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On dimensions modulo a compact metric ANR and modulo a simplicial complex

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ABSTRACT

V.V. Fedorchuk has recently introduced dimension functions K-dim $\leqslant K$-Ind and L-dim $\leqslant L$-Ind, where K is a simplicial complex and L is a compact metric ANR. For each complex K with a non-contractible join $|K| * |K|$ (we write $|K|$ for the geometric realisation of K), he has constructed first countable, separable compact spaces with K-dim $< K$-Ind.

In a recent paper we have combined an old construction by P. Vopěnka with a new construction by V.A. Chatyrko, and have assigned a certain compact space $Z(X, Y)$ to any pair of non-empty compact spaces X, Y. In this paper we investigate the behaviour of the four dimensions under the operation $Z(X, Y)$. This enables us to construct examples of compact Fréchet spaces which have K-dim $< K$-Ind, L-dim $< L$-Ind, or K-Ind $< |K|$-Ind, and (connected) components of which are metrisable. In particular, given a natural number $n \geqslant 1$, an ordinal $\alpha \geqslant n$, and any metric continuum C with $L\text{-dim}\, C = n$, we obtain

- a compact Fréchet space $X_{C,\alpha}$ such that $L\text{-dim}\, X_{C,\alpha} = n$, $L\text{-Ind}\, X_{C,\alpha} = \alpha$, and each component of $X_{C,\alpha}$ is homeomorphic to C.

If $L * L$ is non-contractible, or $n = 1$ and L is non-contractible, then C can be a cube $[0, 1]^m$ for a certain natural number $m = m(n, L)$.

Introduction

All considered topological spaces are T_1 and completely regular. Let K be a fixed (finite) simplicial complex, $|K|$ its geometric realisation, and L a (compact metric) ANR. *We assume that $|K|$ and L are non-contractible.*[1]

Fedorchuk [6–9] has begun the investigation of dimensions[2] $K\text{-dim}\, X$, $L\text{-dim}\, X$, $K\text{-Ind}\, X$, $L\text{-Ind}\, X$ of normal spaces X. There is a far reaching analogy between the theories of K-dim / K-Ind, L-dim / L-Ind, and the classical dim / Ind. In particular, $K\text{-dim}\, X \leqslant K\text{-Ind}\, X$, $L\text{-dim} \leqslant L\text{-Ind}\, X$, $K\text{-dim}\, X = |K|\text{-dim}\, X$, and $K\text{-Ind}\, X \leqslant |K|\text{-Ind}\, X$ if X is normal. Moreover $K\text{-Ind}\, X = |K|\text{-Ind}\, X$ if X is hereditarily normal, and all the four dimensions for K and $|K|$ coincide if X is metrisable. In [8], for each natural number $n \geqslant 2$ and each simplicial complex K with a non-contractible join $|K| * |K|$, Fedorchuk has constructed a first countable, separable compact space X_n such that $K\text{-dim}\, X_n = n < 2n - 1 \leqslant K\text{-Ind}\, X_n \leqslant 2n$.

Henceforth, let K-Ind and L-Ind denote the transfinite extensions of Fedorchuk's K-Ind and L-Ind.

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1 If $|K|$ or L were contractible, then the considered dimension functions would be trivial (they would assign zero to each non-empty normal space).

2 Note that Fedorchuk [6,8] has defined $\mathcal{K}$-dim, $\mathcal{L}$-dim, $\mathcal{K}$-Ind, $\mathcal{L}$-Ind for collections $\mathcal{K}$ and $\mathcal{L}$ which consist of simplicial complexes and ANR's, respectively. However, in the present paper each of $\mathcal{K}$ and $\mathcal{L}$ has exactly one element, K or L, and we write K-dim, L-dim, K-Ind, L-Ind.

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In the joint paper [2] with M.G. Charalambous we have constructed first countable and separable continua $S_{n,\alpha}$ such that K-dim $S_{n,\alpha} = n$ and K-Ind $S_{n,\alpha} = \alpha$, where $n \geqslant 1$ is any natural number, $\alpha \geqslant n$ is any ordinal of cardinality at most $\mathfrak{c}$, and moreover $n = 1$ or the join $|K| * |K|$ is non-contractible. This may be considered as a partial solution to the following.

Problem. Let n be a natural number, α an ordinal, and $1 \leqslant n \leqslant \alpha$.

(a) Under what circumstances do there exist compact spaces with K-dim $= n$ and K-Ind $= \alpha$?
(b) Can all components of such a space be metrisable?
(c) What about L-dim and L-Ind?

In [10] we have combined constructions by P. Vopěnka [12] and V.A. Chatyrko [3], and have assigned a compact space $Z(X, Y)$ to any pair of non-empty compact spaces X, Y. Each component of $Z(X, Y)$ is homeomorphic to a component of X or Y. This has allowed us to construct compact Fréchet spaces $X_{C,\alpha}$ such that $\dim X_{C,\alpha} = n$, $\operatorname{trind} X_{C,\alpha} = \operatorname{trInd} X_{C,\alpha} = \alpha$, and all components of $X_{C,\alpha}$ are homeomorphic to C, where C is any metric continuum with $\dim C = n < \infty$ and $\alpha \geqslant n$ is any ordinal [10, Theorem 5].

In the present paper we investigate the behaviour of Fedorchuk's dimensions under the operation $Z(X, Y)$. We prove that L-dim $Z(X, X) = L$-dim X (the same holds for K), and under mild assumptions on X, L-Ind $Z(X, X) = L$-Ind $X + 1$. We use transfinite induction, and answer the questions (b, c) together by constructing examples of spaces satisfying L-dim $= n$ and L-Ind $= \alpha$ in all cases without obvious obstructions (see the Abstract, Theorem 4.6, and Corollary 4.8).

In the case of K-Ind we encounter serious difficulties because it often happens that K-Ind $Z(X, X) = K$-Ind X. We distinguish two sorts of spaces X which satisfy the equality K-Ind $X = \alpha$ weakly or strongly, and we formalise this by defining the *dimensional strength degree* K-str $X \in \{0, 1\}$. We confine ourselves to the case of $K = \partial\Delta^k$, the simplicial complex that consists of the proper faces of a k-dimensional simplex Δ^k. We show that the spectrum of $\partial\Delta^k$-str on the class of compact metric spaces is $\{0, 1\}$. We prove that if X is a compact Fréchet space with $\partial\Delta^k$-str $X = 1$, then $\partial\Delta^k$-Ind $Z(X, X) = \partial\Delta^k$-Ind $X + 1$.

Our approach enables us to obtain the following examples. Let C be a metric continuum and $n \geqslant 1$. Then there exists

- a compact Fréchet space X_C with $\partial\Delta^k$-dim $X_C = n$, $\partial\Delta^k$-Ind $X_C = n + 1$, and components homeomorphic to C *whenever* $k \geqslant 1$ *and* $\dim C = k(n + 1) - 1$ (in this case $\partial\Delta^k$-dim $C = n$ and $\partial\Delta^k$-str $C = 1$);
- a compact Fréchet space X_C such that $\partial\Delta^k$-dim $X_C = \partial\Delta^k$-Ind $X_C = n$ while $|\partial\Delta^k|$-Ind $X_C = n + 1$, and each component of X_C is homeomorphic to C—*this example needs the assumptions that* $k \geqslant 2$ *and* $\dim C = kn$ (then $\partial\Delta^k$-dim $C = n$ and $\partial\Delta^k$-str $C = 0$).

Using the latter series of examples, we answer Fedorchuk's question [8, Question 3.1] in the negative: the equality K-Ind $=$ $|K|$-Ind is not true outside the class of hereditarily normal spaces.

1. Notation, basic definitions and facts

In this paper *maps* and their *extensions* are meant to be continuous. A *continuum* is a non-empty, connected compact space. By $\mathbb{N}$ we denote the set of natural numbers, and $0 \in \mathbb{N}$ is also the first ordinal. We write $A_{\mathfrak{m}}$ for the one-point compactification of the discrete space of cardinality $\mathfrak{m}$, and $\mu \in A_{\mathfrak{m}}$ is the unique non-isolated point. In most cases we employ the terminology used in R. Engelking's monographs [4,5].

We write K for a (finite) simplicial complex with distinct vertices $e_0, \dots, e_k$ in a Euclidean space, $|K|$ for the geometric realisation of K (the underlying polyhedron), and L for a (compact metric) ANR. *We assume that both* $|K|$ *and* L *are non-contractible.* Δ^k stands for the k-dimensional simplex with vertices $e_0, \dots, e_k$, and $\partial\Delta^k$ for the simplicial complex that consists of all at most $(k - 1)$-dimensional faces of Δ^k. Of course, $|K|$ is always an ANR, and $|\partial\Delta^k|$ is homeomorphic to the sphere S^{k-1}. By $\{0, 1\}$ we denote $\partial\Delta^1$, a simplicial complex that has two vertices and no edge.

