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Wavelength routing in optical networks of diameter two

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Abstract

We consider optical networks with routing by wavelength division multiplexing. We show that wavelength switching is unnecessary in routings where communication paths use at most two edges. We then exhibit routings in some explicit pseudo-random graphs, showing that they achieve optimal performance subject to constraints on the number of edges and the maximal degree. We also observe the relative inefficiency of planar networks. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A *routing* R for a graph G is an assignment, to each pair of vertices a, b, of a path $P(a,b)$ between a and b in G. The *switching number* $s(G;R)$ of the routing is the maximum, over all edges e of G, of the number of paths P in R that contain e. The *switching number* $s(G)$ of the graph G is the minimum of $s(G;R)$ over all possible routings R. The *routing number* $r(G;R)$ of the routing is the minimum number of colours needed so that each path P can be assigned a colour, in such a way that whenever two paths have a common colour then they have no edges in common. The *routing number* $r(G)$ of the graph G is the minimum of $r(G;R)$ over all routings R of G. Clearly $s(G;R) \leqslant r(G;R)$ and $s(G) \leqslant r(G)$.

The perspective of this paper is theoretical, but our initial motivation comes from the problems arising in optical networks, where it is necessary to set up a communication path between each pair of vertices. Messages from a to b are transmitted along the path $P(a,b)$. Each edge of the

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network is a fibre optic link that can support several simultaneous communications by means of *wavelength division multiplexing*; that is, each of the paths $P(a,b)$ using the edge is assigned a different wavelength of light, and the messages are transmitted along the edge using these different wavelengths. In a technically simple network, the message travelling along a given path $P(a,b)$ would use the same wavelength in every edge that it travelled through. But with fancier hardware, *wavelength switching* is possible; in this case, the wavelength used for a given message can be changed when the message passes through a vertex and enters a fresh edge. It can readily be seen that, where wavelength switching is available, communication is possible provided each edge can support $s(G)$ wavelengths. However, if wavelength switching is not available, communication is possible only if edges can support $r(G)$ different wavelengths. The parameter $s(G)$ has been well studied. But for optical networks, the parameter $r(G)$ is important because, in practice, wavelength switching is technically difficult and tends to cause severe degradation of performance due to the conversion of the optical signal to an electrical one and back again. More information about constraints and algorithms for efficient routing is given by Raghavan and Upfal [14].

The routings that are in some simple sense (described in Section 2) the most efficient, are those in which $P(a,b)$ is the edge ab if a and b are adjacent, and otherwise $P(a,b)$ is a path of length two. Our principal observation is that, in such routings, *wavelength switching is unnecessary*.

Theorem 1. *Let R be a routing of a graph G, in which every pair of adjacent vertices is joined by a path of length one* (*namely the edge between them*) *and every nonadjacent pair is joined by a path of length two. Then $s(G;R)=r(G;R)$. In particular, if R satisfies $s(G)=s(G;R)$ then $s(G)=r(G)$.*

The proof of this theorem is given in Section 3.

Another instance where wavelength switching is not really required is when the network is just a ring—that is, G is a cycle [15]. But, in general, it is known that $r(G)$ might be bigger than $s(G)$; for example, Jansen [9] has described a network G for which $r(G) \geqslant 5s(G)/3$.

Random graphs perform well as networks achieving the minimum value of $s(G)$ for a given size (number of edges) and maximum degree. Explicit graphs having similar characteristics are called pseudo-random graphs. In Section 4 we give explicit routings for a few such graphs—these graphs are optimal in terms of minimizing $s(G)$ and, in view of Theorem 1, of minimizing $r(G)$ also (as a function of size and maximum degree). By contrast, in Section 5 we remark on the inefficiency of planar networks.

Throughout the paper we shall, unless stated otherwise, include *multigraphs* in the discussion—that is, graphs in which multiple edges are allowed. The term *simple* graph will refer to a graph without multiple edges. Apart from this exception, our terminology is that of Bollobás [2].

