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Error of Truncated Chebyshev Series and Other Near Minimax Polynomial Approximations

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It is well known that a near minimax polynomial approximation p is obtained by truncating the Chebyshev series of a function f after $n+1$ terms. It is shown that if $f \in C^{(n+1)}[-1, 1]$, then $\|f-p\|$ may be expressed in terms of $f^{(n+1)}$ in the same manner as the error of minimax approximation. The result is extended to other types of near minimax approximation. © 1987 Academic Press, Inc.

1. INTRODUCTION

Bernstein [1] has shown that if $p \in \mathscr{P}_n$ is the minimax approximation on $[-1, 1]$ to $f \in C^{(n+1)}[-1, 1]$, then the error satisfies

$$\|f-p\| = \frac{1}{2^n(n+1)!} |f^{(n+1)}(\xi)|, \tag{1.1}$$

where $\xi \in (-1, 1)$ and $\|\cdot\|$ denotes the Chebyshev norm. Phillips [5] has shown that a similar result holds for other choices of norm and Holland,

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Phillips, and Taylor [3, 4] have extended Bernstein's proof to other approximations.

It is well known that a near minimax approximation is given by the interpolating polynomial $p \in \mathscr{P}_n$ constructed on the zeros of T_{n+1}, the Chebyshev polynomial of degree $n+1$. The error $f-p$ again satisfies (1.1) (except ξ may have a different value).

Recently Phillips and Taylor [6] have shown that (1.1) holds if $p \in \mathscr{P}_n$ is chosen so that $f-p$ equioscillates on the point set consisting of the $n+2$ extrema of T_{n+1}. This is another case of near minimax approximation and is recommended as a means of starting the Remez exchange algorithm for finding the minimax approximation.

A more important type of near minimax approximation is obtained by truncating the Chebyshev series for f after $n+1$ terms. If

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(t)\, T_k(t)}{\sqrt{1-t^2}}\, dt \qquad k = 0, 1, \ldots, \tag{1.2}$$

the truncated series is

$$s_n(x) = \sum_{k=0}^{n}{}' a_k T_k(x), \tag{1.3}$$

where $\sum'$ denotes summation with the first term halved. We will prove that (1.1) holds with $p = s_n$.

In Section 3 we give an alternative proof to that in [6] for the case of equioscillation on the extrema of T_{n+1} and in Section 4 we show that (1.1) holds if p is obtained by "economising" the interpolating polynomial of degree $n+1$ constructed on the zeros of T_{n+2}.

If $f^{(n+1)}$ is approximately constant over the interval, the error formula (1.1) suggests that there will not be a substantial difference in the accuracy of these various types of approximation and confirms the near minimax property.

2. Truncated Chebyshev Series

The usual proof that the truncated Chebyshev series (1.3) is a near minimax approximation to $f \in C[-1, 1]$ uses bounds on the Lebesgue constant (operator norm) for s_n. (See, e.g., Elliott [2] and Rivlin [7].) An intuitive argument that s_n is near minimax is based on the observation that if the Chebyshev series converges rapidly then the error

$$f(x) - s_n(x) = \sum_{k=n+1}^{\infty} a_k T_k(x) \tag{2.1}$$

is dominated by $a_{n+1}T_{n+1}(x)$ which equioscillates $n+2$ times on $[-1, 1]$. We will adopt a different approach to the problem and will assume that $f \in C^{(n+1)}[-1, 1]$. We start by deriving an expression for the a_k which shows that if $a_k = 0$ for $k \geqslant n+2$ then (1.1) holds with $p = s_n$.

LEMMA 2.1. *If $f \in C^k[-1, 1]$, the Chebyshev coefficients defined by* (1.2) *satisfy*

$$a_k = \frac{1}{2^{k-1}\sqrt{\pi}\,\Gamma(k+1/2)} \int_{-1}^{1} f^{(k)}(t)(1-t^2)^{k-1/2}\,dt, \tag{2.2}$$

$$a_{k+1} = \frac{1}{2^{k-1}\sqrt{\pi}\,\Gamma(k+1/2)} \int_{-1}^{1} f^{(k)}(t)\,t(1-t^2)^{k-1/2}\,dt. \tag{2.3}$$

Proof. Use Rodrigue's formula

$$T_k(t) = \frac{(-1)^k\sqrt{\pi}(1-t^2)^{1/2}}{2^k\Gamma(k+1/2)}\frac{d^k}{dt^k}(1-t^2)^{k-1/2}$$

and partial integration noting that for $1 \leqslant j \leqslant k-1$,

$$\left[\frac{d^{k-j}}{dt^{k-j}}(1-t^2)^{k-1/2}\right]_{t=\pm 1} = 0.$$

COROLLARY. *If $f \in C^k[-1, 1]$ then*

$$a_k = \frac{f^{(k)}(\xi)}{2^{k-1}k!}, \tag{2.4}$$

where $\xi \in (-1, 1)$.

