The joint distribution of consecutive patterns and desents in permutations avoiding 3-1-2

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Abstract

We exploit Krattenthaler’s bijection between the set $S_n(3-1-2)$ of permutations in $S_n$ avoiding the classical pattern 3-1-2 and Dyck $n$-paths to study the joint distribution over the set $S_n(3-1-2)$ of a given consecutive pattern of length 3 and of descents. We utilize an involution on Dyck paths due to E. Deutsch to show that these consecutive patterns split into 3 equidistribution classes. In addition, we state equidistribution theorems concerning quadruplets of statistics relative to occurrences of consecutive patterns of length 3 and of descents in a permutation.

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1. Introduction

Let $\sigma \in S_n$ and $\tau \in S_k$, $k \leq n$, be two permutations. We say that $\sigma$ contains the pattern $\tau$ if $\sigma$ contains a subsequence order isomorphic to $\tau$. We say that $\sigma$ avoids $\tau$ if such a subsequence does not exist. The subject of pattern avoiding permutations was initiated by Knuth [15], while the first systematic study of this theme is attributed to Simion and Schmidt [23]. After that, a large literature on this topic has blossomed. Problems treated include counting permutations avoiding a pattern or a set of patterns, or containing patterns a specified number of times.

More recently, Babson and Steingrímsson [2] introduced generalized permutation patterns where two adjacent letters in a pattern may be required to be adjacent in the permutation. A number of interesting results on generalized patterns were obtained by several authors in recent years (for an extensive survey, see [25]).

One particular case of generalized patterns are consecutive patterns. For a subsequence of a permutation to be an occurrence of a consecutive pattern, its elements have to appear in adjacent positions of the permutation. We write a “classical” pattern with dashes between symbols, while...
a consecutive pattern will be written without dashes, accordingly with the most common notation (see [25]). A number of results for the enumeration of permutations and words by consecutive patterns have recently been obtained (see, e.g., [1,9,10,12–14,20,21,27,28]).

Many well-known integer sequences arise in enumerative problems concerning permutations avoiding a pattern \( \tau \) or containing it a fixed number of times. In particular, in [15] Knuth gives a bijection between the set of Dyck paths and the set of permutations avoiding 3–1–2, while in [16] the same author describes a bijection between Dyck paths and permutations avoiding 3–2–1 (see [6] for a detailed description of both bijections). This allows to show that the number of permutations avoiding any classical pattern \( \tau \in S_3 \) equals the \( n \)th Catalan number \( C_n \). Afterward, Krattenthaler [17] described a further bijection between Dyck paths of semilength \( n \) and the set \( S_n(3\cdot1\cdot2) \) of permutations avoiding 3–1–2.

More recently, some authors studied the distribution of the descent statistic over some sets of pattern avoiding permutations (see, e.g., [3,4,26,29]).

In this paper, we merge these two points of view, namely, we study the joint distribution of the five non-trivial consecutive patterns of length 3 and descents over the set of permutations avoiding 3–1–2.

More precisely, for every consecutive pattern \( \tau \) of length 3, \( \tau \neq 312 \), we study the trivariate generating function

\[
A^\prime(x, y, z) = \sum_{n,k,j=0} a^\tau_{n,k,j} x^ny^kz^j,
\]

where \( a^\tau_{n,k,j} \) is the number of permutations in \( S_n(3\cdot1\cdot2) \) containing \( k \) occurrences of the consecutive pattern \( \tau \) and \( j \) descents.

In all cases, the descents of a permutation \( \sigma \) in \( S_n(3\cdot1\cdot2) \) correspond to the occurrences of \( DD \) in the Dyck path \( K(\sigma) \) associated to \( \sigma \) by Krattenthaler’s bijection. In addition, we prove that each occurrence of a consecutive pattern \( \tau \) in \( \sigma \) corresponds bijectively to a peculiar configuration in \( K(\sigma) \). This correspondence allows us to study this problem in terms of Dyck path configurations. We use an involution \( \Delta \) on Dyck paths due to Deutsch [7], that induces an involution \( \hat{\Delta} \) on the set \( S_n(3\cdot1\cdot2) \), and we show that a permutation \( \sigma \) contains \( k \) occurrences of a consecutive pattern \( \tau \) and \( j \) descents if and only if \( \hat{\Delta}(\sigma) \) contains \( k \) occurrences of the pattern \( \hat{\Delta}(\tau) \) and \( n – 1 – j \) descents. This implies that the five non-trivial patterns of length 3 split into 3 classes (i.e., \( \{213\}, \{123, 321\}, \{132, 231\} \)), so that two patterns in the same class are equidistributed over \( S_n(3\cdot1\cdot2) \). These considerations allow us to reduce the study to 3 cases. In each one of these cases, we exploit the described correspondence between consecutive patterns and Dyck path configurations in order to get a functional equation satisfied by the trivariate generating function \( A^\prime(x, y, z) \). These arguments provide some equidistribution results concerning consecutive patterns and descents of a permutation in \( S_n(3\cdot1\cdot2) \).

