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ORIGINAL ARTICLE

Three-dimensional triangular functions and their applications for solving nonlinear mixed Volterra–Fredholm integral equations

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Abstract In this paper, we used the three-dimensional triangular functions (3D-TFs) for the numerical solution of three-dimensional nonlinear mixed Volterra–Fredholm integral equations. First, 3D-TFs and their properties are described. Then the properties of 3D-TFs together with their operational matrix are used to reduce the problem to a nonlinear system of algebraic equations. Furthermore, existence and uniqueness of the solution of three-dimensional nonlinear mixed Volterra–Fredholm integral equations are proved. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique. Also, some interesting comparisons between proposed method, block-pulse functions (BPFs) method and modified block-pulse functions (MBPFs) method are presented.

1. Introduction

Over the last years, the integral equations and differential equations have been used increasingly in different areas of applied sciences. This tendency could be explained by the deduction of knowledge models which describe real physical phenomena. For details, we refer to [1–5]. Various problems in physics, mechanics and biology arise to a multi-dimensional integral equation. Such equations also appear in electromagnetic and electrodynamic, elasticity and dynamic contact, heat and mass transfer, fluid mechanic, acoustic, chemical and electrochemical processes, molecular physics, population, medicine and in many other fields [6–14].

One-dimensional triangular functions were introduced by Deb et al. in [15]. Babolian et al. used these functions for solving variational problems [16], and nonlinear Volterra–Fredholm integro-differential equations [17] in the one-dimensional case. Moreover, Maleknejad et al. applied these functions for solving nonlinear Volterra–Fredholm integral equations [18]. 1D-TFs have been widely used for solving different problems. The new and basic idea in this paper is extending 1D-TFs to 3D-TFs and using them for solving general three-dimensional nonlinear mixed Volterra–Fredholm integral equations of the second kind of the form

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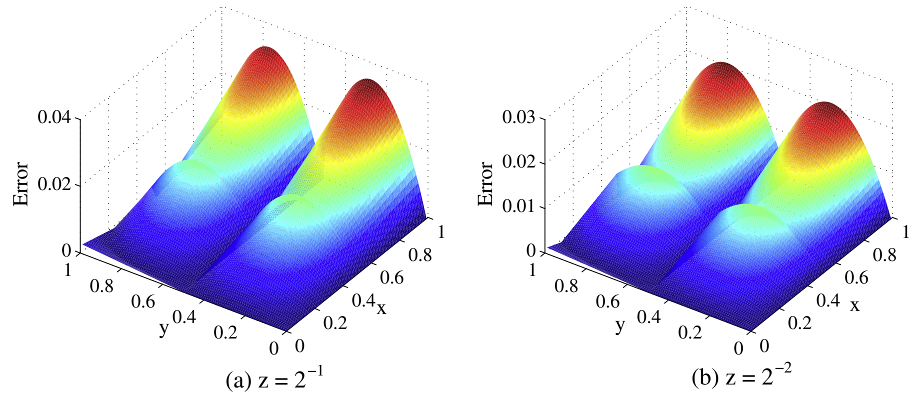

Figure 1 Absolute errors for Example 1, with $m = 2$ and $z = 2^{-1},\ 2^{-2}$.

$$f(s,t,r) = g(s,t,r) + \int_0^s \left(\int \int_\Omega H(s,t,r,x,y,z,f(x,y,z))dzdy \right) dx; \quad (s,t,r) \in [0,1) \times \Omega, \tag{1}$$

where $f(s,t,r)$ is an unknown function, $g(s,t,r)$ and $H(s,t,r,x,y,z,f(x,y,z))$ are analytical functions on $[0,\,1) \times \Omega$ and $([0,\,1) \times \Omega)^2 \times \mathbb{C}$, respectively and Ω is close subset on R^2. The existence and uniqueness of the solution for the two-dimensional model of Eq. (1) are discussed in [19,20]. Equations of this type often arise from the mathematical modeling of the spreading, in space and time, of some contagious disease in a population living in a habitat Ω [21,22], in the theory of nonlinear parabolic boundary value problems [23], and in many physical and biological models. The analytical solution of the multi-dimensional integral equations is usually difficult and in many cases, it is required to approximate the solutions. The analysis of computational methods for multi-dimensional integral equations, especially in the nonlinear case, has started more recently and is not so well developed. To numerically solve integral equations, there are some well-known numerical methods that some of them can be used for solving triple integral equations. Significant progress has been made in numerical analysis linear and nonlinear version of Eq. (1). For the linear case, some methods for numerical treatment are given in [24–26]. For nonlinear two-dimensional mixed Volterra–Fredholm integral equations, the literature of integral equations contains few numerical methods [27,28] for handling Eq. (1).

In the rest of the paper, we assume

$$H(s,t,r,x,y,z,f(x,y,z)) = k(s,t,r,x,y,z)[f(x,y,z)]^p, \tag{2}$$

where p is positive integer. In [29], Eq. (1) with Eq. (2) is solved by MBPF. The TFs method is very simpler and cheaper than MBFs method from the computational point of view. In [29], it was shown that the MBPFs method is convergence of order $O(h)$, but we can't prove the order of convergence for TFs method.

The paper is organized as follows: In Section 2, we study the existence and uniqueness of the solution of Eq. (1). In Sections 3 and 4, we describe 1D-TFs and 3D-TFs and their properties, respectively. In Section 5, we apply 3D-TFs, to solve the three-dimensional nonlinear mixed Volterra–Fredholm integral Eq. (1) with Eq. (2). Numerical results are given in Section 6 to illustrate the efficiency and the accuracy of our algorithm.

2. On the existence of the solution of the three-dimensional nonlinear mixed Volterra–Fredholm integral equations

In this section, we prove an existence and uniqueness theorem for a three-dimensional nonlinear mixed Volterra–Fredholm integral equations.

Consider Eq. (1) on the complete metric space of complex-valued continuous functions as follows:

$$X = (C(S,d)), \quad d(g,w) = \sup\{|g(s,t,r) - w(s,t,r)| : (s,t,r) \in S\},$$

where $S = [0,1] \times [0,1] \times [0,1]$.

Theorem 1. *Let g and H be continuous functions on S and $S \times S \times \mathbb{C}$ respectively and there exists nonnegative constant $L \leqslant 1$ such that*

$$|H(s,t,r,x,y,z,f(x,y,z)) - H(s,t,r,x,y,z,v(x,y,z))| \leqslant L|f(x,y,z) - v(x,y,z)|.$$

Then Eq. (1) has only one solution f on S such that

$$f(s,t,r) = g(s,t,r) + \int_0^s \int_0^1 \int_0^1 H(s,t,r,x,y,z,f(x,y,z))dzdydx,$$

on S.