Let X be a space, $A \subset X$, and $f : A \to L$ a map. An open set $U \subset X$ is called an *L-neighbourhood* of f in X provided that $A \subset U$ and f has an extension from U to L. Then $P = X \setminus U$ is called an *L-partition* in X for f. Since L is an ANR, *every map* $f : F \to L$ *from a closed subset* F *of a normal space* X *has an L-neighbourhood and an L-partition in* X.

We adopt a convention, by which we use calligraphic letters $\mathcal{A}$, $\mathcal{B}$, etc. to denote $(k+1)$-*tuples* $(A_0, \dots, A_k)$, $(B_0, \dots, B_k)$, etc. of subsets of any given space X. A $(k + 1)$-tuple $\mathcal{A}$ of X is said to be *open* [respectively: *closed*] if $A_0, \dots, A_k$ are open [respectively: closed] in X. We write $\operatorname{cl}\mathcal{A} = (\operatorname{cl} A_0, \dots, \operatorname{cl} A_k)$, $\mathcal{A}|E = (A_0 \cap E, \dots, A_k \cap E)$ for any subset E of X, $f(\mathcal{A}) = (f(A_0), \dots, f(A_k))$ for a map f defined on X, etc. A $(k+1)$-tuple $\mathcal{A}$ is called a *K-tuple* provided that, if $I \subset \{0, \dots, k\}$ and $\bigcap_{i\in I} A_i \neq \emptyset$, then $\{e_i \colon i \in I\}$ is the vertex set of a certain simplex in K. We write $\bigcup\mathcal{A} = \bigcup_{i=0}^{k} A_i$. If $\mathcal{A}$ is an open K-tuple of X, we call $P = X \setminus \bigcup\mathcal{A}$ the *K-partition corresponding to* $\mathcal{A}$; if moreover $B_i \subset A_i$ for $i = 0, \dots, k$, we say that $\mathcal{A}$ is a *K-neighbourhood of* $\mathcal{B}$ and P is a *K-partition for* $\mathcal{B}$.

We shall frequently use this simple corollary to [4, Theorem 7.1.4]: *Every closed K-tuple of a normal space has a K-neighbourhood* $\mathcal{U}$ *such that* $\operatorname{cl}\mathcal{U}$ *is a K-tuple.*

Definition 1.1. (Fedorchuk[3] [6, Definition 3.4]) Let M be a simplicial complex. For normal spaces X, the dimension $M\text{-dim}\, X \in \mathbb{N} \cup \{-1, \infty\}$ is defined as follows.

(a) $M\text{-dim}\, X = -1$ iff X is empty.
(b) When $n \in \mathbb{N}$, $M\text{-dim}\, X \leqslant n$ iff for each sequence of closed M-tuples $\mathcal{F}^0, \ldots, \mathcal{F}^n$ of X there are M-partitions P^i for $\mathcal{F}^i$, $i = 0, \ldots, n$, so that $\bigcap_{i=0}^n P^i = \emptyset$.
(c) $M\text{-dim}\, X = \min\{n \in \mathbb{N}\colon\ M\text{-dim}\, X \leqslant n\}$, where $X \neq \emptyset$ and $\min \emptyset = \infty$.

See Fedorchuk [6, Section 1] for information about the join $X * Y$ of compact spaces X, Y. At this place, let us recall these two facts: $(X * Y) * Z$ is $X * (Y * Z)$ up to homeomorphism, and if X, Y are ANR's, then so is $X * Y$. The join $X * \cdots * X$ of n copies of X will be denoted by X^{*n}.

Definition 1.2. (Cf. Fedorchuk [6, Definition 3.9, Corollary 3.13] and [8, Definition 1.14].) Let M be an ANR. Then $M\text{-dim}\, X \in \mathbb{N} \cup \{-1, \infty\}$, where X is any normal space, is defined so that it satisfies the statements (a), (c) of Definition 1.1 and the following statement (b′) instead of (b).

(b′) When $n \in \mathbb{N}$, $M\text{-dim}\, X \leqslant n$ iff every map $f : F \to M^{*(n+1)}$ from a closed subspace F of X has an extension from X to $M^{*(n+1)}$.

Definition 1.3. (Cf. Fedorchuk [8, Definition 2.1] and [9, Definition 2.16].) Let M be a simplicial complex. The inductive dimension[4] $M\text{-Ind}\, X \in \mathit{Ordinals} \cup \{-1, \infty\}$ is defined for normal spaces X as follows.

(a) $M\text{-Ind}\, X = -1$ iff X is empty.
(b) When α is an ordinal, $M\text{-Ind}\, X \leqslant \alpha$ iff for every closed M-tuple $\mathcal{F}$ of X there is an M-partition P such that $M\text{-Ind}\, P < \alpha$.
(c) $M\text{-Ind}\, X = \min\{\alpha\colon\ M\text{-Ind}\, X \leqslant \alpha\}$, where $X \neq \emptyset$ and $\min \emptyset = \infty$.

Definition 1.4. (Cf. Fedorchuk [8, Definition 2.3] and [9, Definition 2.14].) Let M be an ANR. For normal spaces X, the dimension $M\text{-Ind}\, X \in \mathit{Ordinals} \cup \{-1, \infty\}$ is defined so that it satisfies the statements (a, c) of Definition 1.3 and the following statement (b′) instead of (b).

(b′) When α is an ordinal, $M\text{-Ind}\, X \leqslant \alpha$ iff for every map $f : F \to M$ from a closed subset F of X there is an M-partition P such that $M\text{-Ind}\, P < \alpha$.

It is evident that $\{0,1\}\text{-dim}\, X = \dim X$ and $\{0,1\}\text{-Ind}\, X = \operatorname{trInd} X$ for normal spaces X, no matter whether we treat $\{0,1\}$ as a simplicial complex or as an ANR.

Let us recall the following well-known facts on homotopy equivalence.

Theorem 1.5. *(J.E. West [13]) Every compact metric ANR is homotopy equivalent to a compact polyhedron.* □

Theorem 1.6. *(Fedorchuk [6, Proposition 4.5] and [9, Theorem 3.3]) If ANR's L_1 and L_2 are homotopy equivalent, then*

$$L_1\text{-dim}\, X = L_2\text{-dim}\, X \quad \text{and} \quad L_1\text{-Ind}\, X = L_2\text{-Ind}\, X$$

for every normal space X. □

It follows from the foregoing two theorems that when we investigate relations between the four dimensions K-dim, L-dim, K-Ind, and L-Ind, it is sufficient to consider only simplicial complexes K and their geometric realisations $L = |K|$.

Theorem 1.7. *(Fedorchuk [6, Theorem 4.8], [8, Theorems 3.18 and 3.23], and [9, Theorem 2.22]) Suppose that X is a normal space. Then*

$$K\text{-dim}\, X = |K|\text{-dim}\, X \quad \text{and} \quad K\text{-Ind}\, X \leqslant |K|\text{-Ind}\, X.$$

If X is hereditarily normal, then

$$K\text{-Ind}\, X = |K|\text{-Ind}\, X.$$

[3] See the remark in Footnote 2.
[4] Fedorchuk's original $K\text{-Ind}\, X$ and $L\text{-Ind}\, X$ in [8] are natural numbers, -1, or ∞. Following [2], we allow both $K\text{-Ind}\, X$ and $L\text{-Ind}\, X$ to be an infinite ordinal.

If either K-dim X or K-Ind X is finite, then

$$K\text{-}\dim X \leqslant K\text{-}\mathrm{Ind}\, X.$$

If X is metrisable and K-dim X is finite, then all four of the dimensions of X coincide. □

It is worth adding that the inequality $K\text{-}\mathrm{Ind}\, X \leqslant |K|\text{-}\mathrm{Ind}\, X$ also results from [2, Lemma 6] by induction on $\alpha = |K|\text{-}\mathrm{Ind}\, X$. To prove that $|K|\text{-}\mathrm{Ind}\, X \leqslant K\text{-}\mathrm{Ind}\, X$ for any hereditarily normal space X, one can easily apply [2, Lemma 7] (Lemma 2.3 herein) and induction on $\alpha = K\text{-}\mathrm{Ind}\, X$.

The topic of dimension-lowering maps for K-dim and L-dim is more complex than in the case of dim (see [6, Section 7]). However, there is

Theorem 1.8. *(Cf. Fedorchuk [8, Theorem 3.24].) If X is a compact space, then*

$$L\text{-}\dim X = \sup\{L\text{-}\dim P\colon\ P \text{ is a component of } X\}.$$

Proof. By Theorems 1.5 and 1.6, it is sufficient to consider $L = |K|$. Theorem 1.7 yields the equalities $|K|\text{-}\dim X = K\text{-}\dim X$ and $|K|\text{-}\dim P = K\text{-}\dim P$ for each component P of X. Consider the decomposition $\mathcal{D}$ of X into the components of X and the quotient map $q : X \to X/\mathcal{D}$. The quotient space $X/\mathcal{D}$ is compact and $\dim X/\mathcal{D} = 0$ unless X is empty. The requested equality results from Fedorchuk's theorem [8, Theorem 3.24] applied to q. □

Theorem 1.9. *Suppose that $k, n \geqslant 1$ are natural numbers, and X is a metric space. Then*

$$\partial\Delta^k\text{-}\mathrm{Ind}\, X < n \quad \textit{iff} \quad \bigl|\partial\Delta^k\bigr|\text{-}\dim X < n \quad \textit{iff} \quad \dim X < kn.$$

Proof. The former equivalence results from Theorem 1.7. The latter for $n = 1$ is the well-known theorem on extending maps to spheres (see [5, Theorem 3.2.10]).