As a fallout, we get a description of the descent distribution over the set of permutations in \( S_n(3\cdot1\cdot2) \) that avoid a given consecutive pattern \( \tau \in S_3 \).

2. Preliminaries

2.1. Lattice paths

A Dyck path of semilength \( n \) (or Dyck \( n \)-path) is a lattice path starting at \((0, 0)\), ending at \((2n, 0)\), and never going below the \( x \)-axis, consisting of up steps \( U = (1, 1) \) and down steps \( D = (1, -1) \). A Motzkin path of length \( n \) (or Motzkin \( n \)-path) is a lattice path starting at \((0, 0)\), ending at \((n, 0)\), and never going below the \( x \)-axis, consisting of up steps \( U = (1, 1) \), horizontal steps \( H = (1, 0) \), and down steps \( D = (1, -1) \).

Dyck paths of semilength \( n \) are counted by the \( n \)th Catalan number \( C_n \), while Motzkin paths of length \( n \) are counted by the \( n \)th Motzkin number \( M_n \).

A Dyck path can be regarded as a word over the alphabet \( \{U, D\} \) such that any prefix contains at least as many symbols \( U \) as symbols \( D \). A subword of a Dyck path \( P \) is a subsequence of consecutive steps in \( P \).
An irreducible Dyck path is a Dyck path that does not touch the x-axis except for the origin and the final destination. An irreducible component of a Dyck path $P$ is a maximal irreducible Dyck subpath of $P$.

We list some notions on Dyck paths that will be used in the following. A run (respectively fall) of a Dyck path is a maximal subword consisting of up (resp. down) steps. A return of a Dyck path is a down step landing on the x-axis.

We now describe an involution $\Delta$ on Dyck paths due to Deutsch [7]. Consider a Dyck path $P$ and decompose it according to its first return as $P = U ADB$, where $A$ and $B$ are (possibly empty) Dyck paths. Then, the Dyck path $\Delta(P)$ is recursively determined by the following rules (see Fig. 1):

- if $P$ is empty, so is $\Delta(P)$;
- otherwise, $\Delta(P) = U \Delta(B) D \Delta(A)$.

It is easily checked that the two paths $P$ and $\Delta(P)$ have the same length. In Fig. 2 we show how the involution $\Delta$ acts on Dyck paths of semilength 3.

2.2. Restricted permutations

Let $\sigma \in S_n$ and $\pi \in S_k$, $k \leq n$, be two permutations. The permutation $\sigma$ contains the pattern $\pi$ if there exists a subsequence $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ that is order isomorphic to $\pi$. Moreover, if $i_1, i_2, \ldots, i_k$ are consecutive integers, i.e. $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)$ is a subword of $\sigma$, then we say that $\sigma$ contains the consecutive pattern $\pi$. In order to avoid confusion, we write a “classical” pattern with dashes between symbols, while a consecutive pattern will be written without dashes.

For example, the permutation $\sigma = 4317256$ contains 5 occurrences of the pattern 3-1-2 (namely, 412, 312, 725, 726, 756), but only one occurrence of the consecutive pattern 312 (namely, 725).

The permutation $\sigma$ avoids the (consecutive) pattern $\pi$ if $\sigma$ does not contain $\pi$. For example, the permutation $\sigma = 52134$ avoids the consecutive pattern 312, but it does not avoid the pattern 3-1-2, since it contains, for instance, the subsequence 513.

We denote by $S_n(\tau_1, \ldots, \tau_k)$ the set of permutations in $S_n$ that avoid simultaneously the patterns $\tau_1, \ldots, \tau_k$. 
We say that two permutation statistics $f$ and $g$ are equidistributed on a set $A \subseteq S_n$, if
\[ \sum_{\sigma \in A} x^{f(\sigma)} = \sum_{\sigma \in A} x^{g(\sigma)}. \]

In the present paper we are interested in studying the distribution of an arbitrary consecutive pattern of length 3 over the set $S_n(3\text{-}1\text{-}2)$. It is well known that the permutation $\sigma$ avoids 3\text{-}1\text{-}2 if and only if it can be written as follows:
\[ \sigma = m_1 \, w_1 \, m_2 \, w_2 \, \ldots \, m_r \, w_r, \]
where
- the integers $m_i$ are the left-to-right maxima of $\sigma$ (where a left-to-right maximum of a permutation $\sigma$ is an integer $\sigma(i)$ such that $\sigma(i) > \sigma(j)$ for every $j < i$);
- for every symbol $a$ appearing in one of the words $w_i$, consider the suffix $\sigma_a$ of $\sigma$ starting with $a$. Then, $a$ is the greatest symbol in $\sigma_a$ among those that are less than $m_i$.

Some results in this direction can be found in [18], where the author exhibits the generating functions for the number of permutations on $n$ letters avoiding 1\text{-}3\text{-}2 (or containing 1\text{-}3\text{-}2 exactly once) and an arbitrary generalized pattern $\tau$ on $k$ letters, or containing $\tau$ exactly once.