Proof. Consider the iterative scheme

$$f_{n+1}(s,t,r) = g(s,t,r) + \int_0^s \int_0^1 \int_0^1 H(s,t,r,x,y,z,f_n(x,y,z))dzdydx; \quad n = 1,2,\ldots. \tag{3}$$

We have

$$\begin{aligned} |f_{n+1}(s,t,r) - f_n(s,t,r)| &= \left| \int_0^s \int_0^1 \int_0^1 H(s,t,r,x,y,z,f_n(x,y,z))dzdydx \right. \\ &\quad \left. - \int_0^s \int_0^1 \int_0^1 H(s,t,r,x,y,z,f_{n-1}(x,y,z))dzdydx \right| \\ &\leqslant \int_0^s \int_0^1 \int_0^1 |H(s,t,r,x,y,z,f_n(x,y,z)) \\ &\quad - H(s,t,r,x,y,z,f_{n-1}(x,y,z))|dzdydx \\ &\leqslant L \int_0^s \int_0^1 \int_0^1 |f_n(x,y,z) - f_{n-1}(x,y,z)|dzdydx \\ &\leqslant L \int_0^1 \int_0^1 \int_0^1 |f_n(x,y,z) - f_{n-1}(x,y,z)|dzdydx \\ &\leqslant Ld(f_n,f_{n-1}). \quad \Rightarrow d(f_{n+1},f_n) \leqslant Ld(f_n,f_{n-1}). \end{aligned}$$

Hence,

$$d(f_{n+1},f_n) \leqslant L^{n-1} d(f_2,f_1);$$
$$\Rightarrow |f_{n+1}(s,t,r) - f_n(s,t,r)| \leqslant L^{n-1} d(f_2,f_1).$$

Since X is a complete metric space, and $0 \leqslant L \leqslant 1$, then we conclude by using the Weierstrass M-test that

$$\sum_{n=1}^{+\infty} (f_{n+1}(s,t,r) - f_n(s,t,r)),$$

is absolutely and uniformly convergent on S. Due to the fact that $f_n(s,t,r)$ can be written as

$$f_n(s,t,r) = f_1(s,t,r) + \sum_{k=1}^{n-1} (f_{k+1}(s,t,r) - f_k(s,t,r)),$$

so there exists a unique solution $f \in X$ such that $\lim_{n\to+\infty} f_n = f$. Taking limit of both sides of Eq. (3), we obtain

$$\begin{aligned} f(s,t,r) &= \lim_{n\to+\infty} f_{n+1}(s,t,r) = \lim_{n\to+\infty} (g(s,t,r) \\ &+ \int_0^s \int_0^1 \int_0^1 H(s,t,r,x,y,z,f_n(x,y,z)) dz dy dx) = g(s,t,r) \\ &+ \int_0^s \int_0^1 \int_0^1 H(s,t,r,x,y,z, \lim_{n\to+\infty} f_n(x,y,z)) dz dy dx = g(s,t,r) \\ &+ \int_0^s \int_0^1 \int_0^1 H(s,t,r,x,y,z,f(x,y,z)) dz dy dx. \end{aligned}$$

Therefore, the limit function f is the unique solution $f \in X$ such that

$$f(s,t,r) = g(s,t,r) + \int_0^s \int_0^1 \int_0^1 H(s,t,r,x,y,z,f(x,y,z)) dz dy dx. \quad \square$$

3. Definitions of one-dimensional triangular functions

In an m-set of one-dimensional triangular functions (1D-TFs) over interval $[0,1)$, the ith left hand and right hand functions are defined as

$$T_i^1(s) = \begin{cases} 1 - \frac{s-ih}{h}, & ih \leqslant s < (i+1)h, \\ 0, & \text{otherwise}, \end{cases}$$

$$T_i^2(s) = \begin{cases} \frac{s-ih}{h}, & ih \leqslant s < (i+1)h, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 0, 1, 2, \ldots, m-1$ and $h = \frac{1}{m}$.

Also,

$$T_i^1(s) + T_i^2(s) = \phi_i(s),$$

where $\phi_i(s)$ is the ith block-pulse function defined as

$$\phi_i(s) = \begin{cases} 1, & ih \leqslant s < (i+1)h, \\ 0, & \text{otherwise}. \end{cases}$$

It is obvious that $\{T_i^1(s)\}_{i=0}^{m-1}$ and $\{T_i^2(s)\}_{i=0}^{m-1}$ are disjoint. Orthogonality of 1D-TFs is shown in [15], that is,

$$\int_0^1 T_i^p(s) T_j^q(s) ds = \Delta_{p,q} \delta_{i,j},$$

where $\delta_{i,j}$ denotes the Kronecker delta function and

$$\Delta_{p,q} = \begin{cases} \frac{h}{3}, & p = q \in \{1,2\}, \\ \frac{h}{6}, & p \neq q. \end{cases}$$

We can also define

$$\mathbf{T1}(s) = \left[T_0^1(s),\ T_1^1(s), \ldots, T_{m-1}^1(s)\right]^T,$$
$$\mathbf{T2}(s) = \left[T_0^2(s),\ T_1^2(s), \ldots, T_{m-1}^2(s)\right]^T,$$

and

$$\mathbf{T}(s) = \begin{bmatrix} \mathbf{T1}(s) \\ \mathbf{T2}(s) \end{bmatrix}, \tag{4}$$

and the vector $\mathbf{T}(s)$ is called the **1D-TF** vector. Since 1D-TFs are disjoint, we have

$$\mathbf{T}(s).\mathbf{T}^T(s) \simeq diag(\mathbf{T}(s)) = \widetilde{\mathbf{T}}(s),$$

where $\widetilde{\mathbf{T}}(s)$ is a $2m \times 2m$ diagonal matrix [16].

4. Three-dimensional triangular functions and their properties

4.1. Definitions

An $(m_1 \times m_2 \times m_3)$-set of 3D-TFs on the region $D = [0,1) \times [0,1) \times [0,1)$ is defined by

$$T_{i,j,k}^{1,1,1}(s,t,r) = \begin{cases} \left(1 - \frac{s-ih_1}{h_1}\right)\left(1 - \frac{t-jh_2}{h_2}\right)\left(1 - \frac{r-kh_3}{h_3}\right), & \begin{array}{l} ih_1 \leqslant s < (i+1)h_1, \\ jh_2 \leqslant t < (j+1)h_2, \\ kh_3 \leqslant r < (k+1)h_3, \end{array} \\ 0, & otherwise, \end{cases}$$

$$T_{i,j,k}^{1,1,2}(s,t,r) = \begin{cases} \left(1 - \frac{s-ih_1}{h_1}\right)\left(1 - \frac{t-jh_2}{h_2}\right)\left(\frac{r-kh_3}{h_3}\right), & \begin{array}{l} ih_1 \leqslant s < (i+1)h_1, \\ jh_2 \leqslant t < (j+1)h_2, \\ kh_3 \leqslant r < (k+1)h_3, \end{array} \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 0, 1, \ldots, m_1 - 1, j = 0, 1, \ldots, m_2 - 1, k = 0, 1, \ldots, m_3 - 1$ and $h_1 = \frac{1}{m_1}, h_2 = \frac{1}{m_2}, h_3 = \frac{1}{m_3}$ in which m_1, m_2 and m_3 are arbitrary positive integers.