We shall apply this theorem by Fedorchuk [6, Theorem 5.7 and Corollary 5.16]: *A metric space X has L-dim $X \leqslant n \in \mathbb{N}$ iff there are subspaces $X_0, \dots, X_n$ of X such that $X = X_0 \cup \dots \cup X_n$ and L-dim $X_i \leqslant 0$ for $i = 0, \dots, n$.*

Let $n > 1$ and $L = |\partial\Delta^k|$. Then $|\partial\Delta^k|\text{-}\dim X < n$ iff $X = X_0 \cup \dots \cup X_{n-1}$ and $|\partial\Delta^k|\text{-}\dim X_i \leqslant 0$ for $i = 0, \dots, n-1$. These last inequalities are equivalent to $\dim X_i < k$, and in turn, to the statement that $X_i = X_i^0 \cup \dots \cup X_i^{k-1}$ and $\dim X_i^j \leqslant 0$ for $j = 0, \dots, k-1$ (by [5, Theorem 4.1.17]). Thus, $|\partial\Delta^k|\text{-}\dim X < n$ iff X is the union of at most kn subspaces X_i^j with $\dim X_i^j = 0$, i.e. iff $\dim X < kn$ (again by [5, Theorem 4.1.17]). □

Suppose that $\mathcal{U}$ is an open K-tuple of a space X. We say that an element $x \in X$ is a *K-obstruction point for $\mathcal{U}$* provided that $\mathcal{U}$ has no K-neighbourhood $\mathcal{V}$ with $x \in \bigcup\mathcal{V}$. We write $K\text{-}\mathrm{obs}\,\mathcal{U}$ for the set of K-obstruction points for $\mathcal{U}$. Clearly, $K\text{-}\mathrm{obs}\,\mathcal{U}$ does not intersect $\bigcup\mathcal{U}$.

Let us note the following simple observation.

Lemma 1.10. *Consider $K = \partial\Delta^k$. Then*

$$\partial\Delta^k\text{-}\mathrm{obs}\,\mathcal{U} = \bigcap_{0 \leqslant i \leqslant k} \mathrm{cl}\Bigl(\bigcap_{0 \leqslant j \leqslant k,\ j \neq i} U_j\Bigr)$$

for every open $\partial\Delta^k$-tuple $\mathcal{U} = (U_0, \dots, U_k)$.

Proof. Assume that $x \notin \partial\Delta^k\text{-}\mathrm{obs}\,\mathcal{U}$, i.e. $x \in \bigcup\mathcal{V}$ for a certain $\partial\Delta^k$-neighbourhood $\mathcal{V} = (V_0, \dots, V_k)$ of $\mathcal{U}$. If $x \in V_i$, then $x \notin \mathrm{cl}(\bigcap_{0 \leqslant j \leqslant k,\ j \neq i} U_j) \subset \mathrm{cl}(\bigcap_{0 \leqslant j \leqslant k,\ j \neq i} V_j)$ as $\mathcal{V}$ is a $\partial\Delta^k$-tuple. Thus, x does not belong to the intersection of closures.

Assume there is an i such that $x \notin \mathrm{cl}(\bigcap_{0 \leqslant j \leqslant k,\ j \neq i} U_j)$. Then there is a neighbourhood $W \ni x$ disjoint from $\bigcap_{0 \leqslant j \leqslant k,\ j \neq i} U_j$. The union $V_i = U_i \cup W$ and the sets $V_j = U_j$, $j \neq i$, form a $\partial\Delta^k$-neighbourhood $\mathcal{V}$ of $\mathcal{U}$, and hence, $x \notin \partial\Delta^k\text{-}\mathrm{obs}\,\mathcal{U}$. □

Considering the dimension K-Ind, we distinguish two ways, in which a space X may be α-dimensional. We define the *dimensional strength degree* $K\text{-}\mathrm{str}\, X \in \{0, 1\}$ as follows. Let $0 < \alpha = K\text{-}\mathrm{Ind}\, X < \infty$. We put $K\text{-}\mathrm{str}\, X = 0$ (X is *weakly* α-dimensional) when every closed K-tuple of X has a K-neighbourhood $\mathcal{U}$ with $K\text{-}\mathrm{obs}\,\mathcal{U} = \emptyset$ and $K\text{-}\mathrm{Ind}(X \setminus \bigcup\mathcal{U}) < \alpha$. Otherwise, we put $K\text{-}\mathrm{str}\, X = 1$ (i.e. X is *strongly* α-dimensional when $0 < \alpha = K\text{-}\mathrm{Ind}\, X < \infty$ and there is a closed K-tuple whose every K-neighbourhood $\mathcal{U}$ with $K\text{-}\mathrm{Ind}(X \setminus \bigcup\mathcal{U}) < \alpha$ has $K\text{-}\mathrm{obs}\,\mathcal{U} \neq \emptyset$). By abuse of notation, we write $K\text{-}\mathrm{str}\, X = 0$ when α is -1, 0, or ∞.

In the next section we prove that the above distinction is material at least for some K's: if $1 \leqslant n \in \mathbb{N}$ and $2 \leqslant k \in \mathbb{N}$, then—for instance—the following cubes have $\partial\Delta^k\text{-}\mathrm{Ind}[0,1]^{kn} = \partial\Delta^k\text{-}\mathrm{Ind}[0,1]^{k(n+1)-1} = n$, $\partial\Delta^k\text{-}\mathrm{str}[0,1]^{kn} = 0$, and $\partial\Delta^k\text{-}\mathrm{str}[0,1]^{k(n+1)-1} = 1$ (cf. Theorem 1.9 and Propositions 2.7–2.8). On the other hand, in the case when $k = 1$ and

$K = \{0,1\}$, *every normal space* X *with* $\operatorname{trInd} X$ *being a successor ordinal has* $\{0,1\}\text{-str}\, X = 1$. Indeed, let $\alpha = \operatorname{trInd} X > 0$, and suppose on the contrary that $\{0,1\}\text{-str}\, X = 0$. Take arbitrary disjoint closed sets $F_0, F_1 \subset X$. Then, the $\{0,1\}$-tuple (F_0, F_1) has a $\{0,1\}$-neighbourhood (U_0, U_1) with $P = X \setminus (U_0 \cup U_1)$, $\operatorname{trInd} P \leqslant \alpha - 1$, and $\{0,1\}\text{-obs}(U_0, U_1) = \operatorname{cl} U_0 \cap \operatorname{cl} U_1 = \emptyset$. Hence, there exists a partition Q between $\operatorname{cl} U_0$ and $\operatorname{cl} U_1$ with $\operatorname{trInd} Q < \alpha - 1$, and we have shown that $\operatorname{trInd} X \leqslant \alpha - 1$. A contradiction. Therefore $\{0,1\}\text{-str}\, X = 1$. Finally, the Smirnov compactum S_{ω_0} (i.e. the one-point compactification of the topological sum $\bigoplus_{i=1}^{\infty}[0,1]^i$) has $\operatorname{trInd} S_{\omega_0} = \omega_0$ and $\{0,1\}\text{-str}\, S_{\omega_0} = 0$.

Using the definition of K-str, one easily proves the following.

Proposition 1.11. *Suppose that* A *is a closed subspace of a normal space* X*. If* $K\text{-Ind}\, A = K\text{-Ind}\, X$ *and* $K\text{-str}\, X = 0$*, then* $K\text{-str}\, A = 0$. □

2. General lemmas

In this section we collect miscellaneous properties of Fedorchuk's dimensions, prove a combinatorial analogue (Corollary 2.5) of Yu.T. Lisitsa's theorem [11] on partial extensions of maps into spheres (Theorem 2.4 herein), investigate the $\partial\Delta^k$-str of metric spaces, and prove the theorem on the dimensions of a product with a compact discontinuum.

Proposition 2.1. *(Cf. Fedorchuk [7, Theorem 2.5], Charalambous and Krzempek [2, Corollary 2].) Let* $n \geqslant 1$ *be a natural number. If* $n = 1$ *or the join* $L * L$ *is non-contractible, then there is a natural number* m *such that* $L\text{-dim}[0,1]^m = n$.

