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Some properties of E-convex functions[☆]

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Abstract

Recently, it was shown by Youness [E.A. Youness, On E-convex sets, E-convex functions and E-convex programming, Journal of Optimization Theory and Applications, 102 (1999) 439–450] that many results for convex sets and convex functions actually hold for a wider class of sets and functions, called E-convex sets and E-convex functions. We introduce the concept of E-quasiconvex functions and strictly E-quasiconvex functions, and develop some basic properties of E-convex and E-quasiconvex functions. For a real-valued function f defined on a nonempty E-convex set M, we show under the convexity condition of $E(M)$, that f is E-quasiconvex (resp. strictly E-quasiconvex) if and only if its restriction to $E(M)$ is quasiconvex (resp. strictly quasiconvex). Similarly, we show under the convexity condition of $E(M)$, that f is E-convex (resp. strictly E-convex) if and only if its restriction to $E(M)$ is convex (resp. strictly convex). In addition, under the convexity condition of $E(M)$, a characterization of an E-quasiconvex function in terms of the lower level sets of its restriction to $E(M)$ is also given. Finally, examples in nonlinear programming problem are used to illustrate the applications of our results.

Keywords: E-convexity; Generalized convexity; Convexity; Nonlinear programming

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1. Introduction

The concept of convexity and its various generalizations is important for quantitative and qualitative studies in operations research or applied mathematics. Recently, Youness [2] introduced a class of sets and a class of functions called E-convex sets and E-convex functions by relaxing the definitions of convex sets and convex functions. However, as pointed out by Yang [1], some results and proofs in Youness [2] seem to be incorrect.

Motivated both by earlier research works and by the importance of the concept of convexity, we introduce the concept of E-quasiconvex functions and strictly E-quasiconvex functions, and develop some basic properties of E-convex and E-quasiconvex functions. For a real-valued function f defined on a nonempty E-convex set M, we show under the convexity condition of $E(M)$, that f is E-quasiconvex (resp. strictly E-quasiconvex) if and only if its restriction to $E(M)$ is quasiconvex (resp. strictly quasiconvex). Similarly, we show under the convexity condition of $E(M)$, that f is E-convex (resp. strictly E-convex) if and only if its restriction to $E(M)$ is convex (resp. strictly convex). In addition, under the convexity condition of $E(M)$, a characterization of an E-quasiconvex function in terms of the lower level sets of its restriction to $E(M)$ is also given. Finally, examples in nonlinear programming problems are used to illustrate the applications of our results.

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space. We recall:

Definition 2.1 (*Ref. [2, Definition 2.1]*). A set $M \subseteq R^n$ is said to be E-convex if there is a mapping $E : R^n \to R^n$ such that

$$\lambda E(x) + (1 - \lambda)E(y) \in M$$

for all $x, y \in M$ and $\lambda \in [0, 1]$.

Let E be a mapping from R^n to R^n. For any $S \subseteq R^n$, $E(S)$ is defined as follows:

$$E(S) = \{E(x) : x \in S\}.$$

Restricting E to S, for any $\overline{x} \in E(S)$, the pre-image of E at $\overline{x}$, denoted by $E^{-1}(\overline{x})$, is defined as

$$E^{-1}(\overline{x}) = \{x \in S : E(x) = \overline{x}\}.$$

For any $\overline{X} \subseteq E(S)$, $E^{-1}(\overline{X})$, is defined as

$$E^{-1}(\overline{X}) = \bigcup_{\overline{x} \in \overline{X}} E^{-1}(\overline{x}).$$

Lemma 2.1 (*Ref. [2, Proposition 2.2]*). *If a set $M \subseteq R^n$ is E-convex with respect to a mapping $E : R^n \to R^n$, then $E(M) \subseteq M$.*

Theorem 2.1. *If $\{M_j : j \in J\}$ is an arbitrary nonempty collection of E-convex subsets of R^n with respect to a mapping $E : R^n \to R^n$, then the intersection $\cap_{j \in J} M_j$ is an E-convex subset of R^n.*

Proof. Let $\{M_j : j \in J\}$ be a family of E-convex subsets of R^n. If $\cap_{j \in J} M_j$ is an empty set then it is obviously an E-convex subset of R^n. Assume that $x, y \in \cap_{j \in J} M_j$ (x and y may not be distinct), then

$x, y \in M_j$ for each $j \in J$. By the E-convexity of M_j, we have, for each $j \in J$,

$$\lambda E(x) + (1-\lambda)E(y) \in M_j \qquad \text{for each } \lambda \in [0, 1],$$

which implies that

$$\lambda E(x) + (1-\lambda)E(y) \in \bigcap_{j \in J} M_j \qquad \text{for each } \lambda \in [0, 1].$$

Corollary 2.1. *If M_j, $j = 1, 2, \ldots, m$, are E-convex subsets of R^n, then the intersection $\cap_{j=1}^m M_j$ is an E-convex subset of R^n.*

From now on, let E be a mapping from R^n to R^n, and let $M \subseteq R^n$ be a nonempty E-convex set.

