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Suborthogonality and orthocentricity of matrices☆

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Abstract

We present some results on submatrices of orthogonal and unitary matrices and their relation to so called orthocentric matrices. These are then completely characterized.

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1. Introduction

We intend to characterize submatrices of unitary and orthogonal matrices and show a relationship with Euclidean geometry.

We first recall the notion of the *off-diagonal rank* of a square matrix A. It was defined in [3,4] as the order of a maximal nonsingular submatrix of A which does not contain any diagonal entry of A. Let us denote by $w(A)$ the off-diagonal rank of A.

In addition, we denote by $d(A)$ the number

$$d(A) = \min_D \ \text{rank}\{A + D;\ D \text{ diagonal matrix}\}; \tag{1}$$

we call $d(A)$ d-rank of A.

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It is immediate that $w(A) \leqslant d(A)$ for any square matrix A. Equality need not be attained for $n > 3$. The 4×4 irreducible tridiagonal matrix has off-diagonal rank 2 whereas the d-rank is 3. In [3], conditions were found under which equality is attained.

In addition, we say that A has *strict* d-rank $d(A)$ if some matrix D for which the minimum is attained in (1) is nonsingular. Also, in the case that A is symmetric, we say that the $n \times n$ matrix A has *negatively d-rank* r if A has the form

$$A = D - XX^{\mathrm{T}},$$

where D is diagonal and X is a real $n \times r$ matrix of rank r.

It is useful to mention that, contrary to the off-diagonal rank, the d-rank does not enjoy the property that $d(A^{-1}) = d(A)$ if A is nonsingular, not even when A is symmetric. Indeed, the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

has d-rank 1, its inverse

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

has d-rank 2.

However, the following holds:

Theorem 1.1. *If A is a nonsingular matrix with d-rank strict, then $d(A^{-1}) \leqslant d(A)$.*

Proof. Suppose that A is a nonsingular $n \times n$ matrix, $d(A) = r$, and $A = D_0 + XY^{\mathrm{T}}$, where both X and Y are $n \times r$ of rank r and D_0 is nonsingular. Observe first that the $r \times r$ matrix $I + Y^{\mathrm{T}} D_0^{-1} X$ is nonsingular. Indeed, if $(I + Y^{\mathrm{T}} D_0^{-1} X)x = 0$ for some $x \neq 0$, then $(D_0 + XY^{\mathrm{T}}) D_0^{-1} Xx = 0$ so that $Xx = 0$, a contradiction. Then

$$A^{-1} = D_0^{-1} - D_0^{-1} X (I + Y^{\mathrm{T}} D_0^{-1} X)^{-1} Y^{\mathrm{T}} D_0^{-1},$$

so that $d(A^{-1}) \leqslant d(A)$. □

An immediate consequence is

Corollary 1.2. *If A is a positive definite matrix with negatively d-rank, then $d(A^{-1}) \leqslant d(A)$.*

Proof. Indeed, $A = D_0 - XX^{\mathrm{T}}$ implies that the d-rank of A is strict. □

We should also mention that in [3] the following was shown; there, we defined that a matrix has *totally the rank* r if it has rank r and *all* $r \times r$ submatrices are nonsingular. Also, it has *totally off-diagonal rank* r if it has off-diagonal rank r and *all* $r \times r$ off-diagonal submatrices are nonsingular.

Theorem 1.3 [3, Theorem 3.1]. *Let A be an $n \times n$ matrix which has totally off-diagonal rank r, where $n \geqslant 3r$ and $(n, r) \neq (3, 1)$. Then there exists an $n \times n$ matrix $\widetilde{A}$ which has the same off-diagonal entries as A such that $\widetilde{A}$ has totally the rank r.*

Remark 1.4. In our notation, this means that given the conditions stated in Theorem 1.3, then $d(A) = w(A)$.