We describe the bijection $K$ between permutations avoiding 3\text{-}1\text{-}2 and Dyck paths due to Krattenthaler [17].

Given a permutation $\sigma \in S_n(3\text{-}1\text{-}2)$, $\sigma = m_1 \, w_1 \, m_2 \, w_2 \, \ldots \, m_r \, w_r$, the Dyck path $K(\sigma)$ of semilength $n$ is constructed as follows: start with $m_1$ up steps followed by $|w_1| + 1$ down steps. Then add $m_2 - m_1$ up steps followed by $|w_2| + 1$ down steps, and so on.

For example, the 3\text{-}1\text{-}2-avoiding permutation $\sigma = 4 \, 3 \, 6 \, 5 \, 2 \, 7 \, 8 \, 1$ is mapped to the Dyck path in Fig. 3.

Recall that a permutation $\sigma$ has a descent at position $i$ if $\sigma(i) > \sigma(i+1)$. Otherwise, we say that $\sigma$ has an ascent at position $i$. We denote by des$(\sigma)$ the number of descents of the permutation $\sigma$ and by asc$(\sigma)$ the number of ascents of $\sigma$.

It straightforward that the descents of a permutation avoiding 3\text{-}1\text{-}2 are in one-to-one correspondence with the double falls (i.e. occurrences of $DD$) in the associated Dyck path.

3. Consecutive patterns and descents

In this section we describe the trivariate generating function
\[ A^\tau(x, y, z) = \sum_{n,k,j \geq 0} a_{n,k,j}^\tau x^n y^k z^j, \]
where $a_{n,k,j}^\tau$ is the number of permutations in $S_n(3\text{-}1\text{-}2)$ containing $k$ occurrences of a given consecutive pattern $\tau \in S_3$ and $j$ descents. Of course, we do not consider the case $\tau = 312$.

To this aim, we prove that every occurrence of such a pattern in a permutation $\sigma$ corresponds bijectively to a peculiar configuration in the Dyck path $K(\sigma)$. We examine the five non-trivial cases. Recall that a permutation avoiding 3\text{-}1\text{-}2 can be written as
\[ \sigma = m_1 \, w_1 \, m_2 \, w_2 \, \ldots \, m_r \, w_r, \]
where the integers $m_i$ are the left-to-right maxima of $\sigma$ and the (possibly empty) subwords $w_j$ are decreasing.

1. The consecutive pattern $321$ occurs in $\sigma$ if and only if at least one among the subwords $w_j$ has length greater than one. More precisely, if $|w_j| = t > 0$, the subword $m_j \, w_j$ contains $t - 1$ occurrences of $321$, since the elements in $w_j$ appear in decreasing order. Note that a subword $w_j$ of length $t$
corresponds to a fall $F_j$ of length $t + 1$ of the Dyck path $K(\sigma)$. This implies that occurrences of 321 in $m_i w_i$ correspond bijectively to occurrences of DDD in $F_i$.

2. $\sigma \in S_n(3\text{-}1\text{-}2)$ contains an occurrence of the consecutive pattern 123 if and only if it contains two adjacent left-to-right maxima $m_i m_{i+1}$, with $i > 1$. In fact, in this case, $\sigma$ contains the 132-subword $a m_i m_{i+1}$, where $a$ is the symbol preceding $m_i$. This consideration implies that occurrences of 123 in $\sigma$ correspond bijectively to occurrences of DU$^i$DU, with $t > 0$, in the Dyck path $K(\sigma)$.

3. Occurrences of the consecutive pattern 231 in $\sigma$ correspond bijectively to those indices $i$ such that:
   
   (i) $m_{i+1} = m_i + 1$;
   
   (ii) $w_{i+1}$ is nonempty.

   In fact, consider the subword $m_i w_i m_{i+1} w_{i+1}$ and its subword $a m_{i+1} b$, where $a$ is the rightmost symbol in $w_i$ (or $m_i$ if $w_i$ is empty) and $b$ is the leftmost symbol in $w_{i+1}$. Then, $a m_{i+1} b$ is order isomorphic to 231 if and only if $a > b$. This happens whenever the two conditions above hold. Each one of these occurrences corresponds to an occurrence of DUDD in the Dyck path $K(\sigma)$.

4. $\sigma \in S_n(3\text{-}1\text{-}2)$ contains an occurrence of the consecutive pattern 132 if and only if there exists an index $i$, $i < k$, such that the subword $m_i w_i m_{i+1} w_{i+1}$ verify the following conditions:

   (i) $m_{i+1} - m_i > 1$;
   
   (ii) $w_{i+1}$ is nonempty.

   In fact, in this case, $\sigma$ contains the 132-subword $a m_{i+1} b$, where $a$ is the symbol preceding $m_{i+1}$ in $\sigma$ and $b = m_{i+1} - 1$. These remarks imply that occurrences of 132 in $\sigma$ correspond bijectively to occurrences of DUDD, with $t > 1$, in the Dyck path $K(\sigma)$.