2. A simple lower bound

Let the multigraph G have order n and size $e(G)$. We define the edge *density* $p(G)$ of G to be $p(G)=e(G)/\binom{n}{2}$. In any routing R of G, the length of the path $P(a,b)$ is at least $d(a,b)$, the distance in G between the vertices a and b. So the average number of communication paths per edge is at least

$$\mathrm{dist}(G) = \frac{1}{e(G)} \sum_{a,b \in V(G)} d(a,b).$$

Clearly $s(G) \geqslant \mathrm{dist}(G)$. Now $d(a,b) \geqslant 2$ unless ab is an edge of G. Therefore,

$$\mathrm{dist}(G) \geqslant \frac{1}{e(G)} \left\{ 2 \left[\binom{n}{2} - e(G) \right] + e(G) \right\} = \frac{2}{p(G)} - 1.$$

Thus, we obtain what we call the *distance* bound, namely

$$r(G) \geqslant s(G) \geqslant \lceil \mathrm{dist}(G) \rceil \geqslant \left\lceil \frac{2}{p} \right\rceil - 1 \quad \text{where } p = p(G). \tag{†}$$

For a network to come close to achieving $s(G) = 2/p - 1$, it must have a routing in which most paths have length at most two. The problem of finding good sparse networks is closely related to the problem of finding graphs of diameter two having minimal size subject to a bound on the maximum degree. This is a classical problem of extremal graph theory. We shall say more in Section 4.

However, there is a further significance to routings in which paths have length at most two.

3. Paths of length two

It is convenient here to define a *sub-routing* S of G to be a collection of paths $P(a,b)$ between some, though not necessarily all, pairs of vertices a, b of G.

Theorem 2. *Let G be a graph and let k be an integer. Let S be a sub-routing of G in which all paths have length two and such that no edge of G appears in more than k paths of S. Then each path of S can be assigned one of $k+1$ colours so that no two paths sharing a common edge receive the same colour.*

Proof. We construct an ancillary graph H. The vertices of H are the edges of G. Two vertices of H are joined if the two edges of G form a path of S. Then the maximum degree of H is at most k. Moreover, H will be a simple graph even if G itself is not. Therefore, by Vizing's theorem [17,4], the edges of H can be coloured with $k+1$ colours so that no two incident edges receive like colours. Since each edge of H corresponds to a path of S, the colouring of H gives the required colouring of S. □

Proof of Theorem 1. Let S be the sub-routing of R consisting of all the paths of length two. Since each edge of G is a path of R, Theorem 2 applies with $k = s(G;R) - 1$. So colour the paths of S with $s(G;R)$ colours. Since the paths of length two containing a given edge use at most $s(G;R) - 1$ colours, there is a spare colour to colour the path of length one in R which consists of this edge alone. Thus all paths of R have been coloured using only $s(G;R)$ colours, and so $r(G;R) \leqslant s(G;R)$. □

We note that Vizing's theorem, and therefore Theorem 1, is effective insofar as there is a polynomial time algorithm for finding the colouring; hence it is easy to remove wavelength switching from a routing using paths of length two.

In the examples in the next section we shall construct routings R satisfying the requirements of Theorem 1 and having $s(G;R) = \lceil 2/p \rceil - 1$. It follows from (†) that these examples are optimal for their size: they satisfy $r(G) = \lceil 2/p \rceil - 1$.

4. Pseudo-random constructions of efficient networks

For the distance bound (†) to be achieved, the network must have diameter two. For edge densities p greater than $1/\sqrt{n}$ there are (more or less) regular graphs that will work. In particular, Wischik [19] has proved a result for dense graphs, namely, that the routing number $r(G)$ of a sufficiently large random graph G, whose edges are chosen independently with constant probability q, almost certainly satisfies $r(G) = \lceil 2/p \rceil - 1 = \lceil 2/q \rceil - 1$, provided $2/q$ is not an integer (this is a necessary condition).