Proof. Since

$$\int_{-1}^{1} (1-t^2)^{k-1/2}\,dt = \frac{\sqrt{\pi}\,\Gamma(k+1/2)}{k!}, \tag{2.5}$$

(2.4) follows from (2.2) on using the mean value theorem for integrals.

Note that if $f \in C^{(n+1)}[-1, 1]$ then

$$\|a_{n+1}T_{n+1}\| = |a_{n+1}| = \frac{1}{2^n(n+1)!}\,|f^{(n+1)}(\xi)|,$$

where $\xi \in (-1, 1)$.

THEOREM 2.1. *If* $f \in C^{(n+1)}[-1, 1]$ *then*

$$\|f - s_n\| = \frac{1}{2^n(n+1)!} |f^{(n+1)}(\xi)|, \tag{2.6}$$

where $\xi \in (-1, 1)$.

Proof. Substitute (1.2) into (1.3) and interchange the order of the summation and integration to give

$$s_n(x) = \frac{2}{\pi} \int_{-1}^{1} (1-t^2)^{-1/2} f(t) \sum_{k=0}^{n}{}' T_k(t)\, T_k(x)\, dt.$$

From the orthogonality property of Chebyshev polynomials it is clear that for the function $f = 1$ we must have $s_n = 1$ and thus

$$\begin{aligned} r_n(x) &:= f(x) - s_n(x) \\ &= \frac{2}{\pi} \int_{-1}^{1} (1-t^2)^{-1/2}(f(x) - f(t)) \sum_{k=0}^{n}{}' T_k(t)\, T_k(x)\, dt. \end{aligned}$$

Using the Christoffel–Darboux formula to replace the sum, we have

$$\begin{aligned} r_n(x) = \frac{1}{\pi} \int_{-1}^{1} (1-t^2)^{-1/2} &\left(\frac{f(t) - f(x)}{t - x}\right) \\ &\times (T_n(t)\, T_{n+1}(x) - T_{n+1}(t)\, T_n(x))\, dt. \end{aligned} \tag{2.7}$$

Observe that

$$\frac{f(t) - f(x)}{t - x} = \int_0^1 f'((t-x)\, u + x)\, du, \tag{2.8}$$

and thus

$$r_n(x) = \tfrac{1}{2} \int_0^1 (\alpha_n(u)\, T_{n+1}(x) - \alpha_{n+1}(u)\, T_n(x))\, du, \tag{2.9}$$

where $\alpha_n(u)$, $\alpha_{n+1}(u)$ are the Chebyshev coefficients for the function $F_u(t) := f'((t-x)\, u + x)$;

$$\alpha_j(u) = \frac{2}{\pi} \int_{-1}^{1} (1-t^2)^{-1/2} f'((t-x)\, u + x)\, T_j(t)\, dt, \qquad j = n, n+1. \tag{2.10}$$

Now $F_u^{(n)}(t) = u^n f^{(n+1)}((t-x)\, u + x)$ and thus by Lemma 2.1,

$$\begin{aligned} r_n(x) = \frac{1}{2^n \sqrt{\pi}\, \Gamma(n+1/2)} &\int_0^1 u^n \int_{-1}^{1} f^{(n+1)}((t-x)\, u + x)(1-t^2)^{n-1/2} \\ &\times (T_{n+1}(x) - t T_n(x))\, dt\, du. \end{aligned}$$

Since u^n is positive in $(0, 1)$ we may apply the mean value theorem to the integral with respect to u so that for some $\mu \in (0, 1)$,

$$r_n(x) = \frac{1}{2^n \sqrt{\pi}(n+1)\,\Gamma(n+1/2)} \int_{-1}^{1} f^{(n+1)}((t-x)\,\mu + x)(1-t^2)^{n-1/2} \times (T_{n+1}(x) - tT_n(x))\,dt$$

and thus

$$|r_n(x)| \leqslant \frac{M_{n+1}}{2^n \sqrt{\pi}(n+1)\,\Gamma(n+1/2)} \int_{-1}^{1} (1-t^2)^{n-1/2} \times |T_{n+1}(x) - tT_n(x)|\,dt, \tag{2.11}$$

where

$$M_{n+1} = \|f^{(n+1)}\| = \max_{-1 \leqslant x \leqslant 1} |f^{(n+1)}(x)|. \tag{2.12}$$