5. Occurrences of the consecutive pattern 213 in $\sigma$ correspond bijectively to nonempty subwords $w_i$, $i < k$. In fact, if $|w_i| > 0$, $i < k$, then $\sigma$ contains the 213-subword $b a m_{i+1}$, where $a$ is the last element in $w_i$ and $b$ is either the second last element in $w_i$, or $b = m_i$, when $|w_i| = 1$. It is easily seen that every such subword $w_i$, in turn, corresponds to an occurrence of DDD in the Dyck path $K(\sigma)$.

We now observe that the bijection $\Delta$ on Dyck paths induces an involution $\hat{\Delta} = K^{-1} \circ \Delta \circ K$ on the set $S_n(3\text{-}1\text{-}2)$. In particular, $\hat{\Delta}$ acts on $S_3(3\text{-}1\text{-}2)$ as follows:

$\begin{align*}
321 & \overset{\hat{\Delta}}{\rightarrow} 123 \\
231 & \overset{\hat{\Delta}}{\rightarrow} 132 \\
213 & \overset{\hat{\Delta}}{\rightarrow} 213
\end{align*}$

We prove that the action of $\hat{\Delta}$ on $S_3(3\text{-}1\text{-}2)$ reveals to be paradigmatic for the general case, namely, the involution $\hat{\Delta}$ maps an occurrence of a consecutive pattern $\tau \in S_3$ to an occurrence of the consecutive pattern $\hat{\Delta}(\tau)$.

**Proposition 1.** The two statistics “number of occurrences of DDD” and “number of occurrences of DU$^i$DU”, $t > 0$, are equidistributed on Dyck $n$-paths.

**Proof.** Consider Deutsch’s involution $\Delta$ described in Section 2. Denote by $f(P)$ the number of occurrences of the subword DDD in a given Dyck path $P$ and by $g(P)$ the number of occurrences of DU$^i$DU, $t > 0$, in $P$.

We prove that $g(P) = f(\Delta(P))$ by induction on the semilength $n$ of the Dyck path $P$. The assertion is trivially true for $n = 0$. Fix an integer $n > 0$ and assume that the assertion holds for all Dyck paths of semilength less than $n$. Let $P$ be a Dyck $n$-path, and $P = U A D B$ its first return decomposition. It is easy to verify that:

\[
\begin{align*}
f(P) &= \begin{cases} 
   f(A) + f(B) + 1 & \text{if } A \text{ ends with } DD \\
   f(A) + f(B) & \text{otherwise.}
   \end{cases} \\
g(P) &= \begin{cases} 
   g(A) + g(B) + 1 & \text{if } B \text{ begins with } U^i DU \\
   g(A) + g(B) & \text{otherwise.}
   \end{cases}
\end{align*}
\]

Since the semilengths of the two Dyck paths $A$ and $B$ are strictly less than $n$, by induction hypothesis we have $g(A) = f(\Delta(A))$ and $g(B) = f(\Delta(B))$. It is sufficient to show that the involution $\Delta$ acts as
The two statistics “number of occurrences of DUDD” and “number of occurrences of DUDD”, \( t > 1 \), are equidistributed on Dyck \( n \)-paths.

**Proof.** Denote by \( h(P) \) the number of occurrences of the subword DUDD in a given Dyck path \( P \) and by \( l(P) \) the number of occurrences of DUDD, \( t > 1 \), in \( P \).

We prove that \( l(P) = h(\Delta(P)) \), where \( \Delta \) is Deutsch’s involution. We proceed by induction on the semilength \( n \) of the Dyck path \( P \). The assertion is trivially true for \( n = 0 \). Fix an integer \( n > 0 \) and assume that the assertion holds for all Dyck paths of semilength less than \( n \). Let \( P \) be a Dyck \( n \)-path, and \( P = U A D M B \) its first return decomposition. It is easy to verify that:

\[
\begin{align*}
h(P) &= \begin{cases} h(A) + h(B) + 1 & \text{if } A \text{ ends with } UD \\ h(A) + h(B) & \text{otherwise.} \end{cases} \\
l(P) &= \begin{cases} l(A) + l(B) + 1 & \text{if } B \text{ begins with } U^t DD, \ t > 1 \\ l(A) + l(B) & \text{otherwise.} \end{cases}
\end{align*}
\]

Since the semilengths of the two Dyck paths \( A \) and \( B \) are strictly less than \( n \), by induction hypothesis we have \( l(A) = h(\Delta(A)) \) and \( l(B) = h(\Delta(B)) \). It is sufficient to show that the involution \( \Delta \) maps a Dyck path \( P \) that begins with a subword of type \( U^t DD, t > 1 \), to a path \( \Delta(P) \) ending with UD. Consider a Dyck path \( P \) starting with the subword \( U^t DD, t > 1 \). In this case, the last step of the recursive procedure defining the map \( \Delta \) maps the first peak \( UD \) of \( P \) to the last irreducible component of the Dyck path \( \Delta(P) \) (see Fig. 5). Since \( \Delta(UD) = UD \), the Dyck path \( \Delta(P) \) ends with UD, as desired.