To find explicit examples of dense efficient networks we turn to the theory of pseudo-random graphs, as enunciated in [16] and also in [5]. In particular, the Paley graphs are standard examples of pseudo-random graphs with density $\frac{1}{2}$. In Section 4.1, we show that Paley graphs are optimal optical networks.

There is a considerable classical literature devoted to the subject of graphs of diameter two, small size and low maximum degree, beginning with the papers of Erdős and Rényi [7] and Erdős, Rényi and Sós [8]—see Bollobás [2, p.176]. From a technical point of view it might be desirable to keep the maximum degree as small as possible, so that the traffic through a single vertex does not become prohibitively large. (Of course, if this is not a constraint, then the most efficient network is a star.)

The minimum maximal degree of a graph of diameter two is around $\sqrt{n}$. The Erdős–Rényi graphs are the classical extremal examples. In Section 4.2, we show that these graphs too are optimal optical networks.

When the edge density p drops below $1/\sqrt{n}$ it is necessary that the maximum degree increase in order that the graph still have diameter two. Good examples are given by Bollobás [1]. In Section 4.3, we show that these graphs (and, even more so, slight modifications of them) are good optical networks.

Examples of very sparse pseudo-random graphs are given by several authors, such as Margulis [13], Lubotzky, Philips and Sarnak [12] and Lazebnik, Ustimenko and Woldar [10]. These graphs do not have diameter two but they do have small size and small maximal degree—subject to these constraints, they are probably also optimal networks.

4.1. The Paley graph

Let q be a prime power and let $\mathrm{GF}(q)$ denote the field of order q. The *Paley graph* P_q has vertex set $\mathrm{GF}(q)$ where $q \equiv 1 \pmod 4$, two vertices being joined if their difference is a square. The Paley graph is a pseudo-random graph with edge density $1/2$. For the present purpose we show that P_q achieves exactly the distance bound (†).

Theorem 3. *The Paley graph* $G = P_q$ *is a regular graph with edge density* $p(G) = 1/2$. *It satisfies*

$$r(G) = s(G) = 3 = \frac{2}{p} - 1.$$

The proof of this theorem is given in Section 6.1. Only the existence of a routing is proved. However, in half the cases we can give a simple explicit routing for P_q; details are in Section 6.2.

Theorem 4. *If* $q \equiv 1 \pmod 8$ *there is an explicit routing for the Paley graph* P_q *which realises* $r(G) = 3$.

There are variants on the Paley graph with densities less than $\frac{1}{2}$. If q is a prime power with $q \equiv 1 \pmod{2m}$, the graph P_q^m with vertex set $\mathrm{GF}(q)$, in which a is joined to b if $a - b$ is an mth power, is a regular graph of density $1/m$. (Thus $P_q^2 = P_q$.) In many cases, a theorem similar to Theorem 4 can be proved, exhibiting a routing in which each edge lies in $2(m-1)$ paths of length two and so the bound (†) is achieved. Further information is given in Section 6.

4.2. The Erdős–Rényi graph

The *Erdős–Rényi graph* ER_q (first described in [7]) has order $n = q^2 + q + 1$, where q is an odd prime power. Its vertices are the points of the projective geometry $\mathrm{PG}(2, q)$ of dimension two over $\mathrm{GF}(q)$. Two vertices (a, b, c) and (α, β, γ) of ER_q are adjacent if $a\alpha + b\beta + c\gamma = 0$. All the vertices have degree $q+1$ except for the $q+1$ vertices (a, b, c) for which $a^2 + b^2 + c^2 = 0$, which have degree q; so $e(\mathrm{ER}_q) = q(q+1)^2/2$. In particular, the density of ER_q is $(q+1)/n \approx 1/\sqrt{n}$.