Proof. If $L * L$ is non-contractible, we are done by Fedorchuk [7, Theorem 2.5]. Assume that $n = 1$ and $L * L$ is contractible. Then every normal space X has $L\text{-dim}\, X \leqslant 1$ by Fedorchuk [7, Proposition 2.3]. In view of Theorems 1.5 and 1.6, it is sufficient to consider $L = |K| \subset [0,1]^{m-1}$. As $|K|$ is non-contractible, a certain map from $|K| \times \{0,1\}$ to $|K|$ does not have an extension from $|K| \times [0,1]$ to $|K|$. Therefore, $0 < |K|\text{-dim}(|K| \times [0,1]) \leqslant |K|\text{-dim}[0,1]^m \leqslant 1 = n$. □

The following lemma is an L-Ind analogue of [2, Proposition 1].

Lemma 2.2. *Let* X *be a normal space, and* $F \subset X$ *be closed. If* $L\text{-Ind}\, F = 0$ *and* $L\text{-Ind}\, E \leqslant \alpha$ *for each closed subset* $E \subset X$ *disjoint from* F*, then* $L\text{-Ind}\, X \leqslant \alpha$.

Proof. Take any map $g : G \to L$, where $G \subset X$ is closed. Since $L\text{-Ind}\, F = 0$, we infer that g has an extension from $G \cup F$ to L. As L is an ANR, we now obtain a neighbourhood U of $G \cup F$ with an extension $g' : U \to L$ of g. Let $V \subset X$ be an open set with $G \cup F \subset V \subset \operatorname{cl} V \subset U$. Since $L\text{-Ind}(X \setminus V) \leqslant \alpha$, there is an L-partition P in $X \setminus V$ for the restriction $g'|\operatorname{bd} V$, where $L\text{-Ind}\, P < \alpha$. This means that $P \subset X \setminus \operatorname{cl} V$, and $g'|\operatorname{bd} V$ has an extension $g'' : X \setminus (V \cup P) \to L$. Finally, $(g'|\operatorname{cl} V) \cup g'' : X \setminus P \to L$ extends g, and we have shown that $L\text{-Ind}\, X \leqslant \alpha$. □

Recall that any $x \in |K|$ can be uniquely written in the form $x = \sum_{i=0}^{k} x_i e_i$, where the *barycentric coordinates* $x_0, \ldots, x_k$ are non-negative real numbers with $\sum_{i=0}^{k} x_i = 1$. Put $K_i = \{x \in |K|: x_i \geqslant \frac{1}{k+1}\}$, and note that $\mathcal{K} = (K_0, \ldots, K_k)$ is a closed K-cover of $|K|$.

Lemma 2.3. *([2, Lemma 7]) Suppose that* $f : F \to |K|$ *is a map from a closed subset* F *of a normal space* X*. If the* K*-tuple* $f^{-1}(\mathcal{K})$ *has a* K*-neighbourhood that covers* X*, then* f *has an extension from* X *to* $|K|$. □

Theorem 2.4. *(Lisitsa [11]; see also [5, Problem 1.9.D].) Let* $k \geqslant 1$, $m \geqslant -1$ *be integers, and* X *a normal space. If each map* $f : F \to S^{k-1}$ *from any closed subset* F *of* X *has an extension from* $X \setminus P$ *to* S^{k-1}*, where* $P \subset X$ *is closed, does not meet* F*, and* $\dim P \leqslant m$*, then* $\dim X \leqslant k + m$. □

Corollary 2.5. *Let* $k \geqslant 1$, $m \geqslant -1$ *be integers, and* X *a normal space. If every closed* $\partial\Delta^k$*-tuple of* X *has a* $\partial\Delta^k$*-partition* P *such that* $\dim P \leqslant m$ *and the complement* $X \setminus P$ *is a normal space, then* $\dim X \leqslant k + m$.

Proof. In order to use Lisitsa's theorem, take a map $f : F \to |\partial\Delta^k|$, where $F \subset X$ is closed. Consider the closed $\partial\Delta^k$-cover $\mathcal{F} = f^{-1}(\mathcal{K})$ of F. If $\mathcal{F}$ has a $\partial\Delta^k$-neighbourhood $\mathcal{U}$ such that the corresponding $\partial\Delta^k$-partition $P = X \setminus \bigcup \mathcal{U}$ satisfies the inequality $\dim P \leqslant m$ and $U = \bigcup \mathcal{U}$ is normal, then f extends to a map $f' : U \to |\partial\Delta^k|$ by Lemma 2.3. Therefore, $\dim X \leqslant k + m$ by Lisitsa's theorem. □

It is clear why the extension Lemma 2.3 and the upper bound of the covering dimension in Theorem 2.4 need a normality assumption. The natural range of applications of Corollary 2.5 is the class of hereditarily normal spaces. In view of [2, Lemma 6], the corollary implies Lisitsa's theorem for any hereditarily normal space X. They both should be compared with [5, Problem 2.2.B]—it is easily checked that they all three together imply Theorem 1.9. We do not know *if either the hereditary normality or the normality of the complement in the corollary is a necessary assumption.*

Lemma 2.6. *Suppose that X is a metric space, $\mathcal{U}$ is an open K-tuple of X, and $P = X \setminus \bigcup \mathcal{U}$ is the corresponding K-partition. If* $\operatorname{Ind} K\text{-obs}\,\mathcal{U} < \operatorname{Ind} P \in \mathbb{N}$, *then $\mathcal{U}$ has a K-neighbourhood whose corresponding K-partition Q has* $\operatorname{Ind} Q < \operatorname{Ind} P$.

Proof. Write $m = \operatorname{Ind} P$. At first, we shall prove the lemma under the assumption that $K\text{-obs}\,\mathcal{U} = \emptyset$. Then, let

$$W_i = \bigcup \{V_i\colon\ \mathcal{V} = (V_0, \ldots, V_k) \text{ is a } K\text{-neighbourhood of } \mathcal{U}\}$$

for $i = 0, \ldots, k$. Since $K\text{-obs}\,\mathcal{U} = \emptyset$, the sets W_i form an open cover of X, and the cover has a closed shrinking that consists of sets $F_i \subset W_i$. For each i, there exists an open set W_i' such that $F_i \subset W_i' \subset \operatorname{cl} W_i' \subset W_i$ and $\operatorname{Ind}(P \cap \operatorname{bd} W_i') < m$ [5, Theorem 4.1.13]. Let $W_0'' = W_0'$ and $W_i'' = W_i' \setminus \operatorname{cl}(W_0' \cup \cdots \cup W_{i-1}')$ for $0 < i \leqslant k$. From the two facts that the sets W_i' cover X and $\operatorname{bd} W_i'' \subset \operatorname{bd} W_0' \cup \cdots \cup \operatorname{bd} W_i'$, we infer that $Q = P \setminus (W_0'' \cup \cdots \cup W_k'') \subset P \cap (\operatorname{bd} W_0' \cup \cdots \cup \operatorname{bd} W_k')$. We obtain $\operatorname{Ind} Q < m$ by the countable sum theorem [5, Theorem 4.1.9]. As easily checked, the unions $V_i = U_i \cup W_i''$ form a K-neighbourhood $\mathcal{V}$ of $\mathcal{U}$, and $Q = X \setminus \bigcup \mathcal{V}$.

Assume that $\operatorname{Ind} K\text{-obs}\,\mathcal{U} < m$. Let $X_0 = X \setminus K\text{-obs}\,\mathcal{U}$ and $P_0 = P \setminus K\text{-obs}\,\mathcal{U}$. Then, $\mathcal{U}$ has no K-obstruction points in X_0, and by the first part of the proof, there exists a K-neighbourhood $\mathcal{V}$ of $\mathcal{U}$ in X_0 with the corresponding K-partition $Q_0 = X_0 \setminus \bigcup \mathcal{V}$ and $\operatorname{Ind} Q_0 < m$. Now, $Q = Q_0 \cup K\text{-obs}\,\mathcal{U}$ corresponds to $\mathcal{V}$ in X, and $\operatorname{Ind} Q < m$ by the countable sum theorem. □

The foregoing lemma is also true when X is a *strongly hereditarily normal space* (see [5, Definition 2.1]). To prove this, one applies [5, the statements 2.2.4, 2.3.6 and 2.3.7] instead of theorems on dimension in the class of metric spaces.

Proposition 2.7. *Let $k \geqslant 1$ and $m \geqslant 0$. If X is a metric space with $\dim X \geqslant k + m$, then there exists a closed $\partial\Delta^k$-tuple $\mathcal{F}$ of X such that every $\partial\Delta^k$-neighbourhood $\mathcal{U}$ of $\mathcal{F}$ satisfies the following alternative*: $\dim \partial\Delta^k\text{-obs}\,\mathcal{U} = m$ *or the corresponding $\partial\Delta^k$-partition $P = X \setminus \bigcup \mathcal{U}$ has $\dim P > m$.*

In particular, if $n \geqslant 1$ and $\dim X = k(n + 1) - 1$, then $\partial\Delta^k\text{-Ind}\, X = n$ and $\partial\Delta^k\text{-str}\, X = 1$.

Proof. Let X be metric, and $\dim X \geqslant k+m$. By Corollary 2.5, there is a closed $\partial\Delta^k$-tuple $\mathcal{F}$ whose every $\partial\Delta^k$-neighbourhood $\mathcal{U}$ has $\dim(X \setminus \bigcup \mathcal{U}) \geqslant m$. Thus, if $P = X \setminus \bigcup \mathcal{U}$ and $\dim P = m$, then $\dim \partial\Delta^k\text{-obs}\,\mathcal{U} = m$ by Lemma 2.6.