Define the function h_n by

$$h_n(t) := \tfrac{1}{2}(1-t)\,|T_{n+1}(x) + T_n(x)| + \tfrac{1}{2}(1+t)\,|T_{n+1}(x) - T_n(x)|;$$

$h_n(t)$ represents the straight line joining $(-1, |T_{n+1}(x) + T_n(x)|)$ and $(1, |T_{n+1}(x) - T_n(x)|)$ and thus

$$h_n(t) \geqslant |T_{n+1}(x) - tT_n(x)| \qquad \text{for} \quad -1 \leqslant t \leqslant 1.$$

It follows that

$$r_n(x) \leqslant \frac{M_{n+1}}{2^n \sqrt{\pi}(n+1)\,\Gamma(n+1/2)} \int_{-1}^{1} (1-t^2)^{n-1/2}\, h_n(t)\,dt.$$

On using (2.5) with $k = n$ and

$$\int_{-1}^{1} t(1-t^2)^{n-1/2}\,dt = 0,$$

we have

$$|r_n(x)| \leqslant \frac{M_{n+1}}{2^{n+1}(n+1)!}\,(|T_{n+1}(x) + T_n(x)| + |T_{n+1}(x) - T_n(x)|)$$

$$= \frac{M_{n+1}}{2^n(n+1)!} \max\{|T_n(x)|, |T_{n+1}(x)|\}$$

and thus

$$\|r_n\| \leqslant \frac{M_{n+1}}{2^n(n+1)!}. \tag{2.13}$$

Note that we must have strict inequality in (2.11) and (2.13) if the maximum in (2.12) is attained only at $x=1$ and/or $x=-1$.

Since $\|r_n\|$ cannot be less than the error for minimax approximation and this latter error satisfies (1.1), we must have

$$\|r_n\| \geqslant \frac{m_{n+1}}{2^n(n+1)!}, \tag{2.14}$$

where $m_{n+1} = \min_{-1 \leqslant x \leqslant 1} |f^{(n+1)}(x)|$. We must also have strict inequality in (2.14) if the minimum is attained only at $x=1$ and/or $x=-1$.

Combining (2.13) and (2.14), it follows from the continuity of $f^{(n+1)}$ that (2.6) holds.

3. Equioscillation on the Extrema of T_{n+1}

Let $\eta_j = \cos(j\pi/(n+1))$, $j=0,\ldots, n+1$ denote the $n+2$ extrema of T_{n+1} on $[-1, 1]$. It has been shown [6] that if p is chosen so that $f-p$ equioscillates on the point set $H=\{\eta_0,\ldots, \eta_{n+1}\}$ then (1.1) is satisfied. We present an alternative proof based on the method of Section 2. In this case

$$p(x) = \sum_{k=0}^{n}{}' c_k T_k(x), \tag{3.1}$$

where

$$c_k = \frac{2}{n+1} \sum_{j=0}^{n+1}{}'' f(\eta_j)\, T_k(\eta_j). \tag{3.2}$$

($\sum''$ denotes summation with the first and last terms halved.)

We first note that

$$\begin{aligned} \beta_j := \prod_{\substack{i=0 \\ i \neq j}}^{n+1} (\eta_j - \eta_i) &= ((n+1)/2^{(n-1)})(-1)^j \qquad j=0, n+1, \\ &= ((n+1)/2^n)(-1)^j \qquad j=1, 2,\ldots, n. \end{aligned} \tag{3.3}$$

This is proved by defining

$$q(x) := \prod_{i=0}^{n+1} (x-\eta_i) = \frac{(x^2-1)\, U_n(x)}{2^n},$$

where $U_n \in \mathscr{P}_n$ is the Chebyshev polynomial of the second kind and observing that $\beta_j = q'(\eta_j)$, $j=0,\ldots, n+1$.

We now have analogous to Lemma 2.1,

LEMMA 3.1. *c_n, c_{n+1} defined by* (3.2) *satisfy*

$$c_n = \frac{1}{2^n}(f[\eta_0 \cdots \eta_n] + f[\eta_1 \cdots \eta_{n+1}]), \tag{3.4}$$

$$c_{n+1} = \frac{1}{2^{n-1}} f[\eta_0 \cdots \eta_{n+1}] = \frac{1}{2^n}(f[\eta_0 \cdots \eta_n] - f[\eta_1 \cdots \eta_{n+1}]). \tag{3.5}$$

Proof. Use symmetric expansion of the divided differences and (3.3) noting that $\eta_0 = -\eta_{n+1} = 1$.

Note that we also have expressions for c_n and c_{n+1} similar to (2.4).

