_p^p)^{1/p} \qquad (1 < p \leqslant 2,\ p' = p/(p-1));$$

$$\text{(b)}\quad (\|x+y\|_p^p + \|x-y\|_p^p)^{1/p} \leqslant 2^{1/p}(\|x\|_p^{p'} + \|y\|_p^{p'})^{1/p'} \qquad (2 \leqslant p < \infty);$$

$$\text{(c)}\quad \sum_{j,k=1}^{n} c_j c_k \|x_j - x_k\|_p^\lambda \leqslant 2\left(\max_{1\leqslant j\leqslant n}(1-c_j)\right)^{2-\lambda} \sum_{j=1}^{n} c_j \|x_j\|_p^\lambda \qquad \left(1 \leqslant \lambda \leqslant p',\ 2 \leqslant p < \infty,\ c_j \geqslant 0,\ \sum_{j=1}^{n} c_j = 1\right).$$

The technique, as developed in [5, 11], consists of applying an interpolation argument to a contrived matrix operator acting between the direct sum of L^p spaces. For example, the appropriate matrix operator T for inequality (a) is

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \end{bmatrix},$$

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and inequality (b) is dual to (a) in the sense that it expresses the action of T^* between the appropriate conjugate direct sum.

The purpose of this note is to show that there is a unifying approach to the study of such inequalities, and that (a) and its relatives—some old, some new—are in fact special cases of a Hausdorff–Young theorem for vector-valued functions. In addition we shall show that certain inequalities dual to (c), important in the theory of extending Lipschitz–Hölder maps between L^p spaces and previously established with considerable fuss in [11], follow directly by examining adjoints.

2. A Generalized Hausdorff-Young Theorem

The extension we need is a straightforward modification of the standard result [6] and depends only on an appropriate version of the Riesz–Thorin interpolation theorem. Sufficient for our purposes is that to be found in Benedek and Panzone [1] (see also [10, Chap. V]).

Let G be a locally compact abelian group with dual group Γ, and denote by ν and η the Haar measures on G and Γ, respectively, normalized so that the inversion theorem holds. Also let (Ω, Σ, μ) be a σ-finite measure space. We define $L_{p,\lambda}(G)$ $(1 \leqslant p, \lambda \leqslant \infty)$ to be the space of all complex-valued functions x on $G \times \Omega$ that are product measurable and have finite mixed-norm

$$\| x \|_{p,\lambda} = \left(\int_G \| x(t, \cdot)\|_p^\lambda \, d\nu(t)\right)^{1/\lambda} = \left(\int_G \left[\int_\Omega | x(t, s)|^p \, d\mu(s)\right]^{\lambda/p} d\nu(t)\right)^{1/\lambda} \quad (\lambda < \infty),$$
$$= \operatorname*{ess\,sup}_{t\in G} \| x(t, \cdot)\|_p \quad (\lambda = \infty). \tag{1}$$

Under this norm the linear spaces $L_{p,\lambda}(G)$ are Banach spaces with dual $(L_{p,\lambda}(G))^* = L_{p',\lambda'}(G)$ for $1 \leqslant p < \infty$, $1 \leqslant \lambda < \infty$, and p' and λ' the ordinary conjugate indices [1]. Of course, a similar statement applies to the analogously defined spaces $L_{p,\lambda}(\Gamma)$.

Let H denote the linear span of the simple measurable functions on $G \times \Omega$ supported on rectangles and note that H is dense in $L_{p,\lambda}(G)$ $(1 \leqslant p, \lambda < \infty)$. On H we define the Fourier transform operator by

$$\mathscr{F}x(\gamma, s) = \hat{x}(\gamma, s) = \int_G \gamma(t)\, x(t, s)\, d\nu(t) \qquad (\gamma \in \Gamma, s \in \Omega). \tag{2}$$

Clearly $\mathscr{F}x(\gamma, s)$ is product measurable on $\Gamma \times \Omega$. We shall now show that $\mathscr{F}$ has a unique extension to an operator from $L_{p,\lambda}(G)$ into certain of the spaces $L_{q,r}(\Gamma)$, and that that extension, which we shall continue to denote by $\mathscr{F}$, satisfies the following inequalities.

THEOREM 1. *If* $x \in L_{p,\lambda}(G)$, $\lambda' = \lambda/(\lambda - 1)$, *and* $p' = p/(p-1)$, *then*

$$\| \mathscr{F} x \|_{p,\lambda'} \leqslant \| x \|_{p,\lambda} \qquad (1 < \lambda \leqslant p \leqslant 2) \tag{3}$$

and

$$\| \mathscr{F} x \|_{p,\lambda'} \leqslant \| x \|_{p,\lambda} \qquad (1 \leqslant \lambda \leqslant p', 2 \leqslant p < \infty). \tag{4}$$

Proof. The first step is to show that

$$\| \mathscr{F} x \|_{2,2} = \| x \|_{2,2} \tag{5}$$

and

$$\| \mathscr{F} x \|_{p,\infty} \leqslant \| x \|_{p,1} \qquad (1 \leqslant p \leqslant \infty) \tag{6}$$

hold for each $x \in H$. We omit the straightforward verifications which depend, in (5), on the standard Plancherel theorem and, in (6), on the integral form of Minkowskii's inequality.