Recalling that \( \Delta \) is an involution, we have also \( h(P) = l(\Delta(P)) \). Hence, \( \Delta \) maps every occurrence of the subword DUDD, \( t > 1 \), into an occurrence of the subword DUDD, and vice versa.

**Proposition 3.** The Dyck paths \( P \) and \( \Delta(P) \) have the same number of occurrences of DDU.

**Proof.** Denote by \( r(P) \) the number of occurrences of the subword DDU in \( P \). We prove that \( r(P) = r(\Delta(P)) \) by induction on the semilength \( n \) of the Dyck path \( P \). The assertion is trivially true for \( n = 0 \).
Fix an integer \( n > 0 \) and assume that the assertion holds for all Dyck paths of semilength less than \( n \). Let \( P \) be a Dyck \( n \)-path, and \( P = U A D B \) its first return decomposition. It is easy to verify that:

\[
r(P) = \begin{cases} 
    r(A) + r(B) + 1 & \text{if both } A \text{ and } B \text{ are nonempty} \\
    r(A) + r(B) & \text{otherwise.}
\end{cases}
\]

Since the semilengths of the two Dyck paths \( A \) and \( B \) are strictly less than \( n \), by induction hypothesis we have \( r(A) = r(\Delta(A)) \) and \( r(B) = r(\Delta(B)) \). Noting that both \( A \) and \( B \) are nonempty if and only if \( \Delta(A) \) and \( \Delta(B) \) are nonempty, we get the assertion. \( \diamond \)

We now examine the behavior of the involution \( \hat{\Delta} \) with respect of the descent distribution, namely, the behavior of the map \( \Delta \) with respect to occurrences of \( DD \). It is shown in [7] that the involution \( \Delta \) maps a Dyck path with \( j \) double falls into a Dyck path with \( j \) valleys. Since a Dyck path \( P \) with \( j \) double falls has \( n - 1 - j \) valleys, the Dyck path \( \Delta(P) \) has \( n - j - 1 \) double falls. This implies that, if \( \sigma \) has \( j \) descents, the permutation \( \hat{\Delta}(\sigma) \) has \( n - 1 - j \) descents, and hence \( j \) ascents.

These considerations, together with Propositions 1 and 2, imply:

**Theorem 4.** The following equidistribution results hold:
1. \( \sigma \) contains \( k \) occurrences of the pattern \( 321 \) and \( j \) descents if and only if \( \hat{\Delta}(\sigma) \) contains \( k \) occurrences of the pattern \( 321 \) and \( j \) ascents, namely,
   \[
a_{n,k,j}^{321} = a_{n,k,n-1-j}^{123},
   \]
2. \( \sigma \) contains \( k \) occurrences of the pattern \( 231 \) and \( j \) descents if and only if \( \hat{\Delta}(\sigma) \) contains \( k \) occurrences of the pattern \( 132 \) and \( j \) ascents, namely,
   \[
a_{n,k,j}^{231} = a_{n,k,n-1-j}^{132},
   \]
3. \( \sigma \) contains \( k \) occurrences of the pattern \( 213 \) and \( j \) descents if and only if \( \hat{\Delta}(\sigma) \) contains \( k \) occurrences of the pattern \( 213 \) and \( j \) ascents, namely,
   \[
a_{n,k,j}^{213} = a_{n,k,n-1-j}^{213}.
   \]

Let \( \text{occ}_\tau(\sigma) \) be the number of occurrences of the pattern \( \tau \) in the permutation \( \sigma \). The preceding results can be restated as follows:

**Theorem 5.** The quadruplets of statistics
- \((\text{occ}_{321}, \text{occ}_{132}, \text{occ}_{213}, \text{des})\) and \((\text{occ}_{123}, \text{occ}_{231}, \text{occ}_{213}, \text{asc})\)
- \((\text{occ}_{321}, \text{occ}_{231}, \text{occ}_{213}, \text{des})\) and \((\text{occ}_{123}, \text{occ}_{132}, \text{occ}_{213}, \text{asc})\)

are equidistributed on \( S_n(3\text{-}1\text{-}2) \), namely,

\[
\begin{aligned}
\sum_{\sigma \in S_n(3\text{-}1\text{-}2)} x^{\text{occ}_{321}(\sigma)} y^{\text{occ}_{132}(\sigma)} z^{\text{occ}_{213}(\sigma)} t^{\text{des}(\sigma)} &= \sum_{\sigma \in S_n(3\text{-}1\text{-}2)} x^{\text{occ}_{123}(\sigma)} y^{\text{occ}_{231}(\sigma)} z^{\text{occ}_{213}(\sigma)} t^{\text{asc}(\sigma)}, \\
\sum_{\sigma \in S_n(3\text{-}1\text{-}2)} x^{\text{occ}_{321}(\sigma)} y^{\text{occ}_{231}(\sigma)} z^{\text{occ}_{213}(\sigma)} t^{\text{des}(\sigma)} &= \sum_{\sigma \in S_n(3\text{-}1\text{-}2)} x^{\text{occ}_{123}(\sigma)} y^{\text{occ}_{132}(\sigma)} z^{\text{occ}_{213}(\sigma)} t^{\text{asc}(\sigma)}. \diamond
\end{aligned}
\]