2. Submatrices of unitary and orthogonal matrices

Theorem 2.1. *Let U be an $n \times n$ unitary matrix, and let A be its $p \times q$ submatrix. Then A has all singular values less than or equal to one, and the number of singular values less than one (counting also zero singular values) does not exceed $n - \max(p, q)$.*

Proof. Since U has all singular values one, A has singular values at most one (adding a line to a matrix essentially increases the singular values). To complete the proof, we can assume that $p \leqslant q$ and that U is partitioned as

$$U = \begin{bmatrix} A & A_1 \\ A_2 & A_3 \end{bmatrix}.$$

Let

$$A = V \begin{bmatrix} I & 0 & 0 \\ 0 & \Sigma & 0 \end{bmatrix} W$$

be the singular value decomposition of A, in which V and W are unitary and the matrix Σ has order t and nonnegative diagonal entries less than one. Thus the matrix

$$\begin{bmatrix} V^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & \widetilde{A}_{11} \\ 0 & \Sigma & 0 & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \widetilde{A}_{23} & A_3 \end{bmatrix} \begin{bmatrix} W^* & 0 \\ 0 & I \end{bmatrix}$$

is also unitary. The columns of the matrix

$$\begin{bmatrix} \Sigma & 0 & \widetilde{A}_{12} \\ \widetilde{A}_{22} & \widetilde{A}_{23} & A_3 \end{bmatrix}$$

form an orthogonal set. It follows that any two columns of the $(n - p) \times (t + q - p)$-matrix $[\widetilde{A}_{22}, \widetilde{A}_{23}]$ are orthogonal and non-zero, thus linearly independent. Therefore, $\operatorname{rank}[\widetilde{A}_{22}, \widetilde{A}_{23}] = t + q - p \leqslant n - p$ which completes the proof. □

We now reformulate the result and prove a certain converse. This solves the completion problem to find a necessary and sufficient condition for a matrix to be a submatrix of an orthogonal or unitary matrix.

Theorem 2.2. *Every $p \times q$ submatrix of a unitary (or, orthogonal) $n \times n$ matrix has at least $p + q - n$ singular values equal to one and the remaining singular values less than one.*

Conversely, if A is a $p \times q$ matrix that has k singular values equal to one and the remaining $\min(p, q) - k$ singular values less than one, then for every $m \geqslant p + q - k$ there exists a unitary (or, orthogonal) $m \times m$ matrix containing A as a submatrix, and for no m smaller than $p + q - k$ does such matrix exist.

Proof. The first part follows from Theorem 2.1.

Suppose now that A is a $p \times q$ matrix that has k singular values equal to one and the remaining $\min(p, q) - k$ singular values less than one.

Without loss of generality, we can assume that $p \leqslant q$ and that A has already the form

$$A = \begin{bmatrix} I_k & 0 & 0 \\ 0 & \Sigma & 0 \end{bmatrix}$$

with possibly void submatrices, where Σ is a diagonal matrix of order $\min(p, q) - k$ and nonnegative diagonal entries less than one. It is immediate that A is contained in the unitary (even orthogonal) $(p + q - k) \times (p + q - k)$-matrix

$$\begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & \Sigma & 0 & -W \\ 0 & 0 & I_{q-p} & 0 \\ 0 & W & 0 & \Sigma \end{bmatrix},$$

where W is the nonnegative diagonal $(p - k) \times (p - k)$-matrix satisfying $W^2 = I - \Sigma^2$. This completes the proof since such a matrix can be extended to a larger size and by Theorem 2.1 no smaller unitary or orthogonal matrix satisfying the condition exists. □

Corollary 2.3. *Every square submatrix of a unitary or orthogonal matrix has determinant at most one in modulus. If the modulus is one, the matrix is a direct sum of the submatrix and its complement.*

Theorem 2.4. *Any two complementary submatrices of a unitary or orthogonal matrix have the same singular values different from one, even with same multiplicities. The remaining two submatrices (which are also mutually complementary) have singular values which complement the previous singular values to one.*

Proof. Let

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

be a unitary partitioned matrix. Then

$$I - U_{11}U_{11}^* = U_{12}U_{12}^*,$$
$$I - U_{22}^*U_{22} = U_{12}^*U_{12}.$$

Therefore, U_{11} and U_{22} have the same singular values different from one, even with multiplicities. The singular values of U_{11} and U_{12} different from both zero and one complement to one. □

In the following theorem, we call the unitary or orthogonal matrix containing the given sub matrix *minimal* if it has the smallest possible size.