The interpolation theorem from [1] states that if T is a linear operator from H to the measurable functions on $\Gamma \times G$ which satisfies the inequalities

$$\| Tx \|_{q_i,r_i} \leqslant M_i \| x \|_{p_i,\lambda_i} \qquad (i = 1, 2)$$

for every $x \in H$, with $1 \leqslant p_i, \lambda_i, q_i, r_i \leqslant \infty$, and

$$\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}, \qquad \frac{1}{\lambda} = \frac{1-t}{\lambda_1} + \frac{t}{\lambda_2},$$

$$\frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2}, \qquad \text{and} \qquad \frac{1}{r} = \frac{1-t}{r_1} + \frac{t}{r_2}$$

for some $t \in (0, 1)$, then

$$\| Tx \|_{q,r} \leqslant M_1^{1-t} M_2{}^t \| x \|_{p,\lambda} \qquad \text{for every } x \in H.$$

The result of applying this interpolation theorem to (5) and (6) with $p = 2$ is the inequality

$$\| \mathscr{F} x \|_{2,\lambda'} \leqslant \| x \|_{2,\lambda} \qquad (1 \leqslant \lambda \leqslant 2) \tag{7}$$

for every $x \in H$. Now interpolate again between (6), this time with $p = 1$, and (7); the result is (3) for $x \in H$. Interpolating between the same pair, with $p = \infty$ in (6), we obtain inequality (4) for $x \in H$. That the inequalities hold for all x in $L_{p,\lambda}(G)$ is a consequence of the continuity of $\mathscr{F}$ and the fact that H is dense in $L_{p,\lambda}(G)$ for $p \neq \infty$, $\lambda \neq \infty$. Furthermore, inequality (4) with $p = \infty$ is valid for all x in the closure of H in $L_{\infty,1}(G)$.

We note that in case Ω is a single point, p disappears and both (3) and (4) reduce to the standard Hausdorff–Young inequality. Also, if G is compact, (4) holds for $p = \infty$ by direct computation.

3. Clarkson's Inequalities

Consider the special case in Theorem 1 in which G is the cyclic group $\{1, 2, \ldots, n\}$ under addition mod n. The dual group Γ of G is $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, where $\gamma_j(k) = w^{jk}$ and $w = e^{2\pi i/n}$. The Haar measures are the counting measure on G and $1/n$ times the counting measure on Γ. Writing $x_k(s)$ for $x(k, s)$, the integrals in (3) and (4) yield the following relations:

$$\mathscr{F}x(\gamma, s) = \int_G \gamma(t)\, x(t, s)\, d\nu(t) = \sum_{k=1}^{n} \gamma(k)\, x_k(s) \qquad (\gamma \in \Gamma),$$

$$\left(\int_\Omega |\mathscr{F}x(t, s)|^p\, d\mu(s)\right)^{1/p} = \left\| \sum_{k=1}^{n} \gamma(k)\, x_k \right\|_p \qquad (\gamma \in \Gamma),$$

and

$$\| \mathscr{F}x \|_{p,\lambda'} = \left(\int_\Gamma \left\| \sum_{k=1}^{n} \gamma(k)\, x_k \right\|_p^{\lambda'} d\eta(\gamma)\right)^{1/\lambda'} = \left(\sum_{j=1}^{n} \frac{1}{n} \left\| \sum_{k=1}^{n} w^{jk} x_k \right\|_p^{\lambda'}\right)^{1/\lambda'}.$$

Theorem 2. *If $x_1, x_2, \ldots, x_n$ belong to $L^p(\Omega)$ $(1 \leqslant p \leqslant \infty)$, then*

$$\left(\sum_{j=1}^{n} \frac{1}{n} \left\| \sum_{k=1}^{n} w^{jk} x_k \right\|_p^{\lambda'}\right)^{1/\lambda'} \leqslant \left(\sum_{k=1}^{n} \| x_k \|_p^{\lambda}\right)^{1/\lambda} \qquad (1 < \lambda \leqslant \min\{p, p'\}) \tag{8}$$

and

$$\left(\sum_{k=1}^{n} \| x_k \|_p^{\lambda'}\right)^{1/\lambda'} \leqslant \left(\sum_{j=1}^{n} \left\| \sum_{k=1}^{n} w^{jk} x_k \right\|_p^{\lambda}\right)^{1/\lambda} \qquad (1 < \lambda \leqslant \min\{p, p'\}). \tag{9}$$