Other definitions for $T_{i,j,k}^{a,b,c}(s,t,r); a,b,c \in \{1,2\}$ are similar.

It is clear that

$$T_{i,j,k}^{a,b,c}(s,t,r) = T_i^a(s) \cdot T_j^b(t) \cdot T_k^c(r); \quad a,b,c \in \{1,2\}. \tag{5}$$

Furthermore,

$$\sum_{a=1}^{2} \sum_{b=1}^{2} \sum_{c=1}^{2} T_{i,j,k}^{a,b,c}(s,t,r) = \phi_{i,j,k}(s,t,r),$$

where $\phi_{i,j,k}(s,t,r)$ is the $\{m_2 i + m_3 j + k\}$th block-pulse function defined on $ih_1 \leqslant s < (i+1)h_1,\ jh_2 \leqslant t < (j+1)h_2$ and $kh_3 \leqslant r < (k+1)h_3$ as

$$\phi_{i,j,k}(s,t,r) = \begin{cases} 1, & ih_1 \leqslant s < (i+1)h_1,\ jh_2 \leqslant t < (j+1)h_2,\ kh_3 \leqslant r < (k+1)h_3, \\ 0, & \text{otherwise}. \end{cases}$$

It is obvious that each set $\{T_{i,j,k}^{a,b,c}(s,t,r)\};\ a,b,c \in \{1,2\}$ is disjoint

$$T_{i_1,j_1,k_1}^{a_1,b_1,c_1}(s,t,r) T_{i_2,j_2,k_2}^{a_2,b_2,c_2}(s,t,r) = \begin{cases} T_{i_1,j_1,k_1}^{a_1,b_1,c_1}(s,t,r), & \begin{array}{l} a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2, \\ i_1 = i_2,\ j_1 = j_2,\ k_1 = k_2, \end{array} \\ 0, & \text{otherwise}, \end{cases}$$

for $a_1, a_2, b_1, b_2, c_1, c_2 \in \{1,2\},\ i_1, i_2 = 0, 1, \ldots, m_1 - 1,\ j_1, j_2 = 0, 1, \ldots, m_2 - 1$ and $k_1, k_2 = 0, 1, \ldots, m_3 - 1$.

Also 3D-TFs are orthogonal, that is

$$\int_0^1\int_0^1\int_0^1 T^{a_1,b_1,c_1}_{i_1,j_1,k_1}(s,t,r)T^{a_2,b_2,c_2}_{i_2,j_2,k_2}(s,t,r)dsdtdr = \Delta_{a_1,a_2}\delta_{i_1,i_2}.\Delta_{b_1,b_2}\delta_{j_1,j_2}.\Delta_{c_1,c_2}\delta_{k_1,k_2},$$

where δ denotes the Kronecker delta function, and

$$\Delta_{\alpha,\beta}=\begin{cases}\frac{h}{3}, & \alpha=\beta\in\{1,2\},\\ \frac{h}{6}, & \alpha\neq\beta.\end{cases}$$

On the other hand, if

$$\mathbf{Tabc}(s,t,r)=\left[T^{a,b,c}_{0,0,0}(s,t,r),T^{a,b,c}_{0,0,1}(s,t,r),\ldots,T^{a,b,c}_{0,0,m_3-1}(s,t,r),\right.$$
$$\left.T^{a,b,c}_{0,1,0}(s,t,r),\ldots,T^{a,b,c}_{m_1-1,m_2-1,m_3-1}(s,t,r)\right]^T,$$

where $a,b,c\in\{1,2\}$ and $\mathbf{Tabc}(s,t,r)$ is $(m_1m_2m_3\times 1)$-matrix; then $\mathbf{T}(s,t,r)$, the **3D-TF vector**, can be defined as

$$\mathbf{T'1}(s,t,r)=[\mathbf{T111}(s,t,r),\mathbf{T112}(s,t,r),\mathbf{T121}(s,t,r),\mathbf{T122}(s,t,r)],$$
$$\mathbf{T'2}(s,t,r)=[\mathbf{T211}(s,t,r),\mathbf{T212}(s,t,r),\mathbf{T221}(s,t,r),\mathbf{T222}(s,t,r)],$$

$$\mathbf{T}(s,t,r)=[\mathbf{T'1}(s,t,r),\mathbf{T'2}(s,t,r)]^T. \tag{6}$$

It is possible to cancel the (s,t,r) term in $\mathbf{T(s,t,r)},\mathbf{T111(s,t,r)},\mathbf{T112(s,t,r)},\mathbf{T121(s,t,r)},\mathbf{T122(s,t,r)}$, $\mathbf{T211(s,t,r)},\mathbf{T212(s,t,r)},\mathbf{T221(s,t,r)}$ and $\mathbf{T222(s,t,r)}$, for convenience.

From the above representation, it follows that

$$\mathbf{Tabc}\cdot\mathbf{Tabc}^T\simeq\begin{bmatrix} T^{a,b,c}_{0,0,0} & 0 & \ldots & 0 & 0 & \ldots & 0\\ 0 & T^{a,b,c}_{0,0,1} & \ldots & 0 & 0 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & T^{a,b,c}_{0,0,m_3-1} & 0 & \ldots & 0\\ 0 & 0 & \ldots & 0 & T^{a,b,c}_{0,1,0} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & 0 & 0 & \ldots & T^{a,b,c}_{m_1-1,m_2-1,m_3-1}\end{bmatrix}$$
$$=diag(\mathbf{Tabc}),$$

and

$$\mathbf{Tabc}\cdot\mathbf{Ta'b'c'}^T\simeq\mathbf{0}_{m_1m_2m_3\times m_1m_2m_3};\quad a\neq a' \text{ or } b\neq b' \text{ or } c\neq c',$$

where $a,a',b,b',c,c'\in\{1,2\}$. Hence

$$\mathbf{T}\cdot\mathbf{T}^T\simeq\begin{bmatrix} diag(\mathbf{T111}) & \mathbf{0}_{m_1m_2m_3\times m_1m_2m_3} & \ldots & \mathbf{0}_{m_1m_2m_3\times m_1m_2m_3}\\ \mathbf{0}_{m_1m_2m_3\times m_1m_2m_3} & diag(\mathbf{T112}) & \ldots & \mathbf{0}_{m_1m_2m_3\times m_1m_2m_3}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0}_{m_1m_2m_3\times m_1m_2m_3} & \mathbf{0}_{m_1m_2m_3\times m_1m_2m_3} & \ldots & diag(\mathbf{T222})\end{bmatrix}$$
$$=diag(\mathbf{T})=\widetilde{\mathbf{T}}. \tag{7}$$

Also

$$\mathbf{T}(s,t,r)\cdot\mathbf{T}^T(s,t,r)\cdot X\simeq\widetilde{X}\cdot\mathbf{T}(s,t,r), \tag{8}$$

where X is an $8m_1m_2m_3$-vector and $\widetilde{X}=diag(X)$.

