Dynamical Rewiring Processes in Binary Decision Networks

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Abstract

We present a dynamical binary opinion model for the emergence of collective decision making based on a generalization of the majority and the minority principle. The network consists of $N$ interacting agents, whose connectivity structure is dynamic, governed by a rewiring process controlled by the actual network state. With the aid of damage spreading techniques we show how the dynamical stability of these models can be largely enhanced by self-organized feedback loops between the network topology and the intrinsic network dynamics. The model provides possible explanations for social behaviour, in particular for the stability and instability of the individual and global opinion during an electoral campaign.

Keywords: Complex binary decision networks, Opinion dynamics, Damage spreading

1. Introduction

There has always been considerable interest of theoretical physicists in complex phenomena departing from the traditional avenue of physics research. In particular, the application of statistical physics methods to social phenomena such as opinion formation and economic dynamics has been rather fruitful during the last few years [1, 2, 3, 4, 5]. However, the major part of well established earlier work in sociophysics has been devoted to static network architectures, while real-life networks are usually dynamic with respect to their topology [6, 7, 8, 9]. Moreover, topologic dynamics are often network-intrinsic and quite inhomogeneous, where the evolution of time-dependent connectivity structures is in general not that well understood. The present study extends the analysis of the dynamics of randomly assembled threshold networks to social phenomena first introduced by Galam [2, 3] to flexible network architectures as well as to damage spreading techniques successfully applied earlier. Here the existence of two dynamical phases was established: a stable phase, where the system is resistant to damage spreading and a chaotic phase, where an initially small damage can spread all over the system [10, 11]. In the context of sociophysics this may help to understand under which conditions special shocking events or propaganda are able to influence the results of elections. In this model, individuals gather in groups of fixed size $K$, where $K$ randomly chosen agents discuss a topic in order to arrive at a final decision, in favor or against. The model thus describes how people convince each other by expressing their desire to identify themselves with certain social groups or to differentiate themselves from them. In particular, we study the effects of opportunistic and contrarian interactions guided by Galam’s opinion dynamics model, where
in the absence of opportunists (contrarians), the dynamics is ordered and leads to stable configurations. In contrast, coupled systems exhibit a dynamical phase transition to disordered dynamics.

2. A general model of discrete opinions

The model network consists of a population of \( N \) agents of a democratic community who have a specific opinion on a particular subject. The opinion of agent \( i \) is described by an Ising spin variable \( \sigma_i \) only capable to take the value +1 or −1, i.e. to say YES and NO, to buy or not to buy, or to vote Party A or Party B, respectively. We assume that each agent \( i \) can be influenced by \( K \) randomly chosen agents of the system with \( 1 \leq K \leq N \). The dynamical rule governing the time evolution of the network is, whether agent \( i \) will say YES or NO, is given by the linear threshold rule

\[
\sigma_i(t + 1) = \text{sgn} \left( \sum_{j=1}^{N} c_{ij} \sigma_j(t) + h_i \right) \quad i = 1, \ldots, N.
\]  

The coupling coefficients \( c_{ij} \), usually not symmetric with respect to the interchange of the subscripts, define the interaction strength between agent \( i \) and agent \( j \). These “weights” \( c_{ij} \) can be classified according to “Activating” influences (A) \( (c_{ij} > 0) \) associated with opportunistic behaviour, where agent \( i \) is stimulated to adopt the opinion of agent \( j \). On the other hand, “Inhibiting” influences (I) \( (c_{ij} < 0) \) can be associated with contrarian behaviour, where agent \( i \) is stimulated to adopt the opinion opposite to the actual choice of agent \( j \). The coupling coefficient is zero, if agent \( j \) does not influence agent \( i \) at all. The quantity \( h_i \) represents a threshold which can be thought of as an external field acting on agent \( i \) associated with mass media, public relations, targeted propaganda, or personal preferences toward either orientation. A large positive threshold \( h_i \) favours a YES state, while a negative threshold \( h_i \) favours a NO state of agent \( i \). The time evolution of the \( N \)-dimensional opinion vector

\[
\overline{\sigma}(t) = (\sigma_1(t), \sigma_2(t), \ldots, \sigma_N(t)),
\]

could describe the state of a discussion at time \( t \), where each agent expresses its opinion in order to arrive at a final decision, YES or NO.