Theorem 2.5. *Let A be an $m \times m$ matrix with no zero row, let r be a positive integer. Then the following are equivalent:*

1. *There exists a nonsingular diagonal matrix D such that DA is a submatrix of a minimal $(m + r) \times (m + r)$ unitary or orthogonal matrix.*
2. *AA^* has negatively d-rank r.*

Proof. 1.→ 2. Let $DA = U\Sigma V$ be the singular value decomposition of DA. By Theorem 2.2, DA has $m - r$ singular values equal to one and r singular values less than one so that Σ can be written as

$$\Sigma = \begin{bmatrix} I_{m-r} & 0 \\ 0 & \tilde{\Sigma} \end{bmatrix},$$

where $\widetilde{\Sigma}$ has all diagonal entries less than one.

Thus

$$DAA^*D^* = U \begin{bmatrix} I_{m-r} & 0 \\ 0 & \tilde{\Sigma}^2 \end{bmatrix} U^*,$$

which can be written as

$$I - U \begin{bmatrix} 0 & 0 \\ 0 & I - \tilde{\Sigma}^2 \end{bmatrix} U^*.$$

It follows that AA^* can be written as $D^{-1}(D^{-1})^* - XX^*$, where X is an $m \times r$ matrix of rank r. It is easily seen that r is indeed the d-rank of A.

2.→ 1. Suppose $AA^* = D - XX^*$, where D is diagonal and X is $m \times r$ with rank r. Since A has no zero row, AA^* as well as D have all diagonal entries positive. It follows that the matrix $D^{-\frac{1}{2}}A$ has $m - r$ singular values equal to one and r singular values less than one. By Theorem 2.2, it is a submatrix of a minimal orthogonal matrix of order $m + r$. □

From now on we shall restrict ourselves to the real case.

Theorem 2.6. *Let $A = [a_{ir}]$, $i = 1, \ldots, n$, $r = 1, \ldots, n + 1$, be an $n \times (n + 1)$ matrix, $n \geqslant 2$. Then the following are equivalent*:

1. *The matrix $A^{\mathrm{T}}A$ has negatively d-rank one and all entries different from zero.*
2. *There exists a nonsingular diagonal matrix D and a vector $z \in R^{n+1}$ with all coordinates different from zero, such that $[DA^{\mathrm{T}}z]$ is an orthogonal matrix.*
3. *There exist positive numbers $\alpha_1, \ldots, \alpha_{n+1}$ and nonzero numbers $u_1, \ldots, u_{n+1}$, such that for all $x_1, \ldots, x_{n+1}$*

$$\sum_{r=1}^{n+1} \alpha_r \left[\left(\sum_{i=1}^{n} a_{ir} x_i \right) + u_r x_{n+1} \right]^2 = \sum_{r=1}^{n+1} x_r^2. \tag{2}$$

4. *There exists a nonsingular diagonal matrix D such that*

   $$(AD)(AD)^{\mathrm{T}} = I_n$$

   and the nonzero vector y for which $Ay = 0$, has all coordinates different from zero.

Proof. We intend to prove that 1. → 2., 2. ↔ 3., 2. → 4., and 4. → 1.