If $\dim X = k(n + 1) - 1$, then Theorem 1.9 implies that $\partial\Delta^k\text{-Ind}\, X = n$. For $m = kn - 1 \geqslant 0$, let $\mathcal{F}$ be a closed $\partial\Delta^k$-tuple whose every $\partial\Delta^k$-neighbourhood $\mathcal{U}$ satisfies the stated alternative. If $P = X \setminus \bigcup \mathcal{U}$ has $\partial\Delta^k\text{-Ind}\, P < n$, then $\dim P \leqslant kn - 1$ by Theorem 1.9, and $\dim \partial\Delta^k\text{-obs}\,\mathcal{U} = kn - 1$. This means that $\partial\Delta^k\text{-str}\, X = 1$. □

Proposition 2.8. *Let $k \geqslant 2$ and $n \geqslant 1$. If X is a metric space with $\dim X = kn$, then $\partial\Delta^k\text{-Ind}\, X = n$ and $\partial\Delta^k\text{-str}\, X = 0$.*

Proof. If X is metric and $\dim X = kn$, then $\partial\Delta^k\text{-Ind}\, X = n$ by Theorem 1.9.

Take a closed $\partial\Delta^k$-tuple $\mathcal{F}$ of X, and find an open $\partial\Delta^k$-neighbourhood $\mathcal{V}$ of $\mathcal{F}$. There is an open set W with $\bigcup \mathcal{F} \subset W \subset \operatorname{cl} W \subset \bigcup \mathcal{V}$ and $\dim \operatorname{bd} W < kn$. Put $U_i = V_i \cap W$ for $i = 0, \ldots, k - 1$, $U_k = (V_k \cap W) \cup (X \setminus \operatorname{cl} W)$, and $P = \operatorname{bd} W$. Using Lemma 1.10 and the inequality $k \geqslant 2$, one easily checks that $\partial\Delta^k\text{-obs}\,\mathcal{U} = \emptyset$ for $\mathcal{U} = (U_0, \ldots, U_k)$. Finally, we obtain $\partial\Delta^k\text{-Ind}\, P < n$ by Theorem 1.9. Therefore, $\partial\Delta^k\text{-str}\, X = 0$. □

In Propositions 2.7–2.8 we have shown that if $k \geqslant 2$ and $1 \leqslant n \in \mathbb{N}$, then there are two degrees to which a compact metric space X may have $\partial\Delta^k\text{-Ind}\, X = n$. Maybe there are more such (similar) degrees, but at this moment we have neither good motivation nor good examples, which could help us to identify and point out appropriate combinatorial properties of spaces in terms of K-neighbourhoods and K-partitions.

The formulas (a) and (c) in the next statement are generalisations of P. Vopěnka's theorem [12, p. 320] on the classical Ind.

Theorem 2.9. *If X and Y are compact spaces and $\dim X = 0$, then*

(a) $K\text{-Ind}(X \times Y) = K\text{-Ind}\, Y$,
(b) $K\text{-str}(X \times Y) = K\text{-str}\, Y$, *and*
(c) $L\text{-Ind}(X \times Y) = L\text{-Ind}\, Y$.

Proof. (a) Evidently $K\text{-Ind}(X \times Y) \geqslant K\text{-Ind}\, Y$. We prove "$\leqslant$" by induction on $\alpha = K\text{-Ind}\, Y$. If $\alpha = -1$, we are done. Assume that $\alpha \geqslant 0$. Write $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ for the projections. Take a closed K-tuple $\mathcal{F}$ of $X \times Y$. For any point $x \in X$ consider the sets $\pi_Y(F_i \cap \pi_X^{-1}(x)) \subset Y$, $i = 0, \ldots, k$. For this closed K-tuple of Y, take a K-neighbourhood $\mathcal{U}^x$ whose corresponding K-partition P^x has $K\text{-Ind}\, P^x < \alpha$. Since Y is compact, π_X is a closed map and the image $\pi_X(F_i \setminus (X \times U_i^x)) \not\ni x$ is a closed subset of X for each i. Hence, there is a neighbourhood $V^x \ni x$ such that $F_i \cap \pi_X^{-1}(V^x) \subset X \times U_i^x$ for each i. Take a finite clopen refinement $\{W^s\colon\ s \in S\}$ of $\{V^x\colon\ x \in X\}$ consisting of disjoint sets. For each s fix a point x_s with $W^s \subset V^{x_s}$. We have $F_i \cap \pi_X^{-1}(W^s) \subset W^s \times U_i^{x_s}$ for each i and s. The sets $U_i = \bigcup_{s \in S} W^s \times U_i^{x_s}$, $i = 0, \ldots, k$, form a K-neighbourhood $\mathcal{U}$ of $\mathcal{F}$. Note the fact, which will be needed in a while, that

(∗) *if* K-obs$\mathcal{U}^{x_s} = \emptyset$ *for each* $s \in S$, *then* K-obs$\mathcal{U} = \emptyset$.

By the obvious induction hypothesis, $K\text{-Ind}(W^s \times P^{x_s}) < \alpha$ for each s. Finally, $P = (X \times Y) \setminus \bigcup \mathcal{U} = \bigcup_{s \in S}(W^s \times P^{x_s})$ is a K-partition for $\mathcal{F}$, and $K\text{-Ind}\, P < \alpha$. We have shown that $K\text{-Ind}(X \times Y) \leqslant \alpha = K\text{-Ind}\, Y$.

(b) In view of Proposition 1.11, we infer that if $K\text{-str}(X \times Y) = 0$, then $K\text{-str}\, Y = 0$. The converse becomes justified when analysing the proof in the previous paragraph, we moreover consider the implication (∗).

(c) Again $L\text{-Ind}(X \times Y) \geqslant L\text{-Ind}\, Y$. Write $\alpha = L\text{-Ind}\, Y$. If $\alpha = -1$, the equality (c) holds. Assume that $\alpha \geqslant 0$. Consider the Hilbert cube $Q = [-1, 2]^{\aleph_0}$ equipped with the metric $\varrho((s_i)_{i=0}^{\infty}, (t_i)_{i=0}^{\infty}) = \sum_{i=0}^{\infty} 2^{-i}|s_i - t_i|$, and assume that $L \subset [0, 1]^{\aleph_0}$. There exists a neighbourhood $R \subset Q$ of L with a map $r : R \to L$ such that $r(t) = t$ for $t \in L$. Let $\varepsilon = \inf\{\varrho(s, t)\colon\ s \in L,\ t \in Q \setminus R\}$. Take an arbitrary closed set $F \subset X \times Y$ with a map $f : F \to L$. Since L is an ANR, there exists an open neighbourhood U of F with an extension $g : \mathrm{cl}\, U \to L$ of f. For each point $x \in X$, consider the open set $U^x = \pi_Y(U \cap \pi_X^{-1}(x))$, the closed set $G^x = \pi_Y(\mathrm{cl}\, U \cap \pi_X^{-1}(x))$, and the map $g^x : G^x \to L$, $g^x(b) = g(x, b)$ for $b \in G^x$. In Y there is an L-partition P^x for g^x with $L\text{-Ind}\, P^x < \alpha$ and with an extension $\psi^x : Y \setminus P^x \to L$ of g^x. As π_X is a closed map, $\pi_X(F \setminus (X \times U^x)) \not\ni x$ is closed in X. There is a neighbourhood N^x of x with $F \cap \pi_X^{-1}(\mathrm{cl}\, N^x) \subset X \times U^x$. Writing as usually $(s_i)_{i=0}^{\infty} \pm (t_i)_{i=0}^{\infty} = (s_i \pm t_i)_{i=0}^{\infty}$, we set

$$d^x : \pi_X^{-1}(x) \cup \bigl(F \cap \pi_X^{-1}\bigl(\mathrm{cl}\, N^x\bigr)\bigr) \to [-1, 1]^{\aleph_0},$$
$$d^x(a, b) = \begin{cases} g(a, b) - g(x, b) & \text{for } (a, b) \in F \cap \pi_X^{-1}(\mathrm{cl}\, N^x), \\ 0 & \text{for } a = x,\, b \in Y. \end{cases}$$