Definition 2.2 (*Ref. [2, Definition 3.1]*). A real-valued function $f : M \to R^1$ is said to be E-convex if

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda f(E(x)) + (1-\lambda) f(E(y))$$

for all $x, y \in M$ and $\lambda \in [0, 1]$; and strictly E-convex if strict inequality holds for all $x, y \in M$, $E(x) \ne E(y)$ and $\lambda \in (0, 1)$.

It is obvious that any strictly E-convex function is E-convex.

3. Main results

It is known from Lemma 2.1 that $E(M) \subseteq M$. Hence, for any $f : M \to R^1$, we have the following observations:

Observation (a). The function $f \circ E : M \to R^1$ defined by

$$(f \circ E)(x) = f(E(x)) \qquad \text{for all } x \in M$$

is well defined.

Observation (b). The restriction $\tilde{f} : E(M) \to R^1$ of $f : M \to R^1$ to $E(M)$ defined by

$$\tilde{f}(\tilde{x}) = f(\tilde{x}) \qquad \text{for all } \tilde{x} \in E(M)$$

is well defined.

Definition 3.1. Let f be a real-valued function defined on M. For any real number r, the lower level set, $L_r(f \circ E)$, of $f \circ E : M \to R^1$ is defined as

$$L_r(f \circ E) = \{x \in M : (f \circ E)(x) = f(E(x)) \le r\}.$$

The lower level set, $L_r(\tilde{f})$, of $\tilde{f} : E(M) \to R^1$ is defined as

$$L_r(\tilde{f}) = \{\tilde{x} \in E(M) : \tilde{f}(\tilde{x}) = f(\tilde{x}) \le r\}.$$

It is easy to establish that

$$E^{-1}(L_r(\tilde{f})) = L_r(f \circ E) \qquad \text{for each } r \in R^1.$$

Let $r \in R^1$. Since $L_r(\tilde{f}) \subseteq E(M)$, it follows that

$$E^{-1}(L_r(\tilde{f})) = \bigcup_{\overline{x}\in L_r(\tilde{f})} \{x \in M : E(x) = \overline{x}\}$$
$$= \{x \in M : f(E(x)) \le r\}$$
$$= L_r(f \circ E).$$

Next, we introduce the concept of E-quasiconvex functions and strictly E-quasiconvex functions based on the concept of E-convex sets due to Youness [2].

Definition 3.2. A real-valued function $f : M \to R^1$ is said to be E-quasiconvex if

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \max\{f(E(x)),\ f(E(y))\}$$

for all $x, y \in M$ and $\lambda \in [0, 1]$; and strictly E-quasiconvex if strict inequality holds for all $x, y \in M$, $E(x) \neq E(y)$ and $\lambda \in (0, 1)$.

It is obvious that E-quasiconvexity of a function is a generalization of E-convexity, and consequently, also of strict E-convexity. It is also clear that any strictly E-quasiconvex function is E-quasiconvex.

Let $f : M \to R^n$ be E-quasiconvex. It is known from Lemma 2.1 that $E(M) \subseteq M$. It is easy to establish that the restriction, say $\overline{f}$, of f to any nonempty convex subset C of $E(M)$ is a quasiconvex function: Let $C \subseteq E(M)$ be convex, and let $\overline{x}, \overline{y} \in C$ ($\overline{x}$ and $\overline{y}$ may not be distinct). Then there exist x, $y \in M$ such that $\overline{x} = E(x)$ and $\overline{y} = E(y)$. Since $\lambda\overline{x}+(1-\lambda)\overline{y} \in C$, it follows from the E-quasiconvexity of $f : M \to R^1$ that

$$\overline{f}(\lambda\overline{x} + (1-\lambda)\overline{y}) = f(\lambda E(x) + (1-\lambda)E(y))$$
$$\le \max\{f(E(x)),\ f(E(y))\}$$
$$= \max\{\overline{f}(\overline{x}),\ \overline{f}(\overline{y})\}$$

for all $\lambda \in [0, 1]$. Hence, we obtain the following result:

Theorem 3.1. *Let $f : M \to R^1$ be E-quasiconvex (resp. strictly E-quasiconvex). Then the restriction, say $\overline{f} : C \to R^1$, of f to any nonempty convex subset C of $E(M)$ is a quasiconvex (resp. strictly quasiconvex) function.*

Remark 3.1. There is a considerable confusion in the literature concerning the terminology of various families of generalized convex functions. In this paper, a real-valued function f defined on a nonempty convex set $C \subseteq R^n$ is said to be strictly quasiconvex if

$$f(\lambda x + (1-\lambda)y) < \max\{f(x),\ f(y)\}$$

for all $x, y \in C$, $x \neq y$ and $\lambda \in (0, 1)$.