Proof. The inequalities in (8) follow directly from inequalities (3), (4) and the preceeding computations. To obtain (9) we notice that the Fourier transform for the present choice of G is generated by the matrix

$$A = \begin{bmatrix} w & w^2 & \cdots & w^{n-1} & 1 \\ w^2 & w^4 & \cdots & w^{2(n-1)} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix},$$

and that $A^*A = AA^* = nI$ (identity matrix). So if we choose a vector $x = (x_1, \ldots, x_n)$ in $L_{p,\lambda}(G)$ and apply inequality (8) to A^*x, the result is

$$\left(\sum_{k=1}^{n} \frac{1}{n} \| n x_k \|_p^{\lambda'}\right)^{1/\lambda'} \leqslant \left(\sum_{j=1}^{n} \left\| \sum_{k=1}^{n} \bar{w}^{jk} x_k \right\|_p^{\lambda}\right)^{1/\lambda}.$$

This is clearly equivalent to (9).

Special insight into the meaning of inequalities (8) and (9) may be gained by the further specialization of setting $n = 2$. Thus $w = -1$ and the inequalities become

$$(\tfrac{1}{2}\| -x_1 + x_2 \|_p^{\lambda'} + \tfrac{1}{2}\| x_1 + x_2 \|_p^{\lambda'})^{1/\lambda'} \leqslant (\| x_1 \|_p^{\lambda} + \| x_2 \|_p^{\lambda})^{1/\lambda} \qquad (1 \leqslant \lambda \leqslant \min\{p, p'\})$$

and

$$(\| x_1 \|_p^{\lambda'} + \| x_2 \|_p^{\lambda'})^{1/\lambda'} \leqslant (\tfrac{1}{2}\| -x_1 + x_2 \|_p^{\lambda} + \tfrac{1}{2}\| x_1 + x_2 \|_p^{\lambda})^{1/\lambda} \qquad (1 < \lambda \leqslant \min\{p, p'\}).$$

Setting $\lambda = p$ or $\lambda = p'$ gives the classical Clarkson inequalities [2]:

$$(\tfrac{1}{2}\| x_1 + x_2 \|_p^{p'} + \tfrac{1}{2}\| x_1 - x_2 \|_p^{p'})^{1/p'} \leqslant (\| x_1 \|_p^{p} + \| x_2 \|_p^{p})^{1/p'} \qquad (1 < p \leqslant 2); \qquad (10)$$

$$(\tfrac{1}{2}\| x_1 + x_2 \|_p^{p} + \tfrac{1}{2}\| x_1 - x_2 \|_p^{p})^{1/p} \leqslant (\| x_1 \|_p^{p'} + \| x_2 \|_p^{p'})^{1/p} \qquad (2 \leqslant p < \infty); \qquad (11)$$

$$(\| x_1 \|_p^{p'} + \| x_2 \|_p^{p'})^{1/p'} \leqslant (\tfrac{1}{2}\| x_1 + x_2 \|_p^{p} + \tfrac{1}{2}\| x_1 - x_2 \|_p^{p})^{1/p} \qquad (2 \leqslant p < \infty); \qquad (12)$$

and

$$(\| x_1 \|_p^{p} + \| x_2 \|_p^{p})^{1/p} \leqslant (\tfrac{1}{2}\| x_1 + x_2 \|_p^{p'} + \tfrac{1}{2}\| x_1 - x_2 \|_p^{p'})^{1/p'} \qquad (1 < p \leqslant 2). \qquad (13)$$

From the corresponding inequalities for $n = 3$ and $w = e^{2\pi i/3}$, it is easy to see that the following geometric property holds in L^p space $(1 < p < \infty)$: if

$$\| x_n \| = \| y_n \| = \| z_n \| = 1 \qquad \text{and} \qquad \| x_n + y_n + z_n \| \to 3,$$

then

$$\| wx_n + w^2 y_n + z_n \| \to 0 \qquad \text{and} \qquad \| w^2 x_n + wy_n + z_n \| \to 0.$$

This and its obvious generalizations are equivalent to uniform convexity in an arbitrary Banach space.

