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Asymptotic quasinormal frequencies of d-dimensional Schwarzschild black holes

Danny Birmingham [1]

Department of Physics, University of California, Santa Barbara, CA 93106, USA

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Abstract

We determine the quasinormal frequencies for all gravitational perturbations of the d-dimensional Schwarzschild black hole, in the infinite damping limit. Using the potentials for gravitational perturbations derived recently by Ishibashi and Kodama, we show that in all cases the asymptotic real part of the frequency is proportional to the Hawking temperature with a coefficient of $\log 3$. Via the correspondence principle, this leads directly to an equally spaced entropy spectrum. We comment on the possible implications for the spacing of eigenvalues of the Virasoro generator in the associated near-horizon conformal algebra.

1. Introduction

The idea that the horizon area of black holes is quantized in equally spaced units has attracted considerable attention [1–6]. Moreover, the possibility of a connection between the classical ringing tones (the quasinormal frequencies) of black holes and the quantum properties of the entropy spectrum was first observed by Bekenstein [7], and further developed by Hod [8]. In particular, Hod proposed that the real part of the quasinormal frequencies, in the infinite damping limit, might be related via the correspondence principle to the fundamental quanta of mass and angular momentum. For the Schwarzschild black hole in four dimensions, the asymptotic real part of the gravitational quasinormal frequencies is of the form $\omega = T_H \log 3$, where T_H is the Hawking temperature [9]. The suggestion of Hod was to identify $\hbar\omega$ with the fundamental quantum of mass ΔM. This identification immediately leads to an area spacing of the form $\Delta A = 4\hbar G \log 3$. In a separate development, Dreyer [10] showed that the correspondence principle, when applied to loop quantum gravity [11], fixes the Immirzi parameter [12] in such a way that the Bekenstein–Hawking entropy is obtained naturally.

The proposed correspondence between quasinormal frequencies and the fundamental quantum of mass automatically leads to an equally spaced area spectrum. It is therefore clearly of interest to determine the universality of this approach to black hole quantization. Although extensions to other black hole spacetimes have been discussed [13–20], the generic situation is still far from clear. One encouraging piece of evidence comes from an analysis of the BTZ black hole in $(2+1)$ dimensions [21]. In this case, it was

E-mail address: birm@itp.stanford.edu (D. Birmingham).

[1] On leave from: Department of Mathematical Physics, University College Dublin, Ireland.

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shown that the correspondence principle leads directly to the correct quantum behaviour of the asymptotic Virasoro algebra [22].

Our aim here is to discuss the situation for the Schwarzschild black hole in d dimensions. Before considering the correspondence principle, it is first necessary to determine the quasinormal frequencies precisely, in the limit of infinite damping. An elegant approach, based on analytic continuation and computation of the monodromy of the perturbation, was proposed by Motl and Neitzke [23]. For perturbations of the d-dimensional Schwarzschild black hole by a scalar field, it was found the real part of the asymptotic frequencies is again of the form $T_H \log 3$. However, for the case of gravitational perturbations in $d > 4$, progress was impeded by the lack of knowledge of the corresponding potentials. Building on earlier work [24,25], this situation has now been rectified by the formalism of Ishibashi and Kodama [26]. It has been shown that the gravitational perturbations fall into three classes, namely scalar, vector, and tensor perturbations, and the exact form of the potential is determined in each case. In one application, for example, these potentials have been used to establish the stability of the higher-dimensional Schwarzschild black hole [27]. Using the method of Motl and Neitzke, we show that the asymptotic quasinormal frequencies of all gravitational perturbations share the $\log 3$ behaviour, in all dimensions. This verifies the conjecture made in [23], see also [13]. By applying the correspondence principle, one is then led immediately to an entropy spectrum with a universal $\log 3$ spacing. We also comment on the implications of this result for the near-horizon conformal symmetry proposed in [28–30].