THEOREM 3.1. *If $f \in C^{(n+1)}[-1, 1]$ and $p \in \mathscr{P}_n$ is chosen so that $f - p$ equioscillates on the point set $H = \{\eta_0 \cdots \eta_{n+1}\}$ then* (1.1) *holds.*

Proof. (Analogous to that of Theorem 2.1.) Substitute (3.2) in (3.1) and interchange the order of the summations to obtain

$$p(x) = \frac{2}{n+1} \sum_{j=0}^{n+1}{}'' f(\eta_j) \sum_{k=0}^{n}{}' T_k(\eta_j)\, T_k(x). \tag{3.6}$$

Again for the function $f = 1$ we must have $p = 1$. On using the Christoffel–Darboux formula and (2.8) we have

$$f(x) - p(x) = \tfrac{1}{2} \int_0^1 (\gamma_n(u)\, T_{n+1}(x) - \gamma_{n+1}(u)\, T_n(x))\, du,$$

where

$$\gamma_k(u) = \frac{2}{n+1} \sum_{j=0}^{n+1}{}'' f'((\eta_j - x)\, u + x)\, T_k(\eta_j), \qquad k = n, n+1.$$

On applying Lemma 3.1 with f replaced by $F_u(t) = f'((t-x)\, u + x)$ we deduce that

$$f(x) - p(x) = \frac{T_{n+1}(x) + T_n(x)}{2^{n+1}} \int_0^1 F_u[\eta_1 \cdots \eta_{n+1}]\, du$$
$$+ \frac{T_{n+1}(x) - T_n(x)}{2^{n+1}} \int_0^1 F_u[\eta_0 \cdots \eta_n]\, du.$$

Now

$$F_u[\eta_1 \cdots \eta_{n+1}] = \frac{1}{n!} F_u^{(n)}(\zeta) = \frac{u^n}{n!} f^{(n+1)}((\zeta - x)\, u + x),$$

where $\zeta \in (-1, 1)$, with a similar result for $F_u[\eta_0 \cdots \eta_n]$. Hence

$$|f(x) - p(x)| \leqslant \frac{M_{n+1}}{2^{n+1}(n+1)!} (|T_{n+1}(x) + T_n(x)| + |T_{n+1}(x) - T_n(x)|),$$

where $M_{n+1} = \|f^{(n+1)}\|$. The rest of the proof is identical to the latter part of that of Theorem 2.1.

4. Economised Interpolation on the Zeros of T_{n+2}

The method of Sections 2 and 3 can be used to prove that (1.1) holds when p is the interpolating polynomial constructed on the zeros of T_{n+1}. This is of little interest as the usual proof based on the error of interpolation is shorter. However, in [6], it is observed that the polynomial of Section 3 may be obtained by economising the interpolating polynomial (in $\mathscr{P}_{n+1}$) constructed on the $n+2$ extrema of T_{n+1}. We can also prove that (1.1) holds if p is obtained by economising the interpolating polynomial of degree $n+1$ constructed on the $n+2$ zeros of T_{n+2}. In this latter case

$$p(x) = \sum_{k=0}^{n}{}' c_k^* T_k(x),$$

where

$$c_k^* = \frac{2}{n+2} \sum_{j=1}^{n+2} f(x_j)\, T_k(x_j)$$

and $x_j = \cos((2j-1)\pi/(2n+4))$, $j = 1, \ldots, n+2$ are the zeros of T_{n+2}. In place of Lemma (3.1) we have

$$c_n^* = \frac{1}{2^n} (f[x_1 \cdots x_{n+1}] + f[x_2 \cdots x_{n+2}]), \tag{4.1}$$

$$c_{n+1}^* = \frac{1}{2^n} f[x_1 \cdots x_{n+2}] = \frac{1}{2^{n+1} x_1} (f[x_1 \cdots x_{n+1}] - f[x_2 \cdots x_{n+2}]). \tag{4.2}$$

The proof that (1.1) holds is then similar to that of Theorems 2.1 and 3.1. (We also note $2x_1 > 1$.)

All these approximations are special cases of truncating the finite Chebyshev series on point sets consisting of either the zeros or the extrema of T_{n+1} (or T_{n+2}). Work is continuing[1] on extending the proof to

[1] *Note added in proof.* Since submission of this paper a general proof for these cases has been published (H. Brass, Error estimates for least squares approximation by polynomials, *J. Approx. Theory* **41** (1984), 345–349).

polynomials obtained by truncating the finite series after $r+1$ terms for $r \leqslant n-1$. Suitable formulae for c_r to replace (3.4) and (3.5) when $r<n$ (or, for c_r^*, to replace (4.1) and (4.2) when $r<n$) are being sought.

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