4. Inequalities of Klamkin, Hlawka and Khinchin

Another interesting special case of Theorem 1 occurs if $G = \bigoplus_{k=1}^{n} H_k$ is the direct sum of n copies of the two element group $H_k = \{0, 1\}$, the operation being addition mod 2. This group is generated by elements $e_1, e_2, \ldots, e_n$, where $e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ has its only nonzero entry in the jth coordinate. Each function γ from $\{1, 2, \ldots, n\}$ into $\{-1, 1\}$ can be extended uniquely to a character on G, and conversely, each character on G is such an extension. That is, the dual group Γ of G can be identified with the set of functions from $\{1, 2, \dots, n\}$ into $\{-1, 1\}$, where if $g = \sum_{j=1}^{n} a_j e_j$, $a_j \in \{0, 1\}$, and $\gamma \in \Gamma$, then $\gamma(g) = \prod_{j=1}^{n} [\gamma(j)]^{a_j}$. We take Haar measure on G to be the counting measure and Haar measure on Γ to be $1/2^n$ times the counting measure. Under these assumptions, inequalities (3) and (4) become

$$\left(\sum_{\gamma\in\Gamma} (1/2^n) \| \hat{x}(\gamma, \cdot)\|_p^{\lambda'}\right)^{1/\lambda'} \leqslant \left(\sum_{g\in G} \| x(g, \cdot)\|_p^{\lambda}\right)^{1/\lambda} \qquad (1 < \lambda \leqslant \min\{p, p'\}). \tag{14}$$

If we interchange the roles of G and Γ, then (3) and (4) yield

$$\left(\sum_{g\in G} \| \hat{y}(g, \cdot)\|_p^{\lambda'}\right)^{1/\lambda'} \leqslant \left(\sum_{\gamma\in\Gamma} (1/2^n) \| y(\gamma, \cdot)\|_p^{\lambda}\right)^{1/\lambda} \qquad (1 < \lambda \leqslant \min\{p, p'\}). \tag{15}$$

But if $y = \hat{x}$ for some x in $L_{p,\lambda}(G)$, then

$$\begin{aligned} \hat{y}(g, \cdot) &= \sum_{\gamma\in\Gamma} (1/2^n)\, \gamma(g)\, y(\gamma, \cdot) \\ &= \sum_{\gamma\in\Gamma} (1/2^n)\, \gamma(g)\, \hat{x}(\gamma, \cdot) \\ &= \sum_{\gamma\in\Gamma} (1/2^n)\, \gamma(g) \sum_{h\in G} \gamma(h)\, x(h, \cdot) \\ &= \sum_{h\in G} \left(\sum_{\gamma\in\Gamma} (1/2^n)\, \gamma(g)\, \gamma(h)\right) x(h, \cdot). \end{aligned}$$

However if $g \neq h$, exactly one-half of elements of Γ agree at g and h and thus $\sum_{\gamma\in\Gamma} 1/2^n\, \gamma(g)\, \gamma(h) = 0$. Hence $\hat{y}(g, \cdot) = x(g, \cdot)$ and (15) becomes

$$\left(\sum_{g\in G} \| x(g, \cdot)\|_p^{\lambda'}\right)^{1/\lambda'} \leqslant \left(\sum_{\gamma\in\Gamma} (1/2^n) \| \hat{x}(\gamma, \cdot)\|_p^{\lambda}\right)^{1/\lambda} \qquad (1 < \lambda \leqslant \min\{p, p'\}). \tag{16}$$

Now further restrict x to be supported on $\{e_1, e_2, \dots, e_n\}$ and put $x_j(\cdot) = x(e_j, \cdot)$, $j = 1, 2, \dots, n$. Then

$$x(\gamma, \cdot) = \sum_{g\in G} \gamma(g)\, x(g, \cdot) = \sum_{j=1}^{n} \gamma(j)\, x_j$$

and inequalities (14) and (16) yield the following.

THEOREM 3. *Let $x_1, x_2, \dots, x_n$ belong to $L^p(\Omega)$. Then*

$$\left(\sum_{\gamma\in\Gamma} (1/2^n) \left\| \sum_{j=1}^{n} \gamma(j)\, x_j \right\|_p^{\lambda'}\right)^{1/\lambda'} \leqslant \left(\sum_{j=1}^{n} \| x_j \|_p^{\lambda}\right)^{1/\lambda} \tag{17}$$

and

$$\left(\sum_{j=1}^{n} \| x_j \|_p^{\lambda'}\right)^{1/\lambda'} \leqslant \left(\sum_{\gamma\in\Gamma} (1/2^n) \left\| \sum_{j=1}^{n} \gamma(j)\, x_j \right\|_p^{\lambda}\right)^{1/\lambda} \tag{18}$$

for $1 < \lambda \leqslant \min\{p, p'\}$.