2. Gravitational perturbations

To begin, let us recall the essential features of the computation for the case of a perturbation by a scalar field Φ satisfying $\nabla^2\Phi = 0$. The basic equation takes the form [23]

$$\left[-\left(f\frac{\partial}{\partial r}\right)^2 + V(r) - \omega^2\right]\psi(r) = 0, \tag{1}$$

where

$$f = 1 - \frac{\omega_d M}{r^{d-3}}, \tag{2}$$

and $\psi = r^{(d-2)/2}\Phi$. We have defined $\omega_d = 16\pi G \times ((d-2)A_{d-2})^{-1}$ in (2), where $A_{d-2} = 2\pi^{(d-1)/2} \times (\Gamma((d-1)/2))^{-1}$ is the volume of the unit $(d-2)$-dimensional sphere, and we note that M has dimensions of inverse length. The Hawking temperature of the black hole is $T_H = f'(r_+)/4\pi = (d-3)/4\pi r_+$, where the horizon radius is defined by $r_+^{d-3} = \omega_d M$.

The potential for the scalar field perturbation is given by [23]

$$V = \frac{f}{r^2}\left[l(l+d-3) + \frac{(d-2)(d-4)}{4} + \frac{(d-2)^2(\omega_d M)}{4r^{d-3}}\right]. \tag{3}$$

The physical region of interest is $r_+ < r < \infty$, and the quasinormal modes are defined in terms of appropriate boundary conditions at $r = r_+$ and $r = \infty$. However, the proposal of [23], see also [31], is to consider an analytic continuation to the complex r-plane. It is then convenient to introduce the tortoise coordinate z defined by $dz = f^{-1}\,dr$, which can be integrated to give [23,25]

$$z = r + \sum_{n=0}^{d-4} e^{2\pi i n/(d-3)} \frac{r_+}{(d-3)} \times \log\left(1 - \frac{r}{r_+}\, e^{-2\pi i n/(d-3)}\right), \tag{4}$$

where the additive constant is chosen so that $z = 0$ for $r = 0$. Thus, z is a multi-valued function for complex r. The determination of the asymptotic quasinormal frequencies involves a computation of the monodromy of $\psi(r)$ as one travels along a closed contour in the complex r-plane. This computation requires the ability to match solutions in the asymptotic region and the region near the singularity; this matching is possible precisely for the asymptotic frequencies of interest. The result of this local computation of the monodromy can then be compared to the global result which follows by direct application of the quasinormal mode boundary condition at the horizon.

For our purposes here, it is sufficient to highlight the behaviour of V in the neighbourhood of the singularity $r = 0$, namely

$$V \sim -\frac{(d-2)^2(\omega_d M)^2}{4r^{2d-4}}. \tag{5}$$

It is straightforward to check that near $r = 0$, we have

$$z \sim -\frac{r^{d-2}}{(d-2)r_+^{d-3}}. \tag{6}$$

Hence, the leading term in the potential near $z = 0$ is

$$V(z) \sim -\frac{1}{4z^2}, \tag{7}$$

and Eq. (1) reduces to Bessel's equation. By matching the solution in this region to the solution in the asymptotic region, the monodromy can be calculated. Comparison with the global computation of the monodromy then yields the asymptotic quasinormal frequencies [23]

$$e^{\beta\omega} = -3, \tag{8}$$

where β is the inverse Hawking temperature.

As emphasized in [23], the details of the above calculation proceed without hindrance for the case of a potential whose behaviour near $z = 0$ is of the form

$$V \sim \frac{j^2 - 1}{4z^2}. \tag{9}$$

The asymptotic quasinormal frequencies in this case are given by

$$e^{\beta\omega} = -(1 + 2\cos\pi j). \tag{10}$$

With this calculation in hand, we can now proceed to discuss the case of gravitational perturbations. To begin, let us consider the vector perturbation, which is the generalization of the Regge–Wheeler equation in four dimensions [32]. The potential takes the form [26]

$$V_\mathrm{V} = \frac{f}{r^2}\left[l(l+d-3) + \frac{(d-2)(d-4)}{4} - \frac{3(d-2)^2(\omega_d M)}{4r^{d-3}}\right], \tag{11}$$

where $l \geqslant 2$. We immediately notice that the leading order behaviour in the neighbourhood of $r = 0$ is given by

$$V_\mathrm{V} \sim \frac{3(d-2)^2(\omega_d M)^2}{4r^{2d-4}}. \tag{12}$$

Using (6), we see that the potential for the vector perturbation is of the form (9) with $j = 2$. The asymptotic frequencies can then be simply read off from (10), giving $e^{\beta\omega} = -3$.