The disjoint property of $\mathbf{Tabc}(s,t,r)$; $a,b,c\in\{1,2\}$ also implies that for every $(m_1m_2m_3\times m_1m_2m_3)$-matrix B,

$$\mathbf{Tabc}^T(s,t,r)\cdot B\cdot\mathbf{Tabc}(s,t,r)\simeq\widehat{B}\cdot\mathbf{Tabc}(s,t,r),$$

where $\widehat{B}$ is an $m_1m_2m_3$-vector with elements equal to the diagonal entries of B. Thus for every $(8m_1m_2m_3\times 8m_1m_2m_3)$-matrix A,

$$\mathbf{T}^T(s,t,r)\cdot A\cdot\mathbf{T}(s,t,r)\simeq\widehat{A}\cdot\mathbf{T}(s,t,r),$$

in which $\widehat{A}$ is an $8m_1m_2m_3$-vector with elements equal to the diagonal entries of matrix A.

4.2. Functions expansion with 3D-TFs

A function $g(s,t,r)$ defined over D may be extended using 3D-TFs as

$$\begin{aligned} g(s,t,r)\simeq&\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1}c1_{i,j,k}T^{111}_{i,j,k}(s,t,r)\\ &+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1}c2_{i,j,k}T^{112}_{i,j,k}(s,t,r)\\ &+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1}c3_{i,j,k}T^{121}_{i,j,k}(s,t,r)\\ &+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1}c4_{i,j,k}T^{122}_{i,j,k}(s,t,r)\\ &+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1}c5_{i,j,k}T^{211}_{i,j,k}(s,t,r)\\ &+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1}c6_{i,j,k}T^{212}_{i,j,k}(s,t,r)\\ &+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1}c7_{i,j,k}T^{221}_{i,j,k}(s,t,r)\\ &+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{k=0}^{m_3-1}c8_{i,j,k}T^{222}_{i,j,k}(s,t,r)\\ =&\,G1^T.\mathbf{T111}(s,t,r)+G2^T.\mathbf{T112}(s,t,r)\\ &+G3^T.\mathbf{T121}(s,t,r)+G4^T.\mathbf{T122}(s,t,r)\\ &+G5^T.\mathbf{T211}(s,t,r)+G6^T.\mathbf{T212}(s,t,r)\\ &+G7^T.\mathbf{T221}(s,t,r)+G8^T.\mathbf{T222}(s,t,r)\\ =&\,G^T.\mathbf{T}(s,t,r),\end{aligned}$$

where G is an $8m_1m_2m_3$-vector given by

$$G=[G1^T,G2^T,G3^T,G4^T,G5^T,G6^T,G7^T,G8^T]^T,$$

and $\mathbf{T}(s,t,r)$ is defined in Eq. (6).

The 3D-TFs coefficients in $G1,G2,G3,G4,G5,G6,G7$ and $G8$ can be computed by sampling the function $g(s,t,r)$ at grid points s_i, t_j and r_k such that $s_i=ih_1$, $t_j=jh_2$ and $r_k=kh_3$, for various i,j and k. Therefore,

$$Gd_l=cd_{i,j,k}=g(s_{i+\alpha},t_{j+\beta},r_{k+\gamma}),$$

where $\alpha,\beta,\gamma\in\{0,1\}$, $d=4\alpha+2\beta+\gamma+1$, $l=m_2i+m_3j+k$, $i=0,\ldots,m_1-1$, $j=0,\ldots,m_2-1$ and $k=0,\ldots,m_3-1$. The $8m_1m_2m_3$-vector G is called the **3D-TF** coefficient vector.

Let $k(s,t,r,x,y,z)$ be a function of six variable on $(D\times D)$. It can be approximated with respect to 3D-TFs as follows:

$$k(s,t,r,x,y,z)=\mathbf{T}^T(s,t,r)\cdot K\cdot\mathbf{T}(x,y,z),$$

where $\mathbf{T}(s,t,r)$ and $\mathbf{T}(x,y,z)$ are 3D-TFs vectors of dimension $8m_1m_2m_3$ and $8m_4m_5m_6$, respectively, and K is an $(8m_1m_2m_3\times 8m_4m_5m_6)$ 3D-TFs coefficient matrix. This matrix can be represented as

$$K = \begin{bmatrix} K11 & K12 & K13 & K14 & K15 & K16 & K17 & K18 \\ K21 & K22 & K23 & K24 & K25 & K26 & K27 & K28 \\ K31 & K32 & K33 & K34 & K35 & K36 & K37 & K38 \\ K41 & K42 & K43 & K44 & K45 & K46 & K47 & K48 \\ K51 & K52 & K53 & K54 & K55 & K56 & K57 & K58 \\ K61 & K62 & K63 & K64 & K65 & K66 & K67 & K68 \\ K71 & K72 & K73 & K74 & K75 & K76 & K77 & K78 \\ K81 & K82 & K83 & K84 & K85 & K86 & K87 & K88 \end{bmatrix},$$

where each block of K is an $(m_1m_2m_3 \times m_1m_2m_3)$-matrix that can be computed by sampling the $k(s,t,r,x,y,z)$ at grid points $(s_{i_1}, t_{j_1}, r_{k_1}, x_{i_2}, y_{j_2}, z_{k_2})$ such that

$$\begin{aligned} s_{i_1} &= i_1h_1; \quad i_1 = 0,1,\ldots,m_1-1, \quad h_1 = \tfrac{1}{m_1}, \\ t_{j_1} &= j_1h_2; \quad j_1 = 0,1,\ldots,m_2-1, \quad h_2 = \tfrac{1}{m_2}, \\ r_{k_1} &= k_1h_3; \quad k_1 = 0,1,\ldots,m_3-1, \quad h_3 = \tfrac{1}{m_3}, \\ x_{i_2} &= i_2h_4; \quad i_2 = 0,1,\ldots,m_4-1, \quad h_4 = \tfrac{1}{m_4}, \\ y_{j_2} &= j_2h_5; \quad j_2 = 0,1,\ldots,m_5-1, \quad h_5 = \tfrac{1}{m_5}, \\ z_{k_2} &= k_2h_6; \quad k_2 = 0,1,\ldots,m_6-1, \quad h_6 = \tfrac{1}{m_6}. \end{aligned}$$

Hence, let $p = m_2i_1 + m_3j_1 + k_1$ and $q = m_5i_2 + m_6j_2 + k_2$, then

$$Kdd'_{p,q} = k(s_{i_1+\alpha_1}, t_{j_1+\beta_1}, r_{k_1+\gamma_1}, x_{i_2+\alpha_2}, y_{j_2+\beta_2}, z_{k_2+\gamma_2}),$$

where $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in \{0,1\}$, $d = 4\alpha_1 + 2\beta_1 + \gamma_1 + 1$, $d' = 4\alpha_2 + 2\beta_2 + \gamma_2 + 1$.