An application of Hölder's inequality gives the additional inequalities,

$$\left(\sum_{\gamma\in\Gamma}(1/2^n)\left\|\sum_{j=1}^{n}\gamma(j)\,x_j\right\|_p^{\lambda}\right)^{1/\lambda}\leqslant\left(\sum_{\gamma\in\Gamma}(1/2^n)\left\|\sum_{j=1}^{n}\gamma(j)\,x_j\right\|_p^{\lambda'}\right)^{1/\lambda'}\leqslant\left(\sum_{j=1}^{n}\|x_j\|_p\right)^{1/\lambda} \tag{19}$$

and

$$\left(\sum_{j=1}^{n}\|x_j\|_p^{\lambda'}\right)^{1/\lambda'}\leqslant\left(\sum_{\gamma\in\Gamma}(1/2^n)\left\|\sum_{j=1}^{n}\gamma(j)\,x_j\right\|_p^{\lambda}\right)^{1/\lambda}\leqslant\left(\sum_{\gamma\in\Gamma}(1/2^n)\left\|\sum_{j=1}^{n}\gamma(j)\,x_j\right\|_p^{\lambda'}\right)^{1/\lambda'} \tag{20}$$

for $1 < \lambda \leqslant \min\{p, p'\}$.

The outside inequalities in (19) and (20) extend inequalities of Klamkin [7] to L^p spaces. For example, it follows from (19) that

$$\sum\|\pm x_1 \pm x_2 \pm \cdots \pm x_n\|_p^{\lambda} \leqslant 2\sum_{j=1}^{n}\|x_j\|_p^{\lambda} \qquad (0<\lambda\leqslant\min\{p,p'\}),$$

where the summation on the left is taken over all 2^n permutations of the $\pm$ signs. (The result for $0 < \lambda \leqslant 1$ follows by convexity of t^λ.)

Hlawka's inequality [8, p. 171] states that if A, B, C are vectors in E^n, then

$$\|A\| + \|B\| + \|C\| + \|A+B+C\| \geqslant \|A+B\| + \|B+C\| + \|A+C\|.$$

And this result holds in any normed linear space. Suppose we now apply the preceding inequalities with $x_1 = (A+B)/2$, $x_2 = (B+C)/2$, and $x_3 = (A+C)/2$ so that $x_1 + x_2 + x_3 = A + B + C$, $x_1 - x_2 + x_3 = A$, $x_1 - x_2 - x_3 = -C$, and $x_1 + x_2 - x_3 = B$.

THEOREM 4. *If A, B, C belong to $L^p(\Omega)$, then*

$$\begin{aligned}(\|A+B+C\|_p^{\lambda'} + \|A\|_p^{\lambda'} + \|B\|_p^{\lambda'} + \|C\|_p^{\lambda'})^{1/\lambda'}&\\ \leqslant 2^{-1+2/\lambda'}(\|A+B\|_p^{\lambda} + \|B+C\|_p^{\lambda} + \|C+A\|_p^{\lambda})^{1/\lambda}&\end{aligned} \tag{21}$$

and

$$\begin{aligned}2^{-1+2/\lambda}(\|A+B\|_p^{\lambda'} + \|B+C\|_p^{\lambda'} + \|C+A\|_p^{\lambda'})^{1/\lambda'}&\\ \leqslant (\|A+B+C\|_p^{\lambda} + \|A\|_p^{\lambda} + \|B\|_p^{\lambda} + \|C\|_p^{\lambda})^{1/\lambda}&\end{aligned} \tag{22}$$

for $1 < \lambda \leqslant \min\{p, p'\}$.

The connection between inequalities (19), (20) and Khinchin's inequality occurs through the Rademacher functions on [0, 1]. Let $\varphi_n(t) = \operatorname{sign}(\sin 2^n\pi t)$ $(0 \leqslant t \leqslant 1,\ n = 1, 2,\ldots)$, and suppose $a_1, a_2, \ldots, a_n$ are complex numbers. The Khinchin inequality states that

$$\left(\int_0^1\left|\sum_{j=1}^{n}\varphi_j(t)\,a_j\right|^{\lambda}dt\right)^{1/\lambda}\leqslant(\lambda/2+1)^{1/2}\left(\sum_{j=1}^{n}|a_j|^2\right)^{1/2}\qquad(0<\lambda<\infty) \tag{23}$$

and that if $\lambda = 2m$ is an even integer, the constant $(\lambda/2 + 1)^{1/2}$ can be replaced by $m^{1/2}$ [3]. Since Γ is the set of maps from $\{1, 2,..., n\}$ into $\{-1, 1\}$, for each γ in Γ, there is a unique interval I_γ of $[0, 1]$ of length $1/2^n$ such that $\varphi_j(t) = \gamma(j)$, $1 \leqslant j \leqslant n$, $t \in I_\gamma$. Consequently,

$$1/2^n \sum_{\gamma\in\Gamma} \Big\| \sum_{j=1}^{n} \gamma(j)\, x_j \Big\|_p^\lambda = \int_0^1 \Big\| \sum_{j=1}^{n} \varphi_j(t)\, x_j \Big\|_p^\lambda dt.$$