A standard property of ER_q is that every pair of vertices $\{a, b\}$ is joined by exactly one path of length two, with the exception of those $(q+1)q$ pairs where ab is an edge and a has degree q, between which there is no path of length two. It follows that there is a unique routing R using the edges and paths of length two. In R each edge lies in $2q - 1$ paths except those edges incident with vertices of degree q, which lie in $2q$ paths. So we obtain $r(\mathrm{ER}_q) = s(\mathrm{ER}_q) = 2q$. Now for a graph of this density we know by (†) that $s(G) \geqslant \lceil 2/p \rceil - 1 = \lceil 2n/(q+1) \rceil - 1 = 2q$. Therefore the Erdős–Rényi graph is optimal for its density.

4.3. Sparser constructions

In order to reduce the density of our constructions below that of the Erdős–Rényi graph we must allow the maximum degree to rise. An efficient construction was exhibited by Bollobás [1] (see [2, p.178]). The graph ER_q can be regarded as a pseudo-random subgraph of K_t, where $t = |\mathrm{ER}_q|$, and Bollobás's construction is in turn essentially a pseudo-random subgraph of a graph formed by joining extra vertices to every vertex of K_t.

Let $m \geqslant 0$ be an integer. We define the graph $\mathrm{ER}_q(m)$ as follows. Begin with a copy of ER_q. For each vertex $v \in \mathrm{ER}_q$ take a set $\{v_1, \ldots, v_m\}$ of new vertices. These, together with $V(\mathrm{ER}_q)$, will form the vertex set of $\mathrm{ER}_q(m)$; thus $|\mathrm{ER}_q(m)| = (m+1)|\mathrm{ER}_q| = (m+1)(q^2+q+1)$. Join a new vertex v_i to u if vu is an edge of ER_q. Moreover, if v is a vertex of degree q in ER_q join v_i to v; therefore every new vertex has degree $q+1$. Hence $e(\mathrm{ER}_q(m)) = e(\mathrm{ER}_q) + m|\mathrm{ER}_q|(q+1)$, and $\mathrm{ER}_q(0)$ is just ER_q. Clearly $\Delta(\mathrm{ER}_q(m)) = (m+1)(q+1)$.

By our earlier remarks about ER_q it can be seen that every pair of vertices has a common neighbour if $m \geqslant 1$; in particular. $\mathrm{ER}_q(m)$ has diameter two. Therefore, there is a routing in $\mathrm{ER}_q(m)$ in which no path has length greater than two. Given two vertices not both in the original ER_q and not both in the same set $\{v_1, \ldots, v_m\}$ there is a unique shortest path (of length one or two) joining them; we call these the "unique" paths. Two vertices in the original ER_q will be connected by m or $m+1$ two paths; two vertices in a set $\{v_1, \ldots, v_m\}$ will be connected by $q+1$ two paths. Now in any routing by paths of length at most two, an edge not in ER_q will lie in at most $1 + q(m+1)$ unique paths. However, the edges in ER_q may lie in up to $2mq$ unique paths. Having inserted the unique paths, we must now link up pairs of vertices in the sets $\{v_1, \ldots, v_m\}$ and pairs of vertices within ER_q. Pairs of vertices in the sets $\{v_1, \ldots, v_m\}$ can be linked by paths of length two which are spread around evenly so that no edge lies in more than $\lceil (m+1)/(q+1) \rceil$ paths. Non-adjacent pairs of vertices within ER_q can be linked by paths of length two which avoid the edges of ER_q and such that each edge lies in no more than $\lceil (q+1)/m \rceil$ paths. The same applies to adjacent pairs of vertices within ER_q, but it seems more natural to let them communicate via the edge joining them. This gives us a routing with the property that many edges lie in at most $1 + q(m+1) + \lceil (m+1)/(q+1) \rceil + \lceil (q+1)/m \rceil$ paths but some edges lie in up to $2mq + 1$ paths. We could reduce this last expression to $2mq$ if we did not require each adjacent pair of vertices to communicate via the edge joining them.