4.3. The operational matrix for integration in the 3D-TFs domain

First, we attempt to compute the triple integral of each element of $\mathbf{Tabc}(s,t,r)$

$$\begin{aligned} \int_0^r\int_0^t\int_0^s T^{a,b,c}_{i,j,k}(\tau,\zeta,\eta)d\tau d\zeta d\eta &= \int_{kh_3}^r\int_{jh_2}^t\int_{ih_1}^s T^{a,b,c}_{i,j,k}(\tau,\zeta,\eta)d\tau d\zeta d\eta \\ &= \int_{kh_3}^r T^c_k(\eta)d\eta\int_{jh_2}^t T^b_j(\zeta)d\zeta\int_{ih_1}^s T^a_i(\tau)d\tau \\ &= Q_c(r,h_3)\cdot Q_b(t,h_2)\cdot Q_a(s,h_1), \end{aligned}$$

where $a,b,c \in \{1,2\}$ and

$$\begin{aligned} Q_1(x,h) &= \left((x-ih) - \tfrac{(x-ih)^2}{2h}\right)u(x-ih) \\ &\quad - \left((x-ih) - \tfrac{(x-ih)^2}{2h} - \tfrac{h}{2}\right)u(x-(i+1)h), \end{aligned}$$

and

$$Q_2(x,h) = \frac{(x-ih)^2}{2h}u(x-ih) - \left(\frac{(x-ih)^2}{2h} - \frac{h}{2}\right)u(x-(i+1)h),$$

in which u denotes the step function. Now, by using the following approximating formulas

$$\left((x-ih) - \frac{(x-ih)^2}{2h}\right)u(x-ih) \simeq \frac{h}{2}\sum_{p=i}^{\frac{1}{h}-1} T^2_p(x),$$

$$\left(\frac{h}{2} + \frac{(x-ih)^2}{2h} - (x-ih)\right)u(x-(i+1)h) \simeq \frac{h}{2}\sum_{p=i+1}^{\frac{1}{h}-1} T^1_p(x),$$

$$\frac{(x-ih)^2}{2h}u(x-ih) \simeq \frac{h}{2}\sum_{q=i}^{\frac{1}{h}-1} T^2_q(x),$$

and

$$\left(\frac{h}{2} - \frac{(x-ih)^2}{2h}\right)u(x-(i+1)h) \simeq \frac{h}{2}\sum_{q=i+1}^{\frac{1}{h}-1} T^1_q(x),$$

we have

$$Q_1(x,h) \simeq Q_2(x,h) \simeq \frac{h}{2}\sum_{p=i+1}^{\frac{1}{h}-1} T^1_p(x) + \frac{h}{2}\sum_{p=i}^{\frac{1}{h}-1} T^2_p(x).$$

Then

$$\begin{aligned} \int_0^r\int_0^t\int_0^s T^{a,b,c}_{i,j,k}(\tau,\zeta,\eta)d\tau d\zeta d\eta \simeq \frac{h_1h_2h_3}{8}\cdot\Bigg(&\sum_{p=i+1}^{m_1-1}\sum_{q=j+1}^{m_2-1}\sum_{l=k+1}^{m_3-1} T^{1,1,1}_{p,q,l}(s,t,r) \\ &+ \sum_{p=i+1}^{m_1-1}\sum_{q=j+1}^{m_2-1}\sum_{l=k}^{m_3-1} T^{1,1,2}_{p,q,l}(s,t,r) + \sum_{p=i+1}^{m_1-1}\sum_{q=j}^{m_2-1}\sum_{l=k+1}^{m_3-1} T^{1,2,1}_{p,q,l}(s,t,r) \\ &+ \sum_{p=i+1}^{m_1-1}\sum_{q=j}^{m_2-1}\sum_{l=k}^{m_3-1} T^{1,2,2}_{p,q,l}(s,t,r) + \sum_{p=i}^{m_1-1}\sum_{q=j+1}^{m_2-1}\sum_{l=k+1}^{m_3-1} T^{2,1,1}_{p,q,l}(s,t,r) \\ &+ \sum_{p=i}^{m_1-1}\sum_{q=j+1}^{m_2-1}\sum_{l=k}^{m_3-1} T^{2,1,2}_{p,q,l}(s,t,r) + \sum_{p=i}^{m_1-1}\sum_{q=j}^{m_2-1}\sum_{l=k+1}^{m_3-1} T^{2,2,1}_{p,q,l}(s,t,r) \\ &+ \sum_{p=i}^{m_1-1}\sum_{q=j}^{m_2-1}\sum_{l=k}^{m_3-1} T^{2,2,2}_{p,q,l}(s,t,r)\Bigg). \end{aligned}$$

Let $P11$, $P12$, $P13$, $P14$, $P15$, $P16$, $P17$ and $P18$ be the operational matrix for triple integration of $\mathbf{Tabc}(s,t,r)$ with respect to 3D-TF vectors.

Moreover, suppose that Ps, Pt and Pr are the operational matrices for integration with respect to s, t and r in the 1D-TF domain, respectively. From Deb [15], the operational matrices for integration with respect to v are defined as follows:

$$Pv = \begin{bmatrix} Pv1 & Pv2 \\ Pv1 & Pv2 \end{bmatrix}.$$

Therefore

$$P111 = Ps1 \otimes Pt1 \otimes Pr1, \quad P112 = Ps1 \otimes Pt1 \otimes Pr2,$$
$$P121 = Ps1 \otimes Pt2 \otimes Pr1,$$

$$P122 = Ps1 \otimes Pt2 \otimes Pr2, \quad P211 = Ps2 \otimes Pt1 \otimes Pr1,$$
$$P212 = Ps2 \otimes Pt1 \otimes Pr2,$$

$$P221 = Ps2 \otimes Pt2 \otimes Pr1, \quad P222 = Ps2 \otimes Pt2 \otimes Pr2,$$

where notation $\otimes$ denotes Kronecker product.