1. → 2. Suppose 1. Thus $A^{\mathrm{T}}A = D_0 - vv^{\mathrm{T}}$, where D_0 is diagonal and v is a column vector with all coordinates different from zero. Therefore, D_0 has positive diagonal entries and there exists a nonsingular diagonal matrix D satisfying $D^2 = D_0^{-1}$. Define $z = Dv$ so that z has all coordinates different from zero. It follows that

$$[DA^{\mathrm{T}}z]\begin{bmatrix} AD \\ z^{\mathrm{T}} \end{bmatrix} = I_{n+1},$$

so that the matrix $[DA^{\mathrm{T}}z]$ satisfies 2.

2. $\leftrightarrow$ 3. If $C = [c_{ik}]$ is an $(n+1) \times (n+1)$ orthogonal matrix, then for any vector $x = [x_1, \ldots, x_{n+1}]^{\mathrm{T}}$, $x^{\mathrm{T}}CC^{\mathrm{T}}x = x^{\mathrm{T}}x$, and conversely.

Therefore, if 2. is satisfied and for $k = 1, \ldots, n+1$, α_k is the square of the kth diagonal entry of D, then (2) holds for $u = D^{-1}z$, $u = [u_r]$, and conversely.

2. $\rightarrow$ 4. By 2.

$$[DA^{\mathrm{T}}z]\begin{bmatrix} AD \\ z^{\mathrm{T}} \end{bmatrix} = I_{n+1}.$$

Thus also

$$\begin{bmatrix} AD \\ z^{\mathrm{T}} \end{bmatrix}[DA^{\mathrm{T}}z] = I_{n+1},$$

which implies $(AD)(AD)^{\mathrm{T}} = I_n$ and $Ay = 0$ for $y = Dz$.

4. $\rightarrow$ 1. By 4.

$$\begin{bmatrix} AD \\ y^{\mathrm{T}}D^{-1} \end{bmatrix}[DA^{\mathrm{T}}D^{-1}y] = \begin{bmatrix} I_n & 0 \\ 0 & y^{\mathrm{T}}D^{-2}y \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} DA^{\mathrm{T}} \dfrac{D^{-1}y}{\sqrt{y^{\mathrm{T}}D^{-2}y}} \end{bmatrix}\begin{bmatrix} AD \\ \frac{y^{\mathrm{T}}D^{-1}}{\sqrt{y^{\mathrm{T}}D^{-2}y}} \end{bmatrix} = I_{n+1},$$

so that

$$DA^{\mathrm{T}}AD + \frac{1}{y^{\mathrm{T}}D^{-2}y}D^{-1}yy^{\mathrm{T}}D^{-1} = I_{n+1}.$$

It follows that

$$A^{\mathrm{T}}A = D^{-2} - \frac{1}{y^{\mathrm{T}}D^{-2}y}D^{-2}yy^{\mathrm{T}}D^{-2}$$

and 1. is proved. □

Let us call matrices satisfying one, and thus all, of the properties in Theorem 2.6, *orthocentric matrices*. Explanation for this term will be given in the following section.

Theorem 2.7. *Let*

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

be a real matrix. Then A is orthocentric if and only if

$$(a_1a_2 + b_1b_2)(a_2a_3 + b_2b_3)(a_1a_3 + b_1b_3) < 0.$$

Proof. Let first A be orthocentric. Then, by 2. of Theorem 2.6, there exist nonzero numbers d_1, d_2, d_3, z_1, z_2, and z_3, such that the matrix

$$\begin{bmatrix} d_1a_1 & d_1b_1 & z_1 \\ d_2a_2 & d_2b_2 & z_2 \\ d_3a_3 & d_3b_3 & z_3 \end{bmatrix} \tag{3}$$

is orthogonal.

Since any two distinct rows of this matrix are orthogonal

$$\begin{aligned} d_1d_2(a_1a_2+b_1b_2) &= -z_1z_2,\\ d_2d_3(a_2a_3+b_2b_3) &= -z_2z_3,\\ d_1d_3(a_1a_3+b_1b_3) &= -z_1z_3. \end{aligned} \tag{4}$$

Multiplication of these equations yields the condition.