However, we can improve the situation dramatically by adding an extra edge in parallel with each edge of ER_q; call the resultant graph $\mathrm{ER}_q^+(m)$. So $e(\mathrm{ER}_q^+(m)) = e(\mathrm{ER}_q) + e(\mathrm{ER}_q(m)) = (q+1)(|\mathrm{ER}_q^+(m)| - 1)$. The density of $\mathrm{ER}_q^+(m)$ is therefore $p(\mathrm{ER}_q^+(m)) = 2(q+1)/n$ where $n = |\mathrm{ER}_q^+(m)|$, and the distance bound (†) gives $r(G) \geqslant \lceil 2/p \rceil - 1 = (m+1)q - 1 + \lceil (m+1)/(q+1) \rceil$.

Now a routing for $\mathrm{ER}_q^+(m)$ can be constructed in the same way as for $\mathrm{ER}_q(m)$, except that the paths through the edges of ER_q can now be split between the double edges, and the paths joining vertices of ER_q can now be routed via the edges of ER_q. This routing uses no edge more than $1 + q(m+1) + \lceil (m+1)/(q+1) \rceil$ times, which is only two more than our lower bound.

4.4. A numerical example

As an illustration of the various constructions given above, Table 1 gives the parameters of some diameter two optical networks with around 180 vertices, varying from the sparse to the dense. The parameter $\Delta_{\min}$ is the theoretically smallest possible maximal degree in a diameter two graph with the given order and size. The column labelled $\lceil 2/p \rceil - 1$ is, of course, the distance bound (†).

It will be seen that these graphs offer excellent (at times unbeatable) performance for their respective densities. By comparison, an 8-regular graph G with 180 vertices would have 720 edges, about the same as $\mathrm{ER}_3^+(13)$. Each vertex could reach at most 49 other vertices by paths of length two, so $s(G) \geqslant \mathrm{dist}(G) \geqslant 60$. Hence a 21% reduction (at least) in traffic density can be achieved by using $\mathrm{ER}_3^+(13)$ instead of a regular graph of similar size.

5. Planar networks

As well as the distance bound, another straightforward bound on the routing number is the *bottleneck* bound. Let U be a subset of the vertex set $V = V(G)$. In any routing of G there must be at least $|U||V - U|$ communication paths which cross from U to $V - U$. Let $e(U, V - U)$ be the

Table 1
Some optical networks and their efficiency

G	$\|G\|$	$e(G)$	$\Delta_{\min}$	$\Delta(G)$	$\lceil 2/p \rceil - 1$	$r(G)$
$ER_3(13)$	182	700	47	56	47	79
$ER_3^+(13)$	182	724	42	60	45	47
$ER_5(5)$	186	1020	27	36	33	51
$ER_5^+(5)$	186	1110	23	42	30	32
ER_{13}	183	1274	14	14	26	26
$ER_7(2)$	171	1136	14	24	25	29
$ER_7^+(2)$	171	1360	16	32	21	23
P_{181}^6	181	2715	30	30	11	11
P_{181}^5	181	3258	36	36	9	9
P_{181}^3	181	5430	60	60	5	5
P_{181}	181	8145	90	90	3	3

number of edges between U and $V-U$. Then some edge is in at least $|U||V-U|/e(U,V-U)$ paths and this number is thus a lower bound for $s(G)$. We define the *bottleneck number* $b(G)$ of the graph G by

$$b(G) = \max_{U \subset V(G)} \frac{|U||V-U|}{e(U,V-U)}.$$

In a graph of density p there must be a vertex u of degree at most $p(n-1)$. Taking $U = \{u\}$ we then have

$$r(G) \geqslant s(G) \geqslant b(G) \geqslant \frac{1}{p}.$$

The networks described in Section 4 achieved bottleneck numbers of roughly $1/p$. Since $1/p \leqslant 2/p - 1$ it follows that in efficient networks the bottleneck bound is never the deciding factor on performance; the distance bound (†) is more restrictive.