Thus **the operational matrix of integration in the 3D-TFs domain**, P, is a $(8m_1m_2m_3 \times 8m_1m_2m_3)$-matrix as follows:

$$P = \begin{bmatrix} P111 & P112 & P121 & P122 & P211 & P212 & P221 & P222 \\ P111 & P112 & P121 & P122 & P211 & P212 & P221 & P222 \\ P111 & P112 & P121 & P122 & P211 & P212 & P221 & P222 \\ P111 & P112 & P121 & P122 & P211 & P212 & P221 & P222 \\ P111 & P112 & P121 & P122 & P211 & P212 & P221 & P222 \\ P111 & P112 & P121 & P122 & P211 & P212 & P221 & P222 \\ P111 & P112 & P121 & P122 & P211 & P212 & P221 & P222 \\ P111 & P112 & P121 & P122 & P211 & P212 & P221 & P222 \end{bmatrix}.$$

Finally, the triple integral of function $g(s,t,r)$ can be approximated as

$$\begin{aligned} \int_0^r\int_0^t\int_0^s g(\tau,\zeta,\eta)d\tau d\zeta d\eta &\simeq \int_0^r\int_0^t\int_0^s G^T T(\tau,\zeta,\eta)d\tau d\zeta d\eta \\ &\simeq G^T\cdot P\cdot T(s,t,r), \end{aligned}$$

where G is the 3D-TF coefficient vector of $g(s,t,r)$.

In this paper, we suppose that $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m$ for convergence.

5. Solving 3D nonlinear integral equations

In this Section, we solve three-dimensional nonlinear mixed Volterra–Fredholm integral equations of the second kind of the form Eq. (1) with Eq. (2) by using 3D-TFs. We now approximate functions $f(s,t,r), g(s,t,r), [f(s,t,r)]^p$ and $k(s,t,r,x,y,z)$ with respect to 3D-TFs by the way mentioned in Section 4 as

$$\begin{aligned}
&f(s,t,r) \simeq \mathbf{T}^T(s,t,r)F,\\
&g(s,t,r) \simeq \mathbf{T}^T(s,t,r)G,\\
&[f(s,t,r)]^p \simeq \mathbf{T}^T(s,t,r)F_p,\\
&k(s,t,r,x,y,z) \simeq \mathbf{T}^T(s,t,r)K\mathbf{T}(x,y,z),
\end{aligned} \tag{9}$$

where $\mathbf{T}(s,t,r)$ is defined in Eq. (6), the vectors F, G, F_p, and matrix K are 3D-TFs coefficients of $f(s,t,r), g(s,t,r)$, $[f(s,t,r)]^p$ and $k(s,t,r,x,y,z)$ respectively.

Lemma 1. *Let $8m^3$-vectors F and F_p be 3D-TFs coefficients of $f(x,y,z)$ and $[f(x,y,z)]^p$, respectively. If*

$$F = [f_1, \ldots, f_m, \ldots, f_{m^3}, \ldots, f_{8m^3}]^T,$$

then we have

$$F_p = [f_1^p, \ldots, f_m^p, \ldots, f_{m^3}^p, \ldots, f_{8m^3}^p]^T, \tag{10}$$

where $p \geqslant 1$, is a positive integer.

Proof. (By induction) When $p = 1$, Eq. (10) follows at once from $[f(x,y,z)]^p = f(x,y,z)$. Suppose that Eq. (10) holds for p, we shall deduce it for $(p+1)$. Since $[f(x,y,z)]^{p+1} = f(x,y,z)$ $[f(x,y,z)]^p$, from Eqs. (8) and (9) it follows that

$$\begin{aligned}
[f(x,y,z)]^{p+1} &= f(x,y,z)[f(x,y,z)]^p = F^T\mathbf{T}(x,y,z)\mathbf{T}^T(x,y,z)F_p\\
&= F^T\widetilde{F}_p\mathbf{T}(x,y,z).
\end{aligned}$$

Now by using Eq. (10) we obtain

$$F^T\widetilde{F}_p = [f_1^{p+1}, \ldots, f_m^{p+1}, \ldots, f_{m^3}^{p+1}, \ldots, f_{8m^3}^{p+1}]^T,$$

therefore, Eq. (10) holds for $(p+1)$, and the lemma is established. □

From Eq. (9), we can approximate the integral part in Eq. (1) with Eq. (2) as follows:

$$\begin{aligned}
&\int_0^s \int_0^1 \int_0^1 k(s,t,r,x,y,z)[f(x,y,z)]^p dzdydx\\
&\simeq \int_0^s \int_0^1 \int_0^1 \mathbf{T}^T(s,t,r)K\mathbf{T}(x,y,z)\mathbf{T}^T(x,y,z)F_p dzdydx\\
&= \mathbf{T}^T(s,t,r)K\left(\int_0^s \int_0^1 \int_0^1 \mathbf{T}(x,y,z)\mathbf{T}^T(x,y,z)dzdydx\right)F_p.
\end{aligned} \tag{11}$$

Consider R_i as the $(i+1)$th row of the operational matrix Ps ($Ps_{2m\times 2m}$ is operational matrix of 1D-TFs defined over $[0,1)$, see [15]) and considering $\int_0^1 T_j^1(t)dt = \int_0^1 T_j^2(t)dt = \frac{h}{2}$. Now by using Eqs. (5) and (7), we get

$$\int_0^s \int_0^1 \int_0^1 \mathbf{T}(x,y,z)\mathbf{T}^T(x,y,z)dzdydx = \begin{pmatrix} D_0 & 0 & \ldots & 0\\ 0 & D_1 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & D_{2m-1}\end{pmatrix}_{8m^3\times 8m^3}, \tag{12}$$

where

$$D_i = \begin{pmatrix}
\int_0^s\int_0^1\int_0^1 T_i(x)T_0(y)T_0(z)dzdydx & \ldots & 0 & \ldots & 0\\
\vdots & \ddots & \vdots & \ddots & \vdots\\
0 & \ldots & \int_0^s\int_0^1\int_0^1 T_i(x)T_0(y)T_{m-1}(z)dzdydx & \ldots & 0\\
\vdots & \ddots & \vdots & \ddots & \vdots\\
0 & \ldots & 0 & \ldots & \int_0^s\int_0^1\int_0^1 T_i(x)T_m(y)T_m(z)dzdydx
\end{pmatrix}_{4m^2\times 4m^2}$$

$$= \begin{pmatrix}
\frac{h^2}{4}R_i\mathbf{T}(s) & \ldots & 0 & \ldots & 0\\
\vdots & \ddots & \vdots & \ddots & \vdots\\
0 & \ldots & \frac{h^2}{4}R_i\mathbf{T}(s) & \ldots & 0\\
\vdots & \ddots & \vdots & \ddots & \vdots\\
0 & \ldots & 0 & \ldots & \frac{h^2}{4}R_i\mathbf{T}(s)
\end{pmatrix}_{4m^2\times 4m^2},$$

where $\mathbf{T}(s)$ is defined in Eq. (4).