4. Generating functions

In this section we examine the generating functions \( A^{213}(x, y, z) \), \( A^{231}(x, y, z) \), and \( A^{231}(x, y, z) \). We observe that the power series \( A^\tau(x, y, z) \) can be equivalently written as:

\[
A^\tau(x, y, z) = \sum_{n,k,j \geq 0} a_{n,k,j}^\tau x^n y^k z^j = \sum_{P \in \mathcal{D}} x^{sl(P)} y^{\text{occ}_{W(\tau)}(P)} z^{\text{occ}_{DD}(P)},
\]

where \( \mathcal{D} \) denotes the set of Dyck paths, \( sl(P) \) the semilength of \( P \), \( \text{occ}_{W(\tau)}(P) \) the number of occurrences in \( P \) of the subsequence \( W(\tau) \) corresponding to the pattern \( \tau \), and \( \text{occ}_{DD}(P) \) the number of occurrences of \( DD \) in \( P \).
Our approach will be based on the first return decomposition of the Dyck path \( K(\sigma) \), that allows us to compare the generating function \( A^r(x, y, z) \) with the generating function

\[
B^r(x, y, z) = \sum_{n,k,j \geq 0} b_{n,k,j}^r x^n y^k z^j = \sum_{p \in \mathcal{D}} x^{\text{height}(p)} y^{\text{occ}_{\text{WD}}(p)} z^{\text{occ}_{\text{DD}}(p)}
\]

of the analogous distribution over the set \( \mathcal{D} \) of irreducible Dyck paths.

4.1. The pattern 321

**Proposition 6.** For every \( n > 1 \), we have:

\[
b_{n,k,j}^{321} = a_{n-1,k-1,j-1}^{321} - a_{n-2,k-1,j-1}^{321} + a_{n-2,k,j-1}^{321}.
\]

**Proof.** An irreducible Dyck path of semilength \( n \) with \( k \) triple falls and \( j \) double falls can be obtained by prepending \( U \) and appending \( D \) to a Dyck path of semilength \( n-1 \) of one of the two following types:

1. a Dyck path with \( k \) triple falls and \( j-1 \) double falls, ending with \( UD \),
2. a Dyck path with \( k-1 \) triple falls and \( j-1 \) double falls, not ending with \( UD \).

We remark that:

1. the paths of the first kind are in bijection with Dyck paths of semilength \( n-2 \) with \( k \) triple falls and \( j-1 \) double falls, enumerated by \( a_{n-2,k,j-1}^{321} \);
2. in order to enumerate the paths of the second kind we have to subtract from the integer \( a_{n-1,k-1,j-1}^{321} \) the number of Dyck paths of semilength \( n-1 \) with \( k-1 \) triple falls and \( j-1 \) double falls, ending with \( UD \). Dyck paths of this kind are in bijection with Dyck paths of semilength \( n-2 \) with \( k-1 \) triple falls and \( j-1 \) double falls, enumerated by \( a_{n-2,k-1,j-1}^{321} \).

**Proposition 7.** For every \( n > 0 \), we have:

\[
a_{n,k,j}^{321} = \sum_{s=1}^{n} \sum_{h,i>0} b_{s,h,i}^{321} a_{n-s-h,k-h,j-i}^{321}.
\]

**Proof.** Let \( P \) be a Dyck path of semilength \( n \), and consider its first return decomposition \( P = P' P'' \). We have:

- \( \text{occ}_{\text{DDD}}(P) = \text{occ}_{\text{DDD}}(P') + \text{occ}_{\text{DDD}}(P'') \),
- \( \text{occ}_{\text{DD}}(P) = \text{occ}_{\text{DD}}(P') + \text{occ}_{\text{DD}}(P'') \).

Identities (1) and (2) yield the following relations between the two generating functions \( A^{321}(x, y, z) \) and \( B^{321}(x, y, z) \):

**Proposition 8.** We have:

\[
B^{321}(x, y, z) = xyz(A^{321}(x, y, z) - 1) + x^2 z(1 - y)A^{321}(x, y, z) + 1 + x,
\]

\[
A^{321}(x, y, z) = A^{321}(x, y, z)(B^{321}(x, y, z) - 1) + 1.
\]

Observe that formula (1) holds for \( n > 1 \). This fact gives rise to the correction terms of degree less than 2 in formula (3).