However, it has often been noticed (for example, [19]) that for many existing networks $d(G)$ is much smaller than $b(G)$ and $b(G) = s(G)$ holds. We interpret this to mean that existing networks are inefficient, having some small localized area which creates a high bottleneck and dominates the performance of the whole network.

Existing networks tend to be close to planar. The star $K_{1,n-1}$ is planar and very efficient but has a vertex of high degree. If vertices of high degree are not permitted then planar graphs become very inefficient. A *k-edge separator* of a planar graph G of order n is defined to be a set of k edges whose removal disconnects G and leaves no component of order larger than $2n/3$. This means that after the removal of these edges, there is a partition of $V(G)$ into two disjoint sets A, B such that $n/3 \leqslant |A|, |B| \leqslant 2n/3$. Consequently, if G has a k-edge separator then $b(G) \geqslant 2n^2/9k$. Diks, Djidjev, Sykora and Vrto [6] showed that all simple planar graphs have small edge separators.

Proposition 5 (Diks, Djidjev, Sykora and Vrto [6]). *Let G be a simple planar graph of order n and maximum degree Δ. Then G has a $2\sqrt{2\Delta n}$-edge separator.*

Corollary 6. *Let G be a simple planar graph of order n and maximum degree Δ. Then*

$$r(G) \geqslant s(G) \geqslant b(G) \geqslant \frac{1}{9}\sqrt{\frac{n^3}{2\Delta}}.$$

We take Corollary 6 to mean that planar networks, and probably also networks that are close to planar, behave poorly. For example, a simple planar graph G with maximum degree $\Delta(G) \leqslant 8$ will have $r(G) \geqslant n^{3/2}/36$. By comparison, a random 6-regular graph will have good bottleneck properties (see [3]) and $r(G) = \mathrm{O}(n \log n)$ can be obtained by the algorithm in [14].

6. Proofs of Theorems 3 and 4

We give here those proofs that were deferred during the main discussion.

6.1. Proof of Theorem 3

For an excellent discussion of finite fields see Lidl and Niederreiter [11]. The multiplicative character χ on $\mathrm{GF}(q)$ is defined by $\chi(0) = 0$, $\chi(a) = 1$ if a is a non-zero square in $\mathrm{GF}(q)$ and $\chi(a) = -1$ otherwise. If $q \equiv 1 \pmod 4$ then -1 is a square in $\mathrm{GF}(q)$; so the Paley graph is defined by making ab an edge whenever $\chi(a-b) = 1$. Let $a \in \mathrm{GF}(q) = V(G)$ and let d be the degree of the vertex a. Then $d - (q-1-d) = \sum_b \chi(a-b) = \sum_x \chi(x) = 0$, where the sums run over all the elements in $\mathrm{GF}(q)$. Therefore $d = (q-1)/2$ which means that $p(G) = \frac{1}{2}$.

Let g be a primitive root of $\mathrm{GF}(q)$. We define the antilogarithm function l on the group of non-zero elements by $l(g^t) = t$, where we take $1 \leqslant t \leqslant q-1$. Note that $l(-1) = (q-1)/2$, so, for each a, exactly one of $l(a) \leqslant (q-1)/2$ and $l(-a) \leqslant (q-1)/2$ holds.

We shall need a non-square $n \in \mathrm{GF}(q)$ such that $n-1$ is also a nonsquare. (Such an n can always be found for let c be the number of common neighbours of the vertices 0 and 1. Since these vertices are adjacent and both have degree $(q-1)/2$, they have $c+1 > 0$ common nonneighbours. Take n to be any common nonneighbour.) Now, for each ordered pair (a,b) of vertices we define the vertex $v(a,b) = a + (b-a)n = b + (b-a)(n-1)$. Notice that if ab is not an edge and $v = v(a,b)$ then $\chi(v-a) = \chi(b-a)\,\chi(n) = 1$ and $\chi(v-b) = \chi(b-a)\,\chi(n-1) = 1$, so both av and bv are edges.