In four dimensions, the gravitational scalar perturbation is described by the Zerilli equation [33,34]. While the form of the Zerilli potential is considerably more complicated that the Regge–Wheeler potential, the quasinormal modes are identical [35]. In higher dimensions, however, we must treat this case separately [26]. The potential is given by

$$V_\mathrm{S} = \frac{f}{r^2}\frac{Q}{16[c + (d-2)(d-1)x/2]^2}, \tag{13}$$

where

$$\begin{aligned} Q = {} & (d-2)^4(d-1)^2x^3 \\ & + (d-2)(d-1) \\ & \times \big\{4\big[2(d-2)^2 - 3(d-2) + 4\big]c \\ & \quad + (d-2)(d-4)(d-6)(d-1)\big\}x^2 \\ & - 12(d-2)\big\{(d-6)c \\ & \qquad + (d-2)(d-1)(d-4)\big\}cx \\ & + \big\{16c^3 + 4(d-2)dc^2\big\}, \end{aligned} \tag{14}$$

and we have defined $c = [l(l+d-3) - (d-2)]$ and $x = \omega_d M/r^{d-3}$, and again $l \geqslant 2$. As in the previous case, it is only necessary to record the behaviour of the potential near $z = 0$, which takes the form

$$V_\mathrm{S} \sim -\frac{(d-2)^2(\omega_d M)^2}{4r^{2d-4}}. \tag{15}$$

Thus, the gravitational scalar potential is of the form (9) with $j = 0$. Hence, the asymptotic frequencies again satisfy (8). Finally, the gravitational tensor perturbations were already considered in [23,25], where it was noticed that they behave like a scalar field perturbation. In fact, the potential for the tensor perturbation is identical to (3).

In conclusion, we have shown that the asymptotic quasinormal frequencies for gravitational perturbations of the Schwarzschild black hole have a universal form in all dimensions, namely

$$\omega = \pm T_H \log 3 + 2\pi i T_H\left(n + \frac{1}{2}\right), \tag{16}$$

as $n \to \infty$. It would be worthwhile investigating this problem numerically, in order to verify the results of the monodromy computation in this higher-dimensional setting. It would also be interesting to calculate the first order correction terms along the lines discussed in [31,36]. Incidentally, the low lying modes for perturbations by a scalar field have been studied recently in [37,38].

3. Discussion

According to the correspondence principle [8], we should identify the elementary quantum of mass ΔM with the energy of a quantum with frequency $\omega = T_H \log 3$ to a quantization of entropy, via the relation

$$\Delta S = \frac{\Delta M}{T_H} = \log 3. \tag{17}$$

In [28–30], a conformal field theory approach to black hole entropy in arbitrary dimensions has been suggested. By treating the horizon as a boundary, one finds that with a suitable choice of boundary conditions the algebra of diffeomorphisms in the $(r\text{–}t)$-plane near the horizon is a Virasoro algebra. For example, in [29,39], the central charge and Virasoro generator are given by

$$L_0 = \frac{S}{2\pi}, \qquad \frac{c}{6} = \frac{S}{\pi}, \tag{18}$$

where $S = A/4G$ is the black hole entropy. It is then a simple matter to check that the Cardy formula for the entropy of the conformal field theory yields precisely the Bekenstein–Hawking entropy S. This result suggests that conformal symmetry plays a key role in understanding the microscopic properties of black holes. Clearly, the quantization of entropy results in a corresponding spacing of the operator L_0, with spacing

$$\Delta L_0 = \frac{1}{2\pi} \log 3. \tag{19}$$

In [30], the Virasoro generator and central charge are given by

$$L_0 = \frac{1}{4\pi^2 q^2}\left(\frac{d-2}{d-3}\right) S,$$
$$\frac{c}{6} = q^2 \left(\frac{d-3}{d-2}\right) S, \tag{20}$$

where q is an arbitrary parameter. The correspondence principle in this case gives a spacing of L_0 of the form

$$\Delta L_0 = \frac{1}{4\pi^2 q^2}\left(\frac{d-2}{d-3}\right) \log 3. \tag{21}$$

The arbitrary parameter q could be fixed if one demands integer spacing of L_0.

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