Let us now assume that a fraction \( p \) of the agents adopt the opinion contradictory to the prevailing opinions of others and obey a local minority rule, while the fraction \( 1 - p \) of the agents follow the current trend and obey the local majority rule. [2] The latter group can be considered as opportunists or trend followers, while the first group can be thought of as contrarians who are quite common in social societies. The implications of their existence have long been studied in the context of financial market strategies, where this group plays an important role. For this special case, where the interactions consist only of these two fundamental rules of behaviour, we choose \( c_{ij} = 1 \) for all incoming connections for the opportunists, and \( c_{ij} = -1 \) for the contrarians, respectively. The individual thresholds \( h_i \) are set to zero. The majority rule is employed in a number of settings, such as elections, board meeting votes, and legislative votes, where eventually one has only one winner. Many democratic societies use the majority rule in local and international elections.

3. Decisive macroscopic variables

There are three crucial macroscopic variables which characterize the dynamical behaviour of the system. The first one is the public opinion at time \( t \), the density of “ones” which defines the acceptance rate of the YES or NO state given by the total magnetization

\[
m(t) = \frac{1}{2N} \sum_{i=1}^{N} (\sigma_i(t) + 1).
\]  

The second macroscopic variable specifies how small local perturbations influence the time evolution of the network. Such a perturbation could be specified by switching the opinion variable of only one or a few randomly selected agents from YES to NO or vice versa. In order to study these effects, one considers two identical replicas of the system and
compares the time evolution of the two replicas starting with slightly different initial conditions. The normalized Hamming distance between the two opinion vectors at time $t$ by $\sigma^{(1)}(t)$ and $\sigma^{(2)}(t)$ defined as

$$d(t) = \frac{1}{4N} \sum_{\nu=1}^{N} (\sigma^{(1)}_{\nu}(t) - \sigma^{(2)}_{\nu}(t))^2$$

specifies the fraction of agents who differ in their opinion at time $t$. Provided that this probability $d(t)$ is sufficiently low, the system can be placed into the stable phase. While in biological networks the stability is a necessary requirement for the survival of a system, in opinion networks a slight perturbation of a few agents should not affect a substantial part of the other agents. A complementary measure of the opinion stability based on the individual level can be specified by the third macroscopic quantity, the fraction of agents who change their opinion from one time step to the next by the quantity $D(t)$ defined by

$$D(t) = \frac{1}{4N} \sum_{\nu=1}^{N} (\sigma_{\nu}(t) - \sigma_{\nu}(t-1))^2$$

As has been seen in various electoral campaigns up to about one third of the respondents change their opinion at least once during the campaign [12]. However, note that individual changes in one direction are often cancelled by individual changes in the opposite direction. These fundamental macroscopic quantities can be calculated exactly in the thermodynamic limit $N \to \infty$ [4, 5] with the aid of pure statistical methods via mean-field techniques.

Let us first shortly review some results of the basic version of the model [4], where the agents meet in groups of the same size $K$ and the connectivity structure remains fixed. Fig.1 depicts the analytically predicted long-time behaviour of the magnetization $m^*$ (black line) and the distance $d^*$ (red line) for $K = 9$ within meanfield theory [6]. Due to symmetry properties the “fifty-fifty” outcome, the “tie” situation ($m^* = \frac{1}{2}$) is always a natural fixed point of the magnetization $m(t)$, the acceptance rate of the YES or NO. Eventually this stationary solution loses its stability below a critical $p_c^m = \frac{187}{630} \approx 0.297$ via a bifurcation. Accordingly, for $p < p_c^m$, depending on the initial condition, the system can reach one of the two stable fixed points - symmetric to $\frac{1}{2}$, which specify a preference for the YES or the NO state. Note that the critical concentrations for the distance $p_c^d$, when $d^* = 0$ and the system is dynamically
stable, is somewhat lower than the corresponding critical value for the magnetization $p_m^c$ [6, 7]. However, when the system is in the “tie” situation one can show that also the Hamming distance $d(t)$ (Eq. 4) has the stable fixed point $d^* = \frac{1}{2}$ which makes the system unstable with respect to small perturbations [4, 5]. Moreover, provided that only one single agent changes its opinion at an arbitrary time step $t$, this perturbation can propagate such that in the long time limit 50% of the agents will have changed their opinion. Around the tie we find Gaussian fluctuations such that the outcome becomes essentially random. Note however that the magnetization, the acceptance rate of the YES and the NO, is stable, since the individual changes of the votes of the agents cancel out.

It is important to note that also the macroscopic quantity $D(t)$ (Eq. (5)), the distance of the individual opinions with respect to consecutive time steps, follows the same asymptotics as the Hamming distance between the two replicas $d(t)$ such that about 50% percent of the of the agents change their opinion from one time step to the next. We can conclude that in the whole parameter range of the “tie” the model does does not reflect the real life situation, while below the critical concentration $p_m^c$ the model is more realistic [12]. Note that for increasing values of $K$ the critical concentrations shift to higher values of $p$ and $t$, such that the system is globally stable. In the large $K$-limit we find [5]

$$p_m^c(K) \approx \frac{1}{2} - \sqrt{\frac{\pi}{8}} \sqrt{\frac{1}{K}}$$

However, for small group sizes $K$ one needs a substantial fraction of the opportunists in order to stabilize the system.