Conversely, let the condition be fulfilled. Denote as $-u$ the left-hand side so that $u > 0$. Define as d_1 the positive number satisfying

$$d_1^2[(a_2a_3+b_2b_3)^2(a_1^2+b_1^2)+u] = (a_2a_3+b_2b_3)^2$$

and similarly

$$\begin{aligned} d_2^2[(a_1a_3+b_1b_3)^2(a_2^2+b_2^2)+u] &= (a_1a_3+b_1b_3)^2,\\ d_3^2[(a_1a_2+b_1b_2)^2(a_3^2+b_3^2)+u] &= (a_1a_2+b_1b_2)^2 \end{aligned}$$

with d_2, d_3 positive. Finally, define the z_i's as

$$\begin{aligned} z_1 &= \frac{\sqrt{u}}{a_2a_3+b_2b_3}d_1,\\ z_2 &= \frac{\sqrt{u}}{a_1a_3+b_1b_3}d_2,\\ z_3 &= \frac{\sqrt{u}}{a_1a_2+b_1b_2}d_3. \end{aligned}$$

It is then easily seen that the conditions (4) will be fulfilled, as well as the conditions

$$d_i^2(a_i^2+b_i^2)+z_i^2 = 1$$

for $i = 1, 2, 3$.

Therefore, the matrix (3) is orthogonal and by 2. of Theorem 2.6, the matrix A is orthocentric. □

Example 2.8. The well known $(n+1)\times(n+1)$ matrix

$$C_0 = \begin{bmatrix} c_1 & s_1c_2 & s_1s_2c_3 & s_1s_2s_3c_4 & \cdots & s_1s_2\cdots s_{n-1}c_n & s_1s_2s_3\cdots s_{n-1}s_n\\ -s_1 & c_1c_2 & c_1s_2c_3 & c_1s_2s_3c_4 & \cdots & c_1s_2\cdots s_{n-1}c_n & c_1s_2s_3\cdots s_{n-1}s_n\\ 0 & -s_2 & c_2c_3 & c_2s_3c_4 & \cdots & c_2s_3\cdots s_{n-1}c_n & c_2s_3\cdots s_{n-1}s_n\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \cdots & -s_n & c_n \end{bmatrix}$$

is orthogonal if for $i = 1, \ldots, n$, the numbers c_i and s_i are cosines and sines of some angles ϕ_i. If all the c_i's and s_i's are different from zero, the matrix C obtained by deleting the first row from C_0

$$C = \begin{bmatrix} -s_1 & c_1c_2 & c_1s_2c_3 & c_1s_2s_3c_4 & \cdots & c_1s_2\cdots s_{n-1}c_n & c_1s_2s_3\cdots s_{n-1}s_n\\ 0 & -s_2 & c_2c_3 & c_2s_3c_4 & \cdots & c_2s_3\cdots s_{n-1}c_n & c_2s_3\cdots s_{n-1}s_n\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & \cdots & -s_n & c_n \end{bmatrix}$$

satisfies the condition 4. of Theorem 2.6 and is thus orthocentric.

Remark 2.9. Observe that by 4. of Theorem 2.6, all $n \times n$ submatrices of such an orthocentric matrix are nonsingular. In addition, the following is easily seen to hold.

Theorem 2.10. *If A is an $n \times (n+1)$ orthocentric matrix, Q an orthogonal $n \times n$ matrix and D a nonsingular diagonal matrix of order $n+1$, then QAD is also an orthocentric matrix.*

3. Orthocentric matrices

In this section, we intend to relate the result in Theorem 2.6 to geometric objects in a Euclidean space. As usual, we denote by E_n the n-dimensional Euclidean space endowed by the orthonormal system of coordinates. The entries, points, will be real column vectors.