Corollary 3.1. *Let $f : M \to R^1$ be E-quasiconvex (resp. strictly E-quasiconvex). If $E(M) \subseteq M$ is a convex set, then the restriction $\tilde{f} : E(M) \to R^1$ of $f : M \to R^1$ to $E(M)$ is a quasiconvex (resp. strictly quasiconvex) function.*

Let $f : M \to R^1$, and let $E(M)$ be convex. It is easy to establish that the quasiconvexity of $\tilde{f} : E(M) \to R^1$ implies the E-quasiconvexity of $f : M \to R^1$: Let $\tilde{f} : E(M) \to R^1$ be quasiconvex, and let $x, y \in M$. Then $E(x), E(y) \in E(M)$, and by the convexity of $E(M)$ follows $\lambda E(x) + (1-\lambda)E(y) \in E(M)$ for all $\lambda \in [0, 1]$. Since $\tilde{f} : E(M) \to R^1$ is quasiconvex, we have

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \max\{f(E(x)),\ f(E(y))\} \qquad \text{for all } \lambda \in [0, 1],$$

which implies that $f : M \to R^1$ is E-quasiconvex. Then, by Corollary 3.1, we obtain the following result.

Theorem 3.2. *Suppose that $E(M)$ is convex, and that f is a real-valued function defined on M. Then $f : M \to R^1$ is E-quasiconvex (resp. strictly E-quasiconvex) if and only if its restriction, $\tilde{f} : E(M) \to R^1$, to $E(M)$ is a quasiconvex (resp. strictly quasiconvex) function.*

Corollary 3.2. *Suppose that $E(M)$ is convex, and that f is a real-valued function defined on M. Then $f : M \to R^1$ is E-quasiconvex if and only if the lower level set $L_r(\tilde{f})$ of its restriction $\tilde{f} : E(M) \to R^1$ is a convex set for each $r \in R^1$.*

An analogous result to Theorem 3.1 for the E-convex case is the following theorem.

Theorem 3.3. *Let $f : M \to R^1$ be E-convex (resp. strictly E-convex). Then the restriction, say $\overline{f} : C \to R^1$, of f to any nonempty convex subset C of $E(M)$ is a convex (resp. strictly convex) function.*

An analogous result to Theorem 3.2 for the E-convex case is the following theorem.

Theorem 3.4. *Suppose that $E(M)$ is convex, and that f is a real-valued function defined on M. Then $f : M \to R^1$ is E-convex (resp. strictly E-convex) if and only if its restriction, $\tilde{f} : E(M) \to R^1$, to $E(M)$ is a convex (resp. strictly convex) function.*

Direct examination of the definition of E-convex functions shows that the set of E-convex functions on M is closed under addition and nonnegative scalar multiplication. This is formalized in the following theorem.

Theorem 3.5. *Let f and g be E-convex functions on M and let $\alpha > 0$. Then $f + g$ and αf are E-convex functions on M.*

Theorem 3.6. *If $\{f_j : j \in J\}$ is an arbitrary nonempty collection of E-convex functions on M such that for each $x \in M$, $\sup\{f_j(x) : j \in J\}$ exists in R^1, then the real-valued function $f : M \to R^1$, defined by*

$$f(x) = \sup_{j \in J} f_j(x) \qquad \text{for each } x \in M,$$

is E-convex on M.

Proof. If $\{f_j : j \in J\}$ is an arbitrary nonempty collection of E-convex functions on M. It follows from the E-convexity of M that $E(M) \subseteq M$. So, for each $x \in M$, we have

$$f(E(x)) = \sup_{j \in J} f_j(E(x)). \tag{3.1}$$

Recall that: Given nonempty subsets A and B of R^1, if both $\sup A$ and $\sup B$ exist in R^1, then $\sup(A+B)$ exists in R^1 and $\sup(A+B) = \sup A + \sup B$. This observation, combined with (3.1) and the E-convexity of each f_j, implies that for each $x, y \in M$ and $\lambda \in [0, 1]$

$$\begin{aligned} f(\lambda E(x) + (1-\lambda)E(y)) &= \sup\{f_j(\lambda E(x) + (1-\lambda)E(y)) : j \in J\} \\ &\le \sup\{\lambda f_j(E(x)) + (1-\lambda) f_j(E(y)) : j \in J\} \end{aligned}$$

$$= \lambda \left(\sup_{j \in J} f_j(E(x)) \right) + (1-\lambda) \left(\sup_{j \in J} f_j(E(x)) \right)$$
$$= \lambda f(E(x)) + (1-\lambda) f(E(y)).$$

Hence $f : M \to R^1$ is E-convex.