Also by using Eq. (5), Eq. (6) can be reformulated as

$$\mathbf{T}(s,t,r)=\begin{pmatrix} T_0(s) & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & T_0(s) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & T_{m-1}(s) & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & T_{m-1}(s) \end{pmatrix}_{8m^3\times 8m^3} \times [\mathbf{T}(t,r),\mathbf{T}(t,r),\dots,\mathbf{T}(t,r),\mathbf{T}(t,r)]^T_{8m^3\times 1}.$$

So, we have

$$\mathbf{T}^T(s,t,r)K=[\mathbf{T}(t,r),\mathbf{T}(t,r),\dots,\mathbf{T}(t,r),\mathbf{T}(t,r)]_{8m^3\times 1}\times\begin{pmatrix} k_{1,1}T_0^1(s) & \dots & k_{1,4m^2}T_0^1(s) & \dots & k_{1,4m^3}T_0^1(s) & \dots & k_{1,8m^3}T_0^1(s) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{4m^2,1}T_0^1(s) & \dots & k_{4m^2,4m^2}T_0^1(s) & \dots & k_{4m^2,4m^3}T_0^1(s) & \dots & k_{4m^2,8m^3}T_0^1(s) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{4m^2(m-1)+1,1}T_{m-1}^1(s) & \dots & k_{4m^2(m-1)+1,4m^2}T_{m-1}^1(s) & \dots & k_{4m^2(m-1)+1,4m^3}T_{m-1}^1(s) & \dots & k_{4m^2(m-1)+1,8m^3}T_{m-1}^1(s) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{4m^3,1}T_{m-1}^1(s) & \dots & k_{4m^3,m^2}T_{m-1}^1(s) & \dots & k_{4m^3,4m^3}T_{m-1}^1(s) & \dots & k_{4m^3,8m^3}T_{m-1}^1(s) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{8m^3,1}T_{m-1}^2(s) & \dots & k_{8m^3,4m^2}T_{m-1}^2(s) & \dots & k_{8m^3,4m^3}T_{m-1}^2(s) & \dots & k_{8m^3,8m^3}T_{m-1}^2(s) \end{pmatrix}_{8m^3\times 8m^3}. \tag{13}$$

Also, we have

$$R_i\mathbf{T}(s)=\begin{cases} \frac{h}{2}(T_{i+1}^1(s)+\ldots+T_{m-1}^1(s)+T_j^2(s)+\ldots+T_{m-1}^2(s)), & i=0,\dots,m-1, \\ \frac{h}{2}(T_{i-m+1}^1(s)+\ldots+T_{m-1}^1(s)+T_{j-m}^2(s)+\ldots+T_{m-1}^2(s)), & i=m,\dots,2m-1. \end{cases} \tag{14}$$

By using Eqs. (12)–(14), Eq. (11) can be reformulated as

$$[\mathbf{T}(t,r),\mathbf{T}(t,r),\dots,\mathbf{T}(t,r),\mathbf{T}(t,r)]_{8m^3\times 1}\times\begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ A_{00} & \mathbf{0} & \dots & \mathbf{0} & A_{0,m} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A_{11} & \dots & \mathbf{0} & \mathbf{0} & A_{1,m+1} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & A_{m-1,m-1} & \mathbf{0} & \mathbf{0} & \dots & A_{m-1,2m-1} \end{pmatrix}_{8m^3\times 8m^3}\cdot F_p,$$

where

$$A_{i,j}=\frac{h^3}{8}k_{4m^2(m+i)+p,4m^2j+q}T_i^2(s),\quad 1\leqslant p,q\leqslant 4m^2,$$

where $\mathbf{0}$ is a zero matrix. Also

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ A_{00} & \mathbf{0} & \dots & \mathbf{0} & A_{0,m} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A_{11} & \dots & \mathbf{0} & \mathbf{0} & A_{1,m+1} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & A_{m-1,m-1} & \mathbf{0} & \mathbf{0} & \dots & A_{m-1,2m-1} \end{pmatrix}_{8m^3\times 8m^3} = \begin{pmatrix} T_0(s) & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & T_0(s) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & T_{m-1}(s) & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & T_{m-1}(s) \end{pmatrix}_{8m^3\times 8m^3}\cdot U,$$

where

$$U=\begin{pmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ U_{00} & \mathbf{0} & \dots & \mathbf{0} & U_{0,m} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & U_{11} & \dots & \mathbf{0} & \mathbf{0} & U_{1,m+1} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & U_{m-1,m-1} & \mathbf{0} & \mathbf{0} & \dots & U_{m-1,2m-1} \end{pmatrix}_{8m^3\times 8m^3},$$

$$U_{i,j}=\frac{h^3}{8}k_{4m^2(m+i)+p,4m^2j+q},\quad 1\leqslant p,q\leqslant 4m^2.$$

So, we have

$$\int_0^s\int_0^1\int_0^1 k(s,t,r,x,y,z)[f(x,y,z)]^p\,dzdydx \simeq \mathbf{T}^T(s,t,r)UF_p. \tag{15}$$

Substituting Eqs. (9) and (15) into Eq. (1) with Eq. (2) gives

$$\mathbf{T}^T(s,t,r)F=\mathbf{T}^T(s,t,r)G+\mathbf{T}^T(s,t,r)UF_p\ \Rightarrow\ F-UF_p=G.$$

After solving the above nonlinear system by using Newton–Raphson method, we can find F and then

$$f(s,t,r)\simeq F^T\mathbf{T}(s,t,r).$$

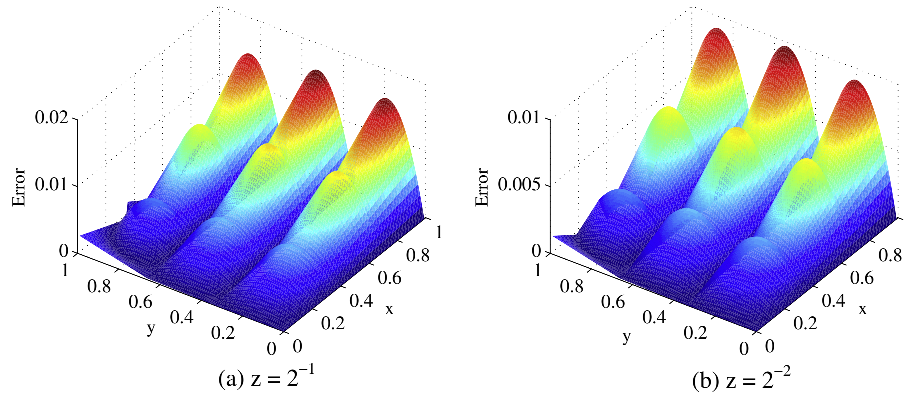

Figure 2 Absolute errors for Example 1, with $m = 3$ and $z = 2^{-1},\ 2^{-2}$.

Table 1 Absolute error for $m = 2, 3$ of $f(s, t, r)$ of Eq. (16).