The function d^x is correctly defined and continuous. Let $e^x : X \times Y \to [-1, 1]^{\aleph_0}$ be an extension of d^x. Consider the point $o = (0, 0, \ldots) \in Q$, the open ball $\mathrm{B}(o, \varepsilon)$, and the closed set $\pi_X[(X \times Y) \setminus (e^x)^{-1}(\mathrm{B}(o, \varepsilon))] \not\ni x$. There is a neighbourhood $V^x \subset N^x$ of x with $\pi_X^{-1}(V^x) \subset (e^x)^{-1}(\mathrm{B}(o, \varepsilon))$. Again, we take a finite clopen refinement $\{W^s\colon\ s \in S\}$ of $\{V^x\colon\ x \in X\}$, where the sets W^s are pairwise disjoint. We fix points x_s with $W^s \subset V^{x_s}$, and we obtain $F \cap \pi_X^{-1}(W^s) \subset W^s \times U^{x_s}$. By the obvious induction hypothesis, $L\text{-Ind}(W^s \times P^{x_s}) < \alpha$ for each s, and $L\text{-Ind}\, P < \alpha$ for $P = \bigcup_{s \in S}(W^s \times P^{x_s})$. There remains to observe that the map

$$\varphi : (X \times Y) \setminus P \to L,$$
$$\varphi(a, b) = r\bigl(\psi^{x_s}(b) + e^{x_s}(a, b)\bigr) \quad \text{for } a \in W_s \text{ and } b \in Y \setminus P^{x_s}$$

is correctly defined and extends f. Indeed, $\psi^{x_s}(b) + e^{x_s}(a, b) \in R$ since $\psi^{x_s}(b) \in L \subset [0, 1]^{\aleph_0}$ and $e^{x_s}(a, b) \in [-1, 1]^{\aleph_0} \cap \mathrm{B}(o, \varepsilon)$. If $(a, b) \in F \cap \pi_X^{-1}(W^s)$, then $b \in U^{x_s} \subset G^{x_s}$, $\psi^{x_s}(b) + e^{x_s}(a, b) = g(x_s, b) + d^{x_s}(a, b) = g(a, b) \in L$ and $\varphi(a, b) = g(a, b) = f(a, b)$. Therefore, P is an L-partition for f, and $L\text{-Ind}(X \times Y) \leqslant \alpha$. □

The foregoing proof also works in the case when X is paracompact and $K\text{-Ind}\, Y$, $L\text{-Ind}\, Y$ are integers (we need a compact Y and $\dim X = 0$, of course).

3. Spreading out compact spaces in a plank

Any suitably chosen subspace of a product or a product itself is sometimes called a plank. We shall additionally compress one of the product's faces into one of the factors.

Suppose that X and Y are non-empty compact spaces. We shall recall the definition of the space $Z(X, Y)$, and investigate its properties (cf. [10]). To begin, write $\mathcal{S}_X$ for the family of all subsets of X that are either finite (so $\emptyset \in \mathcal{S}_X$), or homeomorphic to $A_{\aleph_0}$. Let $\mathfrak{m} \geqslant \max\{\aleph_0, (wX)^+, (wY)^+, \mathrm{card}\, \mathcal{S}_X\}$, where wX and wY denote the weights of X and Y, and put $M = A_{\mathfrak{m}} \times X \times Y$. Let $\pi_1 : M \to N$ be the quotient map that compresses sets $\{(\mu, x, y) \in M\colon\ y \in Y\}$ for all $x \in X$ into points—here N is the compact quotient space.

Given any function $\varphi : A_{\mathfrak{m}} \setminus \{\mu\} \to \mathcal{S}_X$ such that $\mathrm{card}\, \varphi^{-1}(S) = \mathfrak{m}$ for every $S \in \mathcal{S}_X$, we put

$$H(\alpha) = \begin{cases} \pi_1(\{\mu\} \times X \times Y) & \text{for } \alpha = \mu, \\ \pi_1(\{\alpha\} \times \varphi(\alpha) \times Y) & \text{for } \alpha \neq \mu, \end{cases} \quad \text{and} \quad Z(X, Y) = \bigcup_{\alpha \in A_{\mathfrak{m}}} H(\alpha).$$

(We slightly change the notation originating in [10].)

Proposition 3.1. *([10, Section 1]) $Z(X, Y)$ is a compact space. Every component of $Z(X, Y)$ is homeomorphic to some component of X or Y. If X and Y are Fréchet spaces, then so is $Z(X, Y)$.* □

The following results from Theorem 1.8.

Lemma 3.2. $L\text{-dim}\, Z(X, Y) = \max\{L\text{-dim}\, X, L\text{-dim}\, Y\}$. □

Write $\pi_X : Z(X,Y) \to X$ and $\pi_{A_\mathfrak{m}} : Z(X,Y) \to A_\mathfrak{m}$ for projections, i.e. the unique maps such that $\pi_X(\pi_1(\alpha,x,y)) = x$ and $\pi_{A_\mathfrak{m}}(\pi_1(\alpha,x,y)) = \alpha$ for every $(\alpha,x,y) \in \pi_1^{-1}(Z(X,Y))$. Note that $\pi_{A_\mathfrak{m}}^{-1}(\alpha) = H(\alpha)$ for $\alpha \in A_\mathfrak{m}$, the restriction $\pi_X|H(\mu)$ is a homeomorphism onto X, and $H(\alpha)$ is homeomorphic to $\varphi(\alpha) \times Y$ for every $\alpha \neq \mu$. A base of neighbourhoods of a point $\pi_1(\mu,x,y) \in H(\mu)$ consists of sets of the form $\pi_{A_\mathfrak{m}}^{-1}(A) \cap \pi_X^{-1}(U)$, where $\mu \in A \subset A_\mathfrak{m}$, the complement $A_\mathfrak{m} \setminus A$ is finite, and $U \subset X$ is a neighbourhood of x.

The space $Z(X,Y)$ depends on the choice of $\mathfrak{m}$, but this is insignificant in the present paper. The dependence on φ is superficial because another function ψ with $\operatorname{card}\psi^{-1}(S) = \mathfrak{m}$ for $S \in \mathcal{S}_X$ would yield a new space homeomorphic to the former $Z(X,Y)$. Indeed, there would be a bijection $\xi : A_\mathfrak{m} \setminus \{\mu\} \to A_\mathfrak{m} \setminus \{\mu\}$ such that $\varphi = \psi \circ \xi$. The homeomorphism in question would have fixed points of the form $\pi_1(\mu,x,y)$, and would carry

$$H(\alpha) \ni \pi_1(\alpha,x,y) \mapsto \pi_1\big(\xi(\alpha),x,y\big) \in \pi_1\big(\big\{\xi(\alpha)\big\} \times \psi\big(\xi(\alpha)\big) \times Y\big)$$

for every $\alpha \neq \mu$. In particular, when $\mu \in A \subset A_\mathfrak{m}$ and $\operatorname{card}(A_\mathfrak{m} \setminus A) < \mathfrak{m}$, we can think that—roughly speaking—$\pi_{A_\mathfrak{m}}^{-1}(A)$ has the same properties as $Z(X,Y)$. On the other hand, given a non-empty closed set $F \subset X$, we can consider the function $\chi : A_\mathfrak{m} \setminus \{\mu\} \to \mathcal{S}_F$, $\chi(\alpha) = F \cap \varphi(\alpha)$, and it turns out that $\pi_X^{-1}(F)$ has the form of a $Z(F,Y) \subset \pi_1(A_\mathfrak{m} \times F \times Y)$.

The following statement is a simple modification (with the same proof) of [10, Lemma 1].

Lemma 3.3. *If $G \subset Z(X,Y)$ is a G_δ-set (so, also if G is open), then there is a set $A \subset A_\mathfrak{m}$ such that $\mu \in A$, $\operatorname{card}(A_\mathfrak{m} \setminus A) \leqslant \max\{wX, \aleph_0\}$, and*

$$\pi_{A_\mathfrak{m}}^{-1}(A) \cap \pi_X^{-1}\big(\pi_X\big(G \cap H(\mu)\big)\big) \subset G. \qquad \square$$

4. Compact spaces with *L*-dim < *L*-Ind, where *L* is an ANR

We go on to investigate the behaviour of L-Ind under the operation $Z(X,Y)$.

Lemma 4.1. $L\text{-Ind}\, Z(X,Y) \leqslant \max\{L\text{-Ind}\, X + 1, L\text{-Ind}\, Y\}$.