Probably the simplest objects in E_n are n-simplexes, essentialy determined as the set of $n+1$ linearly independent points. In simplex geometry, orthocentric simplexes form one of the most interesting classes of special simplexes. They are defined by the fact that the altitudes meet in one point, the orthocenter. Egerváry [1] described their basic properties, later several authors found more. In this paper, we intend to study the matrix-theoretical aspects of these simplexes.

We shall restrict ourselves to the case that the orthocenter of such n-simplex (with $n+1$ vertices) is an interior point of the simplex. We then call the orthocentric simplex *positive*. In fact, it is not too restrictive. Unless the orthocenter is on the boundary of the n-simplex, the orthocenter forms with the $n+1$ vertices a set of $n+2$ points all $(n+1)$-tuples of which are vertices of an orthocentric n-simplex whose orthocenter is the remaining point. However, exactly one of these orthocentric simplexes is positive.

We now recall a characteristic property of the positive orthocentric n-simplex [1, Theorem 1], using the *Menger matrix*, i.e. the matrix the entries of which are *squares* of the Euclidean distances between pairs of vertices.

Theorem 3.1. *An $(n+1) \times (n+1)$ symmetric matrix $M = [m_{ik}]$ is the Menger matrix of a positive orthocentric n-simplex if and only if there exist positive numbers $\lambda_1, \ldots, \lambda_{n+1}$ (we call them parameters), such that*

$$m_{ii} = 0, \quad i = 1, \ldots, n+1, \qquad m_{ik} = \lambda_i + \lambda_k \ \text{for}\ i, k = 1, \ldots, n+1, \ i \neq k. \tag{5}$$

Remark 3.2. To give the parameters a simple geometric meaning, we suggest the following construction:

In an $(n+1)$-dimensional Euclidean space, let O be the origin and x_i, $i = 1, \ldots, n+1$, the positive halfaxes of a usual cartesian coordinate system.

If A_i, $i = 1, \ldots, n+1$, is the point of the halfaxis x_i at distance $\sqrt{\lambda_i}$ from the origin, then, by the Pythagorean theorem, the points A_i form vertices of a positive orthocentric n-simplex with parameters λ_i. One can then easily show that the orthocenter is then the foot of the perpendicular from O on the hyperplane determined by the vertices A_i. In fact, the construction suggests that if we have a positive orthocentric n-simplex and form in a one dimension larger space E_{n+1} the perpendicular p in the orthocenter to the given E_n, then there is a point O on P for which the rays OA_i are mutually orthogonal.

We shall now use the matrix-theoretic method developed in [2] and surveyed in [5,6]. The column vector denoted as **1** is the vector of all ones.

Theorem 3.3. *Let M be the $(n+1)\times(n+1)$ Menger matrix of an n-simplex Σ. Then the $(n+2)\times(n+2)$ matrix*

$$M_0 = \begin{bmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ \mathbf{1} & M \end{bmatrix}$$

is nonsingular and the matrix Q_0 satisfying $M_0Q_0 = -2I$ partitioned in the same way as M_0

$$Q_0 = \begin{bmatrix} q_{00} & q_0^{\mathrm{T}} \\ q_0 & Q \end{bmatrix}$$

has the following properties:

1. *The matrix Q is the Gramian of the simplex Σ, i.e. the Gram matrix of the outward normals of Σ (in a certain way normalized so that if A_i are the vertices and n_i are the normals, then $\sum_i n_i = 0$ and the inner product of n_1 and $A_1 - A_{n+1}$ is one).*
2. *The coordinates of the vector q_0 are (-2)-multiples of the barycentric coordinates of the circumcenter of Σ.*
3. *The number $\frac{1}{2}\sqrt{q_{00}}$ is the radius of the circumsphere of Σ.*

We apply now the general Theorem 3.3 to our case.