4. History-dependent rewiring procedure

The first decisive step towards a “dynamical” network model with activity-dependent coupling coefficients was Hebb’s classical articulation of the concept of “correlated self-organization” in biological neural networks [13]. His classical postulate that the efficacy of an excitatory synapse increases when the two neurons it links fire simultaneously, was later translated into various mathematical formulations, specifying the modification of a synaptic weight in proportion to the correlation of the values of the two units. In summary the conclusion was that self-organization, adaption as well as learning effects might be based on the same mechanism, which can be formulated in terms of a few simple rules formally closely related to those postulated by Hebb. Following the principle “what fires together will wire together” we suggest the following possible rewiring rule designed to demonstrate its effectiveness in gradually leading
to networks showing more stable behaviour [8, 10, 11]. Let us first introduce the pair correlation coefficient \( C_{ij}(T) \) of two agents \( i \) and \( j \) measured within \( T \) consecutive time steps via

\[
C_{ij}(T) = \frac{1}{T} \sum_{t=1}^{T} \sigma_i(t)\sigma_j(t)
\]  

with \(-1 \leq C_{ij}(T) \leq 1\). The rewiring procedure is then specified as follows:

- Choose randomly a pair of agents \( i \) and \( j \) which are not connected
- provided that \(|C_{ij}(T)|\) exceeds a threshold value after \( T \) time steps, create a new directed bond \( c_{ij} \) linking agent \( j \) to agent \( i \) with probability \( q \). Here the sign of the pair correlation specifies the the sign of new efficiency \( c_{ij} = \text{Sign}(C_{ij}(T)) \).
- Delete randomly an existing bond \( c_{ij} \) with probability \( 1 - q \)

Note that the rewiring parameter \( q \) allows to let the average connectivity of the network grow or decrease. The choice \( q = \frac{1}{2} \) does not affect the average connectivity. Positive pair correlations \( C_{ij}(T) \) induce activating couplings, while negative correlations induce inhibiting couplings, respectively. The network dynamics now has two timescales: a slow local change of the connectivity compared to the faster time evolution of the states of the agents. Such a rewiring step is local in space, since the change depends only on the states of the agents \( i \) and \( j \), whereas with respect to time a rewiring step depends strongly on the states of the two agents \( i \) and \( j \) in the past. Initially, a specimen net is assembled with random connectivity structure which places it in the disordered phase, while the interactions are based only on the two fundamental rules, the majority and the minority rule. Fig. 2 depicts the magnetization \( m(t) \) and the Hamming distance \( d(t) \) as a function of the number of the rewiring steps for \( N = 1000 \) agents and control parameter \( T = 50 \), which specifies the timescale of the rewiring process. Note however, that the simulations reveal that the results are not that much sensitive to the choice of \( T \). We observe that with increasing number of rewiring steps the Hamming distance \( d(t) \) gradually decreases and that there exists a critical number of rewiring steps, where a phase transition
from disordered to ordered behaviour occurs, when the distance eventually reaches the value zero. Fig. 3 depicts the time evolution of the global correlation coefficient $C(t)$ specifying the average over the magnitudes of all local correlation coefficients $C_{ij}(T)$

$$C(T) = \frac{2}{N(N-1)} \sum_{i<j} |C_{ij}(T)|$$  

(8)

with $0 \leq C(T) \leq 1$. Notice that with increasing number of rewiring steps the correlation coefficient $C(T)$ gradually increases and finally tends to approach the value one signalling that the time evolution of the evolving networks might reach a fixed point such that the systems are well stabilized. Fig. 4 shows the final in-degree distribution as well as the out-degree distribution of the couplings after the rewiring process. Starting from a Dirac-like distribution, where all units have the same number of neighbors, we observe that with increasing rewiring steps the in-degree as well as the out-degree distribution broadens remarkably, reminiscent of a heavier tail after sufficiently many rewiring steps.

5. Summary

We have shown that the stability of a binary network model governed by a subtle balance of opportunists and contrarians can be largely enhanced by a local rewiring strategy motivated by generalized Hebbian rules widely used in neural network models. We infer that this mechanism might also be applicable to various biological systems such as genetic and population dynamical networks. Finally the “tie” phenomenon associated with instability, usually present in static networks, could be overcome by an activity dependent rewiring process such that these pathologic networks can even be stabilized in a fifty-fifty situation.

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7. References