We now define a routing R in G as follows. If ab is an edge the communication path $P(a,b)$ is just the edge ab. If ab is not an edge and $l(a-b) \leqslant (q-1)/2$ then $P(a,b)$ is the path $av(a,b)b$ of length two. Since exactly one of $l(a-b) \leqslant (q-1)/2$ and $l(b-a) \leqslant (q-1)/2$ holds, the routing is well defined.

Let us now check that $s(G) = 3$. We know by the distance bound (†) that $s(G) \geqslant 3$ so we need only check that each edge cd of G lies in at most two paths of R of length two (the path of length one consisting of cd itself being also in R). Suppose $d = v(c,x)$ for some vertex x. Then $d = c + (x-c)n$ so $x = c + (d-c)/n$. Likewise if $c = v(d,y)$ for some y then $y = d + (c-d)/n$. Since $c - x = -(d-y)$ exactly one of $l(c-x) \leqslant (q-1)/2$ and $l(d-y) \leqslant (q-1)/2$ holds, so R contains exactly one of the paths cdx and dcy. Likewise there is exactly one path in R for which either $d = v(x',c)$ or $c = v(y',d)$ for some vertices x' and y', the place of n in the argument being taken by $1-n$.

The routing R therefore contains exactly three paths through each edge. Since none of these paths has length greater than two, Theorem 1 implies there is a wavelength assignment via R using only three wavelengths and avoiding wavelength switching. □

6.2. Proof of Theorem 4

We adopt the notation of the previous proof and make the extra constraint on the non-square n that $l(n) \equiv -l(n-1) \pmod 4$. Such an n can always be found. For let k be the number of vertices joined to 0 but not to g^2. Since the vertices all have the same degree there are also exactly k vertices joined to g^2 but not to 0; thus $\sum_x \chi(x)\chi(g^2 - x) = -2k + (q-2-2k) = q-2-4k$. But $\sum_x \chi(x)\chi(a-x) = -1$ for any nonzero a (see [11]) so $k = (q-1)/4$. Now $\mathrm{GF}(q)$ contains only $(q-1)/4$ elements v with $l(v) \equiv 2$ and one of these is g^2; thus there must be some vertex z with $l(z) \not\equiv 2$ which is joined to 0 but not to g^2. Hence $l(z) \equiv 0$ and $g^2 - z = w$ where w is a nonsquare. Take $n = g^2 w^{-1}$ and $n - 1 = zw^{-1}$; then $l(n) \equiv -l(n-1)$, as desired.

We now colour the paths of R as follows. All edges are coloured red. The path $av(a,b)b$ with $l(a-b) \leqslant (q-1)/2$ is coloured blue if $l(a-b) \equiv 1$ and green if $l(a-b) \equiv -1$ (remember that $l(a-b)$ is odd or else ab would be an edge).

Consider the paths of length two containing the edge cd. These paths are exactly one of cdx and dcy where $c - x = -(d-y) = (c-d)/n$ and one of cdx' and dcy' where $c - x' = -(d-y') = (c-d)/(1-n)$. Now $l(n) \equiv -l(n-1)$, and $l(-1) \equiv 0$ (because $q \equiv 1 \pmod 8$), so one of these paths will be blue and the other green, completing the proof. □

As for the graphs P_q^m, we remark that if numbers n_i, $1 \leqslant i < m$ can be found such that $l(n_i) \equiv l(n_i - 1) \equiv i \pmod m$ then P_q^m has diameter two and a routing in P_q^m analogous to that for P_q can be constructed in a manner similar to that above. It is straightforward to prove the existence of the numbers n_i for large q by making use of Weil's estimates [18] for the number of solutions of certain equations over $\mathrm{GF}(q)$, but for a given value of q it is much quicker to check the existence by explicit calculation.

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