Proof. Take a closed subset F of $Z = Z(X,Y)$ and a map $f : F \to L$. Since L is an ANR, there exists a neighbourhood U of F with an extension $g : U \to L$ of f. The restriction $\pi_X|H(\mu)$ is a homeomorphism onto X, and hence, there are open subsets V_0, V_1 of X such that

$$\pi_X\big(F \cap H(\mu)\big) \subset V_0 \subset \operatorname{cl} V_0 \subset V_1 \subset \operatorname{cl} V_1 \subset \pi_X\big(U \cap H(\mu)\big).$$

Observe that $F \setminus \pi_X^{-1}(V_0)$ and $\pi_X^{-1}(\operatorname{cl} V_1) \setminus U$ are closed subsets of Z, and none of them meets $H(\mu)$. Their images under $\pi_{A_\mathfrak{m}}$ do not contain μ, and being closed, are finite. Thus,

$$A = A_\mathfrak{m} \setminus \big[\pi_{A_\mathfrak{m}}\big(F \setminus \pi_X^{-1}(V_0)\big) \cup \pi_{A_\mathfrak{m}}\big(\pi_X^{-1}(\operatorname{cl} V_1) \setminus U\big)\big] \ni \mu$$

is clopen in $A_\mathfrak{m}$. Moreover

$$F \cap \pi_{A_\mathfrak{m}}^{-1}(A) \subset \pi_X^{-1}(V_0) \quad \text{and} \quad \pi_X^{-1}(\operatorname{cl} V_1) \cap \pi_{A_\mathfrak{m}}^{-1}(A) \subset U.$$

For each $S \in \mathcal{S}_X$, let $x^S \in S$ be the limit of S whenever S is infinite. Choose a point $l_0 \in L$. For each $\alpha \in A \setminus \{\mu\}$, we shall define an extension $g'_\alpha : H(\alpha) \to L$ of the restriction $g|(\pi_X^{-1}(V_0) \cap H(\alpha))$. Consider $S = \varphi(\alpha)$. There are two cases. (1) If $\varphi(\alpha)$ is finite or $x^{\varphi(\alpha)} \in V_1$, then $V_1 \cap S$ is clopen in S and $W_\alpha = H(\alpha) \cap \pi_X^{-1}(V_1)$ is clopen in Z. (2) If $x^{\varphi(\alpha)} \notin V_1$, then $V_0 \cap S$ is clopen in S, and we put $W_\alpha = H(\alpha) \cap \pi_X^{-1}(V_0)$. Since $W_\alpha \subset U$ in both cases, we can set

$$g'_\alpha(z) = \begin{cases} g(z) & \text{for } z \in W_\alpha, \\ l_0 & \text{for } z \in H(\alpha) \setminus W_\alpha. \end{cases}$$

$L\text{-Ind}\, H(\alpha) = L\text{-Ind}\, Y$ for $\alpha \neq \mu$ by Theorem 2.9(c). If $\alpha \in A_\mathfrak{m} \setminus A$, then in $H(\alpha)$ we take an L-partition P_α with $L\text{-Ind}\, P_\alpha < L\text{-Ind}\, Y$ for the restriction $f|(F \cap H(\alpha))$. This means that $F \cap H(\alpha) \subset H(\alpha) \setminus P_\alpha$ and there is an extension $f'_\alpha : H(\alpha) \setminus P_\alpha \to L$ of $f|(F \cap H(\alpha))$.

Since $A_\mathfrak{m} \setminus A$ is finite, the set

$$P = \big(H(\mu) \setminus \pi_X^{-1}(V_0)\big) \cup \bigcup_{\alpha \in A_\mathfrak{m} \setminus A} P_\alpha \tag{†}$$

is closed in Z and $L\text{-Ind}\, P < \max\{L\text{-Ind}\, X + 1, L\text{-Ind}\, Y\}$. It is an L-partition for f because the function

$$f'(z) = \begin{cases} g(z) & \text{for } z \in \pi_{A_\mathfrak{m}}^{-1}(A) \cap \pi_X^{-1}(V_0), \\ g'_\alpha(z) & \text{for } z \in H(\alpha), \text{ where } \mu \neq \alpha \in A, \text{ and} \\ f'_\alpha(z) & \text{for } z \in H(\alpha) \setminus P_\alpha, \text{ where } \alpha \in A_\mathfrak{m} \setminus A, \end{cases}$$

is correctly defined on $Z \setminus P$, continuous, and extends f. $\square$

Lemma 4.2. *Suppose that X is a non-empty, compact Fréchet space, $F \subset B \subset X$ are closed, and $f : F \to L$ is a map that does not extend to a map from B to L. Let $G = \pi_X^{-1}(F) \cap H(\mu)$ and $g = f \circ (\pi_X|G) : G \to L$. If P is an L-partition in $Z = Z(X,Y)$ for g, then one of the following conditions is satisfied*:

(a) $B \cap \operatorname{int} \pi_X(P \cap H(\mu)) \neq \emptyset$;
(b) *there is an $\alpha \neq \mu$ such that $\varphi(\alpha) \in \mathcal{S}_X$ is infinite and $\pi_{A_\mathfrak{m}}^{-1}(\alpha) \cap \pi_X^{-1}(x^{\varphi(\alpha)}) \subset P$, where $x^{\varphi(\alpha)} \in B \cap \varphi(\alpha)$ is the limit point of $\varphi(\alpha)$ (and the intersection of the point-inverses is homeomorphic to Y).*

Proof. We need Borsuk's homotopy extension theorem in the following formulation: *Suppose that $f_1, f_2 : F \to L$ are homotopic maps from a closed subspace F of a compact space B into an ANR L. Then f_1 has an extension from B to L iff f_2 has such an extension* (cf. [5, Lemma 1.9.7 and its proof]).

By West's Theorem 1.5, there exists a polyhedron $|K|$ with maps $\gamma_1 : L \to |K|$, $\gamma_2 : |K| \to L$ such that $\gamma_2 \circ \gamma_1 \simeq \mathrm{id}_L$ (the composition is homotopic to the identity id_L on L) and $\gamma_1 \circ \gamma_2 \simeq \mathrm{id}_{|K|}$. Evidently $f \simeq \gamma_2 \circ \gamma_1 \circ f$. It follows from the homotopy extension theorem that $\gamma_1 \circ f$ does not extend to a map from B to $|K|$. Moreover, each L-partition in Z for g is a $|K|$-partition for $\gamma_1 \circ f \circ (\pi_X|G)$. Thus, we can *assume without loss of generality that $L = |K|$, and f, g are maps into $|K|$.*

Consider the closed K-cover $\mathcal{K}$ of $|K|$ (see the definition before Lemma 2.3), and take an open swelling $\mathcal{U}$ of $\mathcal{K}$ such that $\operatorname{cl}\mathcal{U}$ is a K-tuple of $|K|$.

Take any $|K|$-partition $P \subset Z \setminus G$ for g, and assume that the interior $\operatorname{int} \pi_X(P \cap H(\mu))$ does not meet B. Let $g' : Z \setminus P \to |K|$ be an extension of g. Consider the open K-cover $\mathcal{V} = g'^{-1}(\mathcal{U})$ of $Z \setminus P$. Remembering that $\pi_X|H(\mu)$ is a homeomorphism onto X, write $\mathcal{W} = \pi_X(\mathcal{V}|H(\mu))$ and note that $\mathcal{W}$ is a K-neighbourhood of $f^{-1}(\mathcal{K})$. In B choose an open swelling $\mathcal{H}$ of $(\operatorname{cl}\mathcal{W})|B$. We have $B = \bigcup \mathcal{H}$ since $B \subset \operatorname{cl} \pi_X(H(\mu) \setminus P) = \bigcup \operatorname{cl} \mathcal{W}$. It follows that $(\operatorname{cl}\mathcal{W})|B$ is not a K-tuple (in the other case, $\mathcal{H}$ would be a K-neighbourhood of $f^{-1}(\mathcal{K})$, and f would have an extension from B to $|K|$ by Lemma 2.3). Therefore, there is an $x_0 \in B \cap \bigcap_{i \in I} \operatorname{cl} W_i$, where $I \subset \{0, \dots, k\}$ and the simplex with vertices e_i, $i \in I$, does not belong to K. Write z_0 for the unique point in $H(\mu) \cap \pi_X^{-1}(x_0)$. If x_0 were in some W_i, then we would obtain $z_0 \in (\bigcap_{i \in I} \operatorname{cl} V_i) \setminus P$ and $g'(z_0) \in \bigcap_{i \in I} \operatorname{cl} U_i$, which would contradict the fact that $\operatorname{cl}\mathcal{U}$ is a K-tuple. Therefore $x_0 \notin \bigcup_{i=0}^k W_i$. For each $i \in I$, take a sequence $S_i \subset W_i$ converging to x_0 (X is Fréchet), and put $S = \{x_0\} \cup \bigcup_{i=0}^k S_i$. By Lemma 3.3, there is a set $A \subset A_\mathfrak{m}$ with $\mu \in A$, $\operatorname{card}(A_\mathfrak{m} \setminus A) < \mathfrak{m}$, and $\pi_{A_\mathfrak{m}}^{-1}(A) \cap \pi_X^{-1}(W_i) \subset V_i$ for each $i \in I$. As $\operatorname{card} \varphi^{-1}(S) = \mathfrak{m}$, we can find an $\alpha \in A \setminus \{\mu\}$ such that $\varphi(\alpha) = S$.