These identifications allow us to rewrite inequalities (19) and (20) in the form

$$\left(\int_0^1 \Big\| \sum_{j=1}^{n} \varphi_j(t)\, x_j \Big\|_p^\lambda dt\right)^{1/\lambda} \leqslant \left(\int_0^1 \Big\| \sum_{j=1}^{n} \varphi_j(t)\, x_j \Big\|_p^{\lambda'} dt\right)^{1/\lambda'} \leqslant \left(\sum_{j=1}^{n} \| x_j \|_p^\lambda\right)^{1/\lambda} \tag{24}$$

and

$$\left(\sum_{j=1}^{n} \| x_j \|_p^{\lambda'}\right)^{1/\lambda'} \leqslant \left(\int_0^1 \Big\| \sum_{j=1}^{n} \varphi_j(t)\, x_j \Big\|_p^\lambda dt\right)^{1/\lambda} \leqslant \left(\int_0^1 \Big\| \sum_{j=1}^{n} \varphi_j(t)\, x_j \Big\|_p^{\lambda'} dt\right)^{1/\lambda'} \tag{25}$$

for $1 < \lambda \leqslant \min\{p, p'\}$. With $p = 2$ and $x_j = a_j$ we obtain Khinchin-like inequalities.

THEOREM 5. *If $a_1, a_2, ..., a_n$ are complex numbers and $\varphi_j(t) = \operatorname{sign}(\sin 2^n \pi t)$, then*

$$\left(\int_0^1 \Big| \sum_{j=1}^{n} \varphi_j(t)\, a_j \Big|^\lambda dt\right)^{1/\lambda} \leqslant \left(\sum_{j=1}^{n} | a_j |^\lambda\right)^{1/\lambda} \qquad (0 < \lambda \leqslant 2), \tag{26}$$

$$\left(\int_0^1 \Big| \sum_{j=1}^{n} \varphi_j(t)\, a_j \Big|^\lambda dt\right)^{1/\lambda} \leqslant \left(\sum_{j=1}^{n} | a_j |^{\lambda'}\right)^{1/\lambda'} \qquad (2 \leqslant \lambda < \infty), \tag{27}$$

$$\left(\sum_{j=1}^{n} | a_j |^{\lambda'}\right)^{1/\lambda'} \leqslant \left(\int_0^1 \Big| \sum_{j=1}^{n} \varphi_j(t)\, a_j \Big|^\lambda dt\right)^{1/\lambda} \qquad (1 < \lambda \leqslant 2), \tag{28}$$

and

$$\left(\sum_{j=1}^{n} | a_j |^{\lambda}\right)^{1/\lambda} \leqslant \left(\int_0^1 \Big| \sum_{j=1}^{n} \varphi_j(t)\, a_j \Big|^\lambda dt\right)^{1/\lambda} \qquad (2 \leqslant \lambda < \infty). \tag{29}$$

Proof. Inequality (26) with $0 < \lambda < 1$ follows from the convexity of t^λ. The remaining inequalities are restatements of (24) and (25), where the roles of λ and λ' have been reversed by (27) and (29).

It follows directly from the Khinchin inequality that

$$\left(\int_0^1 \Big| \sum_{j=1}^{n} \varphi_j(t)\, a_j \Big|^\lambda dt\right)^{1/\lambda} \leqslant (\lambda/2 + 1)^{1/2} \left(\sum_{j=1}^{n} | a_j |^2\right)^{1/2}$$

$$\leqslant (\lambda/2 + 1)^{1/2} \left(\sum_{j=1}^{n} | a_j |^\lambda\right)^{1/\lambda} \qquad (0 < \lambda \leqslant 2)$$

and

$$\left(\int_0^1 \left| \sum_{j=1}^n \varphi_j(t)\, a_j \right|^\lambda dt\right)^{1/\lambda} \leqslant (\lambda/2+1)^{1/2} \left(\sum_{j=1}^n |a_j|^2\right)^{1/2}$$

$$\leqslant (\lambda/2+1)^{1/2} \left(\sum_{j=1}^n |a_j|^{\lambda'}\right)^{1/\lambda'} \qquad (2 \leqslant \lambda < \infty).$$

Hence inequalities (26) and (27) are stronger than the corresponding results which can be derived from Khinchin's inequality.