Nodes $(s,\ t,\ r)$	Method of [29]	$(k = 1)$	Method of [29]	$(k = 2)$	Present method	
$s = t = r = 2^{-l}$	$m = 2$	$m = 3$	$m = 2$	$m = 3$	$m = 2$	$m = 3$
$l = 1$	0.2656154	0.0023276	0.1354830	0.0709824	0.0000597	0.0077834
$l = 2$	0.0013014	0.0028775	0.0326229	0.0032986	0.0051785	0.0021225
$l = 3$	0.0049635	0.0007847	0.0025225	0.0003024	0.0009898	0.0005116
$l = 4$	0.0051924	0.0010135	0.0027513	0.0005313	0.0001636	0.0000798
$l = 5$	0.0052067	0.0010278	0.0027656	0.0005456	0.0000275	0.0000111
$l = 6$	0.0052076	0.0010287	0.0027665	0.0005465	0.0000050	0.0000015

Table 2 Absolute error for $m = 2, 3$ of $f(s, t, r)$ of Eq. (17).

Nodes $(s,\ t,\ r)$	Method of [29]	$(k = 1)$	Method of [29]	$(k = 2)$	Present method	
$s = t = r = 2^{-l}$	$m = 2$	$m = 3$	$m = 2$	$m = 3$	$m = 2$	$m = 3$
$l = 1$	0.1868866	0.0010831	0.0923437	0.0638287	0.000254292	0.003112872
$l = 2$	0.0006351	0.0332912	0.0780022	0.0053354	0.001785479	0.000599173
$l = 3$	0.0444188	0.0117627	0.0221892	0.0015860	0.000333834	0.000195857
$l = 4$	0.0560233	0.0233672	0.0337937	0.0131905	0.000046763	0.000032094
$l = 5$	0.0589458	0.0262897	0.0367162	0.0161130	0.000005678	0.000004488
$l = 6$	0.0596778	0.0270217	0.0374482	0.0168450	0.000000578	0.000000591

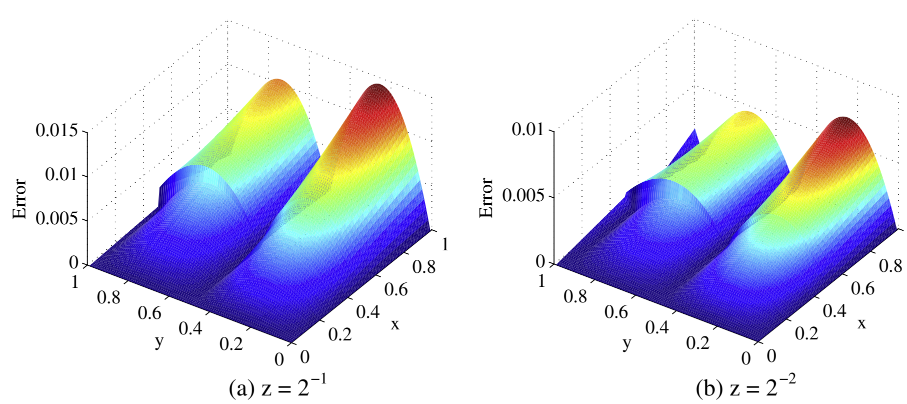

Figure 3 Absolute errors for Example 2, with $m = 2$ and $z = 2^{-1},\ 2^{-2}$.

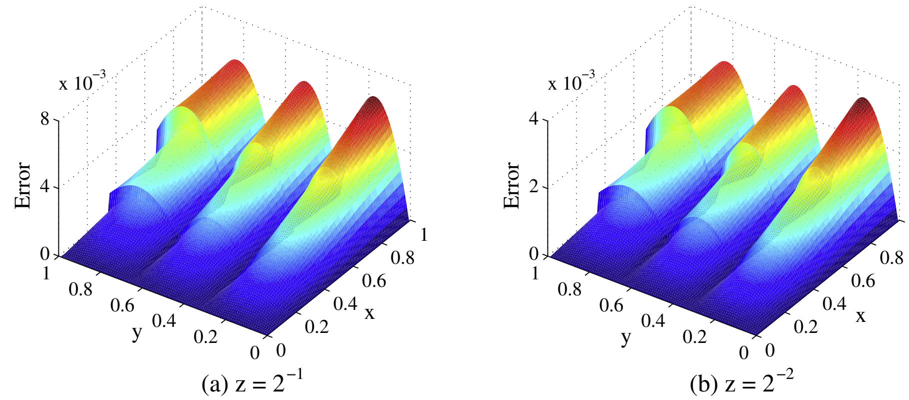

Figure 4 Absolute errors for Example 2, with $m = 3$ and $z = 2^{-1},\ 2^{-2}$.

6. Numerical examples

In this section, numerical examples are given to certify the convergence and error bound of the presented method. All results are computed by using a program written in the Matlab. The numerical experiments are carried out for the selected grid point which are proposed as $(2^{-l};\ l = 1, 2, 3, 4, 5, 6)$ and m terms of the 3D-TFs series.

Example 1. Consider the following three-dimensional nonlinear mixed Volterra–Fredholm integral equation:

$$f(s,t,r) = g(s,t,r) + \frac{1}{4}\int_0^s \int_0^1 \int_0^1 (s+x)(t^2+z)ryf^2(x,y,z)dzdydx, \tag{16}$$

where $(s,t,r) \in D$ and

$$g(s,t,r) = s^2tr - \frac{11}{5760}s^6r(3+4t^2).$$

The exact solution is $f(s,t,r) = s^2tr$. The error results for proposed method besides the error for method of Mirzaee and Hadadiyan [29] are tabulated in Table 1. Figs. 1 and 2 illustrate the error results for this example.

Example 2. Consider the following three-dimensional nonlinear mixed Volterra–Fredholm integral equation:

$$f(s,t,r) = g(s,t,r) + \int_0^s \int_0^1 \int_0^1 s^2tryzsin(x)f^3(x,y,z)dzdydx, \tag{17}$$

where $(s,t,r) \in D$ and

$$g(s,t,r) = \frac{tr}{100}(s^2\cos^4(s) - s^2 + 100\cos(s)).$$

The exact solution is $f(s,t,r) = tr\cos(s)$. The error results for proposed method besides the error for method of Mirzaee and Hadadiyan [29] are tabulated in Table 2. Figs. 3 and 4 illustrate the error results for this example.

7. Conclusion

In this paper, we used 3D-TFs and operational matrix to solve three-dimensional nonlinear Volterra–Fredholm integral equations. One of the benefits of 3D-TFs method is lower cost of setting up the system of equations without applying any projection method such as Collocation, Galerkin, etc., and any integration. Moreover, operational matrix P can be computed at once for large values of m_1, m_2 and m_3, and stored for using in various problems. Therefore, the final nonlinear system is set up only by sampling f and k in grid points, and also computing G. Thus the computational cost of operations is low. These advantages make the method very simple and cheap from the computational point of view. The accuracy and applicability were checked by some examples. Furthermore, the current method can be run with increasing m_1, m_2 and m_3 until the results settle down to an appropriate accuracy. It is to be noted that this method can be easily extended and applied to a system of three-dimensional nonlinear Volterra–Fredholm integral equations.

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