Combining formulae (3) and (4) we obtain the following:

**Theorem 9.** The series \( A^{321}(x, y, z) \) satisfies the following functional equation:

\[
xz(y + x - xy) (A^{321}(x, y, z))^2 + (x - xyz - 1)A^{321}(x, y, z) + 1 = 0 \]

The series \( A^{321}(x, y, z) \) specializes into some well-known generating functions. For example, \( zA^{321}(x, 1, z) \) is the generating function of the Narayana numbers (see, e.g., [11]), and \( A^{321}(x, y, 1) \) is the generating function of the distribution of \( \text{DDD} \) on Dyck paths. This last distribution has been
studied in [22], where the authors deduce the following expression for the coefficients of the series $A^{321}(x, y, 1)$:

$$
\sum_{i=0}^{n-1} a_{n,k,i}^{321} = \frac{1}{n+1} \sum_{j=0}^{k} (-1)^{k-j} \binom{n+j}{n} \binom{n+1}{k-j} \sum_{i=j}^{n+j} \binom{n+j+1-k}{i+1} \binom{n-i}{i-j}.
$$

We deduce that:

**Corollary 10.** The number of permutations in $S_n$ that avoid both the pattern 3-1-2 and the consecutive pattern 321 is

$$|S_n(3-1-2, 321)| = M_n,$$

where $M_n$ is the $n$th Motzkin number. \(\diamondsuit\)

In fact, a bijection $\nu$ between permutations in $S_n(3-1-2, 321)$ and Motzkin paths of length $n$ can be obtained as the composition of the map $K$ with the well-known bijection between Dyck $n$-paths with no $DDD$ and Motzkin $n$-paths, defined by replacing each $UDD$ with $D$ and each remaining $UD$ with a horizontal step $H$ (see e.g. [5]).

It is easy to check that the descents of a permutations $\sigma \in S_n(3-1-2, 321)$ correspond bijectively to the $D$ steps of the associated Motzkin path. This implies that the integer $a_{n,0,j}^{321}$ is the number of Motzkin paths of length $n$ with $j$ $D$ steps (see seq. A055151 in [24]) Fig. 6.

We point out that the arguments used in the proof of Proposition 1 can be iterated to get the following more general result:

**Proposition 11.** We have

$$a_{n,k,j}^{12-\cdots-p} = a_{n,k,n-1-j}^{p-21}.$$

4.2. The pattern 231

**Proposition 12.** For every $n > 1$, we have:

$$b_{n,k,j}^{231} = a_{n-1,k,j-1}^{231} - a_{n-2,k,j-1}^{231} + a_{n-2,k-1,j-1}^{231}.$$  \(6\)

**Proof.** An irreducible Dyck path of semilength $n$ with $k$ occurrences of $D U D D$ and $j$ double falls can be obtained by prepending $U$ and appending $D$ to a Dyck path of semilength $n-1$ of one of the two following types:

1. a Dyck path with $k-1$ occurrences of $D U D D$ and $j-1$ double falls, ending with $U D$,
2. a Dyck path with $k$ occurrences of $D U D D$ and $j-1$ double falls, not ending with $U D$.

We remark that:

1. the paths of the first kind are in bijection with Dyck paths of semilength $n-2$ with $k-1$ occurrences of $D U D D$ and $j-1$ double falls, enumerated by $a_{n-2,k-1,j-1}^{231}$;
2. in order to enumerate the paths of the second kind we have to subtract from the integer $a_{n-1,k,j-1}^{231}$ the number of Dyck paths of semilength $n-1$ with $k$ occurrences of $D U D D$ and $j-1$ double falls,
ending with $UD$. Dyck paths of this kind are in bijection with Dyck paths of semilength $n - 2$ with $k$ occurrences of $D Ud D$ and $j - 1$ double falls, enumerated by $a_{n-2,k,j-1}^{231}$. ◦

**Proposition 13.** For every $n > 0$, we have:

$$a_{n,k,j}^{231} = \sum_{i=1}^{n} \sum_{h,i>0} b_{i,h}^{231} a_{n-h,k-i}^{231}.$$

(7)

**Proof.** Let $P$ be a Dyck path of semilength $n$, and consider its first return decomposition $P = P' P''$. We have:

- $occ_{D Ud D}(P) = occ_{D Ud D}(P') + occ_{D Ud D}(P'')$,
- $occ_{D D}(P) = occ_{D D}(P') + occ_{D D}(P'')$. ◦

Identities (6) and (7) yield the following relations between the two generating functions $A^{231}(x, y, z)$ and $B^{231}(x, y, z)$:

**Proposition 14.** We have:

$$B^{231}(x, y, z) = xz(1 - x + xy)(A^{231}(x, y, z) - 1) + 1 + x.$$  

(8)

$$A^{231}(x, y, z) = A^{231}(x, y, z)(B^{231}(x, y, z) - 1) + 1. \quad \diamond$$

(9)

Observe that formula (6) holds for $n > 1$. This fact gives rise to the correction terms of degree less than 2 in formula (8).