5. Dual Inequalities

The duality relation between inequalities (8) and (9), (17) and (18), as well as some pairs to follow, becomes more apparent in a slightly altered setting. Corresponding to a sequence of positive weights $\alpha_1, \alpha_2, \ldots, \alpha_n$, define $L_{p,\lambda}(\alpha)$ to be the set of all vectors $x = (x_1, \ldots, x_n)$ of measurable functions on Ω with finite norm

$$\| x \|_{p,\lambda} = \left(\sum_{k=1}^n \alpha_k \| x_k \|_p^\lambda\right)^{1/\lambda} \qquad (1 \leqslant p \leqslant \infty, 1 \leqslant \lambda < \infty),$$

$$= \max_{1 \leqslant k \leqslant n} \| x_k \|_p \qquad (1 \leqslant p \leqslant \infty, \lambda = \infty).$$

Every matrix $T = (a_{jk})_{j=1,k=1}^{m,n}$ of complex numbers defines a bounded linear operator from $L_{p,\lambda}(\alpha)$ to $L_{p,q}(\beta)$ (β denotes another sequence of positive weights $\beta_1, \beta_2, \ldots, \beta_m$) and the adjoint map $T^*: L_{p',q'}(\beta) \to L_{p',\lambda'}(\alpha)$ clearly corresponds to the matrix adjoint. We have the following result for such matrix operators.

THEOREM 6. *If*

$$\| Tx \|_{p_i,q_i} \leqslant M_i \| x \|_{p_i,\lambda_i} \qquad (i = 1, 2)$$

and

$$\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}, \qquad \frac{1}{\lambda} = \frac{1-t}{\lambda_1} + \frac{t}{\lambda_2},$$

$$\frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2} \qquad (0 < t < 1),$$

then

$$\left(\sum_{j=1}^m \beta_j \left\| \sum_{k=1}^n a_{jk} x_k \right\|_p^q\right)^{1/q} \leqslant M_1^{1-t} M_2^{\,t} \left(\sum_{k=1}^n \alpha_k \| x_k \|_p^\lambda\right)^{1/\lambda}, \tag{30}$$

$$\left(\sum_{k=1}^n \alpha_k \left\| \sum_{j=1}^m \overline{a_{jk}} x_j \right\|_{p'}^{\lambda'}\right)^{1/\lambda'} \leqslant M_1^{1-t} M_2^{\,t} \left(\sum_{j=1}^m \beta_j \| x_j \|_{p'}^{q'}\right)^{1/q'}, \tag{31}$$

and if there is a constant c such that $T^*T = cI$ (I = *identity matrix*), *then*

$$c\left(\sum_{k=1}^{n} \alpha_k \| x_k \|_{p'}^{\lambda'}\right)^{1/\lambda'} \leqslant M_1^{1-t} M_2^{\,t} \left(\sum_{j=1}^{m} \beta_j \left\| \sum_{k=1}^{n} a_{jk} x_k \right\|_{p'}^{q'}\right)^{1/q'}. \tag{32}$$

Proof. Inequality (30) is a consequence of interpolation [5], (31) is the statement $\| T^*x \|_{p',\lambda'} \leqslant M_1^{1-t} M_2^{\,t} \| x \|_{p',q'}$, and (32) is equivalent to

$$\| cx \|_{p',\lambda'} = \| T^*Tx \|_{p',\lambda'} \leqslant M_1^{1-t} M_2^{\,t} \| Tx \|_{p',q'} .$$

Inequalities (a) and (b) of the introduction are dual in the sense of (30) and (31), whereas inequalities (8) and (9), (10) and (12), (11) and (13), and (17) and (18) are dual in the sense of (30) and (32). As a further example, consider the following important inequality of Harris [4].

THEOREM 7. *Let* n *be a positive integer. For* $1 \leqslant p \leqslant \infty$, *put* $K_p = [4^n \binom{2n}{n}^{-1}]^{|p-2|/p}$ *and* $w = e^{\pi i/n}$. *Then if* $x, y \in L^p(\Omega)$,

$$\sum_{k=1}^{2n} \| x + w^k y \|_p^{2n} \leqslant n \binom{2n}{n} K_p (\| x \|_p^{2n} + \| y \|_p^{2n}) \tag{33}$$

and

$$\begin{aligned}(2n)^{2n/(2n-1)} &(\| x \|_{p'}^{2n/(2n-1)} + \| y \|_{p'}^{2n/(2n-1)}) \\ &\leqslant \left[n \binom{2n}{n} K_p \right]^{1/(2n-1)} \sum_{k=1}^{2n} \| x + w^k y \|_{p'}^{2n/(2n-1)}.\end{aligned} \tag{34}$$

Proof. Inequality (33) has the same form as inequality (8) provided the $2n$ dimensional vector in (8) has only two nonzero entries. However, by restricting the operator to vectors having only two components,

$$T\colon (x, y) \to (x + wy, x + w^2 y, \ldots, x + w^{2n-1} y, x + y),$$

one can derive the sharper estimates

$$\| T(x, y) \|_{2,2n} \leqslant n^{1/2^n} \binom{2n}{n}^{1/2^n} \| (x, y) \|_{2,2n}$$

and

$$\| T(x, y) \|_{p,2n} \leqslant 2n^{1/2^n} \| (x, y) \|_{p,2n} \qquad (p = 1, +\infty).$$

See [4] or [10] for proofs. Inequality (33) is the result of interpolating between these two pairs. Inequality (34) follows from an application of (32) since, in this case, $T^*T = 2nI$.