Theorem 3.4. *In the case that Σ is a positive orthocentric n-simplex with positive parameters λ_i, the matrix Q_0 has the following entries*:

$$\begin{aligned}
\rho q_{00} &= \sum_{k=1}^{n+1}\lambda_k \sum_{k=1}^{n+1}\frac{1}{\lambda_k} - (n-1)^2,\\
\rho q_{0i} &= \frac{n-1}{\lambda_i} - \sum_{k=1}^{n+1}\frac{1}{\lambda_k},\\
\rho q_{ii} &= \frac{1}{\lambda_i}\left(\sum_{k=1}^{n+1}\frac{1}{\lambda_k} - \frac{1}{\lambda_i}\right),\\
\rho q_{ij} &= -\frac{1}{\lambda_i\lambda_j} \quad \textit{for } i \neq j,
\end{aligned} \tag{6}$$

where $\rho = \sum_{k=1}^{n+1}\frac{1}{\lambda_k}$.

Proof. The formula $M_0Q_0 = (-2)I$ for M in (5) and Q_0 in (6) is then easily checked. □

In the following theorem, we identify the class of matrices satisfying the conditions in Theorem 2.6 with geometric objects related to orthocentric simplexes.

Theorem 3.5. *A real $n \times (n+1)$ matrix A satisfies conditions in Theorem* 2.6 *if and only if it is a matrix of coordinates of $n+1$ directions, listed as columns, of the $n+1$ normals to $(n-1)$-dimensional faces of some positive orthocentric n-simplex.*

Proof. Let a positive orthocentric n-simplex in E_n be given. By 1. of Theorem 3.3 and the last two equations in (6), the Gramian of the outward normals of the simplex has negatively d-rank one and all entries different from zero. The coordinates of the directions thus satisfy the condition 1. of Theorem 2.6.

Suppose now that A is an $n \times (n+1)$ matrix satisfying the condition 3. in Theorem 2.6. We shall show that the n-simplex Σ whose $(n-1)$-dimensional faces ω_r, $r = 1, \ldots, n+1$, have in cartesian coordinates equations

$$\left(\sum_{i=1}^{n} a_{ir} x_i\right) + u_r = 0$$

and thus have the column vectors of A as normals, is positive orthocentric.

Observe that the hyperplanes of the faces have by 3. of Theorem 2.6 no point in common. Denote as $\overset{r}{A}_1, \ldots, \overset{r}{A}_n$ the cartesian coordinates of the vertices $\overset{r}{A}$ of Σ so that the values

$$\omega_r(\overset{s}{A}) = \left(\sum_{i=1}^{n} a_{ir} \overset{s}{A}_i\right) + u_r$$

are equal to zero if $r \neq s$, $r, s = 1, \ldots, n+1$, and different from zero if $r = s$. By the condition in 3. of Theorem 2.6, the square of the distance between the vertex $\overset{s}{A}$ and $\overset{t}{A}$ for $s \neq t$ is equal to

$$\sum_{r=1}^{n+1} \alpha_r \left[\left(\sum_{i=1}^{n} a_{ir} \overset{s}{A}_i\right) + u_r - \left(\left(\sum_{i=1}^{n} a_{ir} \overset{t}{A}_i\right) + u_r\right)\right]^2,$$

which is easily seen to be $\lambda_s + \lambda_t$ for $\lambda_r = \alpha_r \omega_r^2(\overset{r}{A})$. By Theorem 3.1, Σ is positive orthocentric. □

This justifies calling these matrices orthocentric matrices.

In the conclusion, we shall mention properties of so called *partial orthocentric matrices*. These are square matrices which can be completed by a column to an orthocentric matrix. They are characterized in the following theorem.