Combining formulae (8) and (9) we obtain the following:

**Theorem 15.** The series $A^{231}(x, y, z)$ satisfies the following functional equation:

$$xz(1 - x + xy) \left(A^{231}(x, y, z)^2 + (x - xz + x^2z - x^2yz - 1)A^{231}(x, y, z) + 1 \right) = 0 \quad \diamond$$

(10)

Also in this case, the series $A^{231}(x, y, z)$ is the generating function of the distribution of $D Ud D$ on Dyck paths, that was studied in [19] and [22]. In [22], the authors determine an expression for the coefficients of $A^{231}(x, y, z)$:

$$\sum_{i=0}^{n-2} a_{n,k,i}^{231} = \sum_{j=k}^{n+1} (-1)^{j-k} \binom{n-j}{k} \binom{n-j}{j} \binom{2n-3j}{n-j+1}.$$  

(11)

We deduce the following:

**Corollary 16.** The number of permutations in $S_n$ that avoid both the pattern 3-1-2 and the consecutive pattern 231 is

$$|S_n(3-1-2, 231)| = \sum_{j=0}^{n+1} (-1)^j \binom{n-j}{j} \binom{2n-3j}{n-j+1}.$$  

(12)

4.3. The pattern 213

In this case, it is straightforward to check that

$$b_{n,k,j}^{213} = a_{n-1,k,j-1}^{213}.$$  

(11)

Moreover, we have:

**Proposition 17.** For every $n > 0$, we have:

$$a_{n,k,j}^{213} = a_{n-1,k,j}^{213} + a_{n-1,k,j-1}^{213} + \sum_{s=2}^{n-1} \sum_{h=0}^{k} \sum_{l=1}^{i} a_{s-1,h,l-1}^{213} a_{n-h,k-h-1,j-i}^{213}.$$  

(12)
Let $P$ be a Dyck path of semilength $n$, and consider its first return decomposition $P = P' P''$. If $P''$ is empty, the Dyck path $P = P'$ is irreducible. An irreducible Dyck path of semilength $n$ with $k$ occurrences of $DDU$ and $j$ double falls can be obtained by prepending $U$ and appending $D$ to a Dyck path of semilength $n-1$ with $k$ occurrences of $DDU$ and $j-1$ double falls, that are counted by $a_{n-1,k,j-1}^{213}$. Otherwise, if $P''$ is not empty, we have:

- $occ_{DDU}(P) = occ_{DDU}(P') + occ_{DDU}(P'') + 1$,
- $occ_{DD}(P) = occ_{DD}(P') + occ_{DD}(P'')$,

unless $P' = UD$. In this last case $P$ and $P''$ have the same number of occurrences of $DDU$ and $DD$. Hence, Dyck path of semilength $n$ with $k$ occurrences of $DDU$ and $j$ double falls starting with $UD$ are counted by $a_{n-1,j,k}^{213}$.

Identity (12) yields the following:

**Proposition 18.** The series $A^{213}(x, y, z)$ satisfies the functional equation

$$A^{213}(x, y, z) = 1 + xA^{213}(x, y, z) + xz(A^{213}(x, y, z) - 1) + xyz(A^{213}(x, y, z) - 1)^2,$$

that is equivalent to

$$xyz(A^{213}(x, y, z))^2 + (x + xz - 2xyz - 1)A^{213}(x, y, z) + 1 - xz + xyz = 0.$$

Once again, the series $A^{213}(x, y, 1)$ is the generating function of the distribution of $DDU$ on Dyck paths. This distribution has been studied in [8], where the author deduces the following explicit expression for the coefficients of its generating function $A^{213}(x, y, 1)$:

$$
\sum_{i=0}^{n-1} a_{n,k,i}^{213} = 2^{n-2k-1} C_k \binom{n-1}{2k},
$$

where $C_k$ is the $k$th Catalan number.

We now analyze the distribution of descents on the set of permutations in $S_n(3-1-2, 213)$.

**Proposition 19.** The number of permutations in $S_n(3-1-2, 213)$ with $j$ descents is

$$a_{n,0,j}^{213} = \binom{n-1}{j}.$$

**Proof.** It is easy to check that a permutation $\sigma \in S_n(3-1-2, 213)$ with $j$ descents has the following structure:

$$\sigma = m_1 m_2 \ldots m_{r-1} n \ w_f,$$

with $|w_f| = j$. This implies that a permutation $\sigma \in S_n(3-1-2, 213)$ with $j$ descents is assigned as soon as we choose the subset of symbols in $w_k$, that is an arbitrary subset of cardinality $j$ of $\{1, 2, \ldots, n-1\}$. ⊠

As an immediate consequence, we have:

**Proposition 20.** The number of permutations in $S_n$ that avoid both the pattern 3-1-2 and the consecutive pattern 213 is

$$|S_n(3-1-2, 213)| = 2^{n-1}.$$ ⊠

5. Final remarks

In Krattenthaler’s paper [17] a further bijection $K'$ between permutations avoiding 1-2-3 and Dyck paths is described. Nevertheless, a study of the joint distribution of consecutive patterns and descents over $S_n(1-2-3)$ cannot be carried out with the techniques described in the present paper, since Deutsch’s involution $\Delta$ does not behave properly in this case.
In fact, recall that $\Delta$ maps the Dyck path $U^2D^3$ into $(UD)^3$ (see Fig. 2). These paths correspond via $K'$ to the permutations 132 and 321, respectively. However, it can be checked that the two consecutive patterns 132 and 321 are not equidistributed over $S_4(1\rightarrow 2\rightarrow 3)$.

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**References**