The idea leading to inequality (32) provides an easy proof to an important inequality first established in [11].

THEOREM 8. *Choose functions $x_1, x_2, \ldots, x_n$ in $L^p(\Omega)$ and nonnegative numbers $c_1, c_2, \ldots, c_n$ such that $\sum_{j=1}^n c_j = 1$. Then*

$$(\max_{1\leqslant j\leqslant n}(1-c_j))^{\lambda-2} \sum_{j,k=1}^n \| x_j - x_k \|_p^\lambda \geqslant 2 \sum_{j=1}^n c_j \left\| x_j - \sum_{k=1}^n c_k x_k \right\|_p^\lambda \qquad (p' \leqslant \lambda, 1 < p \leqslant 2) \qquad (35)$$

and

$$(\max_{1\leqslant j\leqslant n}(1-c_j))^{\lambda-2} \sum_{j,k=1}^n \| x_j - x_k \|_p^\lambda \geqslant 2 \sum_{j=1}^n c_j \left\| x_j - \sum_{k=1}^n c_k x_k \right\|_p^\lambda \qquad (p \leqslant \lambda, 2 \leqslant p < \infty). \qquad (36)$$

Proof. These inequalities are essentially dual to the following classical inequalities of Schoenberg [9]:

$$\sum_{j,k=1}^n c_j c_k \| x_j - x_k \|_p^\lambda \leqslant 2(\max_{1\leqslant j\leqslant n}(1-c_j))^{2-\lambda} \sum_{k=1}^n c_k \| x_k \|_p^\lambda \qquad (1 \leqslant \lambda \leqslant p, 1 \leqslant p \leqslant 2) \qquad (37)$$

and

$$\sum_{j,k=1}^n c_j c_k \| x_j - x_k \|_p^\lambda \leqslant 2(\max_{1\leqslant j\leqslant n}(1-c_j))^{2-\lambda} \sum_{k=1}^n c_k \| x_k \|_p^\lambda \qquad (1 \leqslant \lambda \leqslant p', 2 \leqslant p < \infty). \qquad (38)$$

Inequalities (37) and (38) assert that the matrix map $T(x_1, x_2, \ldots, x_n) = (x_j - x_k)_{j,k=1}^n$ from $L_{p,\lambda}(c_j)$ to $L_{p,\lambda}(c_j c_k)$ is continuous with norm $\leqslant 2^{1/\lambda}(\max_{1\leqslant j\leqslant n}(1-c_j))^{2/\lambda-1}$ for $\lambda \leqslant \min\{p, p'\}$. Then $T^*: L_{p',\lambda'}(c_j c_k) \to L_{p',\lambda'}(c_j)$ is continuous with norm $\leqslant 2^{1/\lambda}(\max_{1\leqslant j\leqslant n}(1-c_j))^{2/\lambda-1}$. If $y \in L_{p',\lambda'}(c_j c_k)$ and $x \in L_{p,\lambda}(c_j)$ we have

$$\begin{aligned}\langle T^*y, x\rangle = \langle y, Tx\rangle &= \sum_{j,k=1}^n c_j c_k \langle y_{jk}, x_j - x_k\rangle \\ &= \sum_{j,k=1}^n c_j c_k \langle y_{jk}, x_j\rangle - \sum_{j,k=1}^n c_j c_k \langle y_{jk}, x_k\rangle \\ &= \sum_{j=1}^n c_j \left\langle \sum_{k=1}^n c_k (y_{jk} - y_{kj}), x_j \right\rangle,\end{aligned}$$

where $\langle , \rangle$ denotes the pairing between a space and its dual. It follows that

$$T^*y = \left\{ \sum_{k=1}^n c_k (y_{jk} - y_{kj}) \right\}_{j=1}^n .$$

Hence

$$T^*(Tx) = \left\{ \sum_{k=1}^{n} c_k(x_j - x_k - (x_k - x_j)) \right\}_{j=1}^{n}$$

$$= \left\{ 2\left(x_j - \sum_{k=1}^{n} c_k x_k \right) \right\}_{j=1}^{n}.$$

Therefore

$$\left(\sum_{j=1}^{n} c_j \left\| 2\left(x_j - \sum_{k=1}^{n} c_k x_k \right) \right\|_{p'}^{\lambda'} \right)^{1/\lambda'}$$

$$= \| T^*Tx \|_{p',\lambda'} \leqslant 2^{1/\lambda} (\max_{1 \leqslant j \leqslant n} (1 - c_j))^{2/\lambda - 1} \left(\sum_{j,k=1}^{n} c_j c_k \| x_j - x_k \|_{p'}^{\lambda'} \right)^{1/\lambda},$$

which reduces to (35) and (36).

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