Theorem 3.7. *If $\{f_j : j \in J\}$ is an arbitrary nonempty collection of E-quasiconvex functions on M such that for each $x \in M$, $\sup\{f_j(x) : j \in J\}$ exists in R^1, then the real-valued function $f : M \to R^1$, defined by*

$$f(x) = \sup_{j \in J} f_j(x) \qquad \text{for each } x \in M,$$

is E-quasiconvex on M.

Proof. If $\{f_j : j \in J\}$ is an arbitrary nonempty collection of E-quasiconvex functions on M. It follows from the E-convexity of M that $E(M) \subseteq M$. So, for each $x \in M$, we have

$$f(E(x)) = \sup_{j \in J} f_j(E(x)). \tag{3.2}$$

It follows from (3.2) and the E-quasiconvexity of each f_j that for each $x, y \in M$ and $\lambda \in [0, 1]$

$$\begin{aligned} f(\lambda E(x) + (1-\lambda)E(y)) &= \sup\{f_j(\lambda E(x) + (1-\lambda)E(y)) : j \in J\} \\ &\leq \sup_{j \in J} \max\{f_j(E(x)), f_j(E(y))\} \\ &= \max\left\{\sup_{j \in J} f_j(E(x)), \sup_{j \in J} f_j(E(y))\right\} \\ &= \max\{f(E(x)), f(E(y))\}. \end{aligned}$$

Hence $f : M \to R^1$ is E-quasiconvex.

4. Application to nonlinear programming

In this section, we consider the following nonlinear programming problem (P):

$$\begin{aligned} &\text{minimize} \quad f(\overline{x}) \\ &\text{s.t.} \quad \overline{x} \in \{\overline{x} \in E(R^n) : g_j(\overline{x}) \leq b_j,\ j = 1, \ldots, m\}, \end{aligned}$$

where $\overline{x} \in R^n$, $f : R^n \to R^1$, $g_j : R^n \to R^1$, $j = 1, \ldots, m$, and $b_j \in R^1$, $j = 1, \ldots, m$. We assume throughout that $g_j : R^n \to R^1$, $j = 1, \ldots, m$, are E-quasiconvex and that $E(R^n)$ is a convex subset of R^n.

Denote the feasible set by $\overline{X}$, where

$$\overline{X} = \bigcap_{j=1}^{m} \overline{X}_j = \bigcap_{j=1}^{m} \{\overline{x} \in E(R^n) : g_j(\overline{x}) \leq b_j\}.$$

It can be easily checked that $\{\overline{x} \in E(R^n) : g_j(\overline{x}) \leq b_j\}$ is a convex set for each $j = 1, \ldots, m$: notice that the set $\{\overline{x} \in E(R^n) : g_j(\overline{x}) \leq b_j\}$ is the lower level set of the restriction of $g_j : R^n \to R^1$ to $E(R^n)$ for each $j = 1, \ldots, m$. Since $E(R^n)$ is convex and $g_j : R^n \to R^1$, $j = 1, \ldots, m$, are E-quasiconvex (by assumption), it follows from Corollary 3.2 that $\{\overline{x} \in E(R^n) : g_j(\overline{x}) \leq b_j\}$, $j = 1, \ldots, m$, are convex. Then by the known fact that the intersection of an arbitrary nonempty collection of convex sets is also a convex set, we have the following:

Lemma 4.1. *$\overline{X}$ is a convex subset of $E(R^n)$.*

According to Lemma 4.1 and Theorem 3.1, we have the following result.

Theorem 4.1. *Suppose that $f : R^n \to R^1$ is E-quasiconvex, then the set of solutions of problem (P) is convex.*

Recall that a strict local minimizer of a quasiconvex function is also a strict global minimizer. Then by Theorem 3.1, we have the following:

Theorem 4.2. *Suppose that*

(1) *$f : R^n \to R^1$ is E-quasiconvex;*
(2) *$\overline{x}_*$ is a strict local minimizer of (P).*

Then $\overline{x}_*$ is also a strict global minimizer of (P).

Due to the convexity property, Lemma 4.1 and Theorem 3.3, we can obtain the following:

Theorem 4.3. *Suppose that*

(1) *$f : R^n \to R^1$ is E-convex;*
(2) *$\overline{x}_*$ is a local minimizer of (P).*

Then $\overline{x}_*$ is also a global minimizer of (P).

Due to the strict quasiconvexity property, Lemma 4.1 and Theorem 3.1, we can obtain the following:

Theorem 4.4. *Suppose that $f : R^n \to R^1$ is strictly E-quasiconvex.*

(1) *If $\overline{x}_*$ is a local minimizer of (P), then it is also a global minimizer.*
(2) *f attains it minimum over $\overline{X}$ at no more than one point.*

References

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