Theorem 3.6. *Let* $A = [a_{ir}]$, $i, r = 1, \ldots, n$, *be a nonsingular real matrix. Then the following are equivalent*:

1. *The matrix* $A^{\mathrm{T}}A$ *has negatively* d*-rank one and all off-diagonal entries different from zero.*
2. *There exists a nonsingular diagonal matrix* D *such that* AD *can be completed to an* $(n+1) \times (n+1)$ *orthogonal matrix whose last added row has no zero entry.*
3. *There exists a column vector which completes* A *to an orthocentric matrix.*

Proof. We shall prove that 1. → 2., 2. → 3., and 3. → 1.

1. → 2. By 1., $A^{\mathrm{T}}A = D_0 - uu^{\mathrm{T}}$, where D_0 is diagonal and u a column vector with all entries different from zero. By positive definiteness of $A^{\mathrm{T}}A$, D_0 has all diagonal entries positive and there exists a nonsingular diagonal matrix D for which $D^2 = D_0^{-1}$. Thus

$$DA^{\mathrm{T}}AD = I_n - (Du)(Du)^{\mathrm{T}},$$

so that

$$[DA^{\mathrm{T}} Du] \begin{bmatrix} AD \\ u^{\mathrm{T}}D \end{bmatrix} = I_n.$$

By Theorem 2.2, the matrix

$$\begin{bmatrix} AD \\ u^{\mathrm{T}}D \end{bmatrix}$$

can be completed by a column into an orthogonal matrix. By the nonsingularity of A and D, the last entry of this orthogonal matrix is different from zero and 2. is proved.

2. $\to$ 3. By 2., there exists a nonsingular diagonal matrix D and a completion

$$\begin{bmatrix} AD & y \\ z_0^{\mathrm{T}} & \xi \end{bmatrix}$$

of AD into an orthogonal matrix in which, in addition, $\xi \neq 0$ and all entries of z_0^{T} are different from zero. But this means that the matrix

$$\begin{bmatrix} DA^{\mathrm{T}} & z_0 \\ y^{\mathrm{T}} & \xi \end{bmatrix}$$

is orthogonal and A can be completed by the column vector y so that and the resulting $n \times (n+1)$ matrix is orthocentric by condition 2. of Theorem 2.6.

3. $\to$ 1. Immediate since by removing the column, the matrix $A^{\mathrm{T}}A$ will be a principal submatrix and by condition 1. of Theorem 2.6, 1. will be satisfied. □

Let us observe a simple fact.

Theorem 3.7. *There exists an upper triangular partially orthocentric $n \times n$ matrix for every $n \geqslant 2$.*

Proof. Indeed, the matrix obtained from the matrix C in Example 2.8 by removing the last column has this property. □

By the well known Gram–Schmidt process, every real square matrix can be written as a product of an orthogonal and an upper triangular matrix. On the other hand, similarly to Theorem 2.10, the following holds.

Theorem 3.8. *If A is a partially orthocentric matrix, Q an orthogonal matrix, and D a nonsingular diagonal matrix, all of the same order $n \geqslant 2$, then QAD is also a partially orthocentric matrix.*

We even have the following complete characterization of partially orthocentric matrices.

Theorem 3.9. *Every partially orthocentric $n \times n$ matrix can be written in the form QUD, where Q is orthogonal, D a nonsingular diagonal matrix, and U an upper triangular matrix of the form*

$$U = \begin{bmatrix} -s_1 & c_1c_2 & c_1s_2c_3 & c_1s_2s_3c_4 & \cdots & c_1s_2\cdots s_{n-1}c_n \\ 0 & -s_2 & c_2c_3 & c_2s_3c_4 & \cdots & c_2s_3\cdots s_{n-1}c_n \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -s_n \end{bmatrix}$$

for some positive numbers c_i and s_i, $i = 1, \ldots, n$, satisfying $c_i^2 + s_i^2 = 1$.

Proof. Follows from Theorem 3.8, the Gram–Schmidt process and the fact that up to eventual multiplication of one row or one column by -1, every indecomposable orthogonal matrix in

Hessenberg form can be expressed as a matrix C_0 in Example 2.8. This can be easily proved by induction. □

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