If $i \in I$, then every point $\pi_1(\alpha, x_0, y) \in \pi_1(\{\alpha\} \times \{x_0\} \times Y)$ is the limit of the sequence $\pi_1(\{\alpha\} \times S_i \times \{y\}) \subset V_i = g'^{-1}(U_i)$. If we had $\pi_1(\alpha, x_0, y) \notin P$, then we would obtain $g'(\pi_1(\alpha, x_0, y)) \in \operatorname{cl} U_i$ for $i \in I$, and $\bigcap_{i \in I} \operatorname{cl} U_i$ would be non-empty. As $\operatorname{cl}\mathcal{U}$ is a K-tuple, we infer that $\pi_{A_\mathfrak{m}}^{-1}(\alpha) \cap \pi_X^{-1}(x_0) = \pi_1(\{\alpha\} \times \{x_0\} \times Y) \subset P$. Finally, we can write $x^{\varphi(\alpha)} = x_0$. $\square$

Let X be a normal space and $b \in X$. Bearing in mind the convention that ∞ is bigger than any ordinal, we define

$$L\text{-}\mathrm{Ind}_{b+} X = \min\{\alpha\colon \text{there is a neighbourhood } U \text{ of } b \text{ with } L\text{-}\mathrm{Ind} \operatorname{cl} U \leqslant \alpha\}.$$

Note that if $B \subset X$ is closed and $b \in B$, then $L\text{-}\mathrm{Ind}_{b+} B \leqslant L\text{-}\mathrm{Ind}_{b+} X \leqslant L\text{-}\mathrm{Ind}\, X$.

Lemma 4.3. *Suppose that X is a non-empty, compact Fréchet space, and B is a closed subspace of X. Let $z \in H(\mu)$ be any point such that $c = \pi_X(z) \in B$ and $L\text{-}\mathrm{Ind}_{c+} B \geqslant 1$. If $L\text{-}\mathrm{Ind}_{b+} X \geqslant \alpha$ for each $b \in B$, then*

$$L\text{-}\mathrm{Ind}_{z+} Z(X,Y) \geqslant \min\{\alpha, L\text{-}\mathrm{Ind}\, Y\} + 1.$$

Proof. It suffices to show that $L\text{-}\mathrm{Ind}(\pi_{A_\mathfrak{m}}^{-1}(A) \cap \pi_X^{-1}(\operatorname{cl} U)) \geqslant \min\{\alpha, L\text{-}\mathrm{Ind}\, Y\} + 1$ for any base neighbourhood $\pi_{A_\mathfrak{m}}^{-1}(A) \cap \pi_X^{-1}(U)$ of z, where $\mu \in A \subset A_\mathfrak{m}$, $A_\mathfrak{m} \setminus A$ is finite, and U is a neighbourhood of c. Let $V \ni c$ be open in X and $\operatorname{cl} V \subset U$. We have $L\text{-}\mathrm{Ind}(B \cap \operatorname{cl} V) \geqslant 1$ as $L\text{-}\mathrm{Ind}_{c+} B \geqslant 1$, and there is a closed set $F \subset B \cap \operatorname{cl} V$ with a map $f : F \to L$ that does not have an extension from $B \cap \operatorname{cl} V$ to L. Let $G = \pi_X^{-1}(F) \cap H(\mu)$ and $g = f \circ (\pi_X|G)$. Take an arbitrary L-partition P for g in $\pi_{A_\mathfrak{m}}^{-1}(A) \cap \pi_X^{-1}(\operatorname{cl} U)$, which has the form of $Z(\operatorname{cl} U, Y) \subset \pi_1(A \times \operatorname{cl} U \times Y)$. By Lemma 4.2, two cases may arise. (1) Some $b \in B \cap \operatorname{cl} V$ is an interior point of $\pi_X(P \cap H(\mu))$ in $\operatorname{cl} U$. Then there is a neighbourhood $W \subset U \cap \pi_X(P \cap H(\mu))$ of b in X. In consequence, $L\text{-}\mathrm{Ind}\, P \geqslant L\text{-}\mathrm{Ind}(\pi_X^{-1}(\operatorname{cl} W) \cap H(\mu)) = L\text{-}\mathrm{Ind} \operatorname{cl} W \geqslant \alpha$ because $L\text{-}\mathrm{Ind}_{b+} X \geqslant \alpha$. (2) P contains a homeomorphic copy of Y, and then $L\text{-}\mathrm{Ind}\, P \geqslant L\text{-}\mathrm{Ind}\, Y$. Thus, $L\text{-}\mathrm{Ind}\, P \geqslant \min\{\alpha, L\text{-}\mathrm{Ind}\, Y\}$ in both cases, which proves the lemma. $\square$

For any normal space X, let us write

$$K(X) = \{b \in X\colon L\text{-}\mathrm{Ind}_{b+} X = L\text{-}\mathrm{Ind}\, X\}.$$

Observe that $K(X)$ is a closed subset of X.

Theorem 4.4. *Suppose that X and Y are non-empty compact spaces, and X is Fréchet. If $L\text{-}\mathrm{Ind}\, X = L\text{-}\mathrm{Ind}\, Y$ and $L\text{-}\mathrm{Ind}\, K(X) \geqslant 1$, then*

$$L\text{-}\mathrm{Ind}\, Z(X,Y) = L\text{-}\mathrm{Ind}\, X + 1.$$

Proof. The inequality "$\leqslant$" results from Lemma 4.1.

Assume that $L\text{-Ind}\,K(X) \geqslant 1$. The equality $L\text{-Ind} = 0$ is equivalent to $L\text{-dim} = 0$. We claim that *there is a point $c \in K(X)$ with $L\text{-Ind}_{c+} K(X) \geqslant 1$*. In the other case, using the compactness of $K(X)$, we could cover $K(X)$ by sets $U_1, \ldots, U_n$ open in $K(X)$ and such that $L\text{-Ind}\,\mathrm{cl}\,U_i = 0$ for $i = 1, \ldots, n$. By the countable sum theorem for L-dim (Fedorchuk [6, Proposition 5.1]), we would obtain the equalities $L\text{-dim}\,K(X) = 0 = L\text{-Ind}\,K(X)$ and a contradiction. Therefore, we can put $B = K(X)$ and apply Lemma 4.3. □

Lemma 4.5. *If X is a separable metric space with $L\text{-Ind}\,X = n \in \mathbb{N}$, then $K(X)$ is non-empty, and $L\text{-Ind}_{b+} K(X) = n$ for each $b \in K(X)$.*

Proof. Theorem 1.7 implies that $L\text{-dim} = L\text{-Ind}$ for closed subspaces of X. X has a countable base $\mathcal{B}$, and $X \setminus K(X)$ is the union of a sequence $\mathrm{cl}\,U_i$, where $U_i \in \mathcal{B}$ and $L\text{-Ind}\,\mathrm{cl}\,U_i < n$ for $i = 0, 1, \ldots$. If we had $L\text{-Ind}\,K(X) < n$, then we would obtain $L\text{-Ind}\,X < n$ by the countable sum theorem for L-dim (Fedorchuk [6, Proposition 5.1]). Thus $L\text{-Ind}\,K(X) = n$.

Let $b \in K(X)$, and U be a neighbourhood of b in $K(X)$. Using the hereditary normality of X, one can find a neighbourhood V of b in X such that $U = V \cap K(X)$ and $\mathrm{cl}\,U = \mathrm{cl}\,V \cap K(X)$. Then $L\text{-Ind}\,\mathrm{cl}\,V = n$. By the same argument as in the first paragraph, we infer that $L\text{-Ind}\,\mathrm{cl}\,U = n$. Therefore $L\text{-Ind}_{b+} K(X) \geqslant n$. □

Theorem 4.6. *Let L be a compact metric ANR. Suppose that C is a metric continuum with $1 \leqslant n = L\text{-dim}\,C < \infty$. For each ordinal $\alpha \geqslant n$, there exists a compact Fréchet space $X_{C,\alpha}$ such that*

(a) *$L\text{-dim}\,X_{C,\alpha} = n$,*
(b) *$L\text{-Ind}\,X_{C,\alpha} = \alpha$, and*
(c) *each component of $X_{C,\alpha}$ is homeomorphic to C.*

Proof. $K(C)$ is closed in C, and $n = L\text{-Ind}_{b+} K(C) \leqslant L\text{-Ind}_{b+} C \leqslant n$ for each $b \in K(C)$ (Lemma 4.5). By transfinite induction on α, we shall construct compact Fréchet spaces $X_{C,\alpha}$, $\alpha \geqslant n$, and closed subspaces $B_\alpha \subset X_{C,\alpha}$ such that

(a) every component of $X_{C,\alpha}$ is homeomorphic to C;
(b) B_α is homeomorphic to $K(C)$;
(c) $L\text{-Ind}\,X_{C,\alpha} \leqslant \alpha$; and
(d) $L\text{-Ind}_{b+} X_{C,\alpha} \geqslant \alpha$ for each $b \in B_\alpha$.

For $\alpha = n$, let $X_{n,n} = C$ and $B_n = K(C)$. Assume $X_{C,\alpha} \supset B_\alpha$ are compact, Fréchet, and satisfy (a)–(d). Let $X = Y = X_{C,\alpha}$, $\mathfrak{m} = \max\{(wX_{C,\alpha})^+, \mathrm{card}\,\mathcal{S}_{X_{C,\alpha}}\}$, $X_{C,\alpha+1} = Z(X_{C,\alpha}, X_{C,\alpha})$, and $B_{\alpha+1} = H(\mu) \cap \pi_X^{-1}(B_\alpha)$. By Proposition 3.1, $X_{C,\alpha+1}$ is Fréchet, and each of its components is homeomorphic to C. The restriction $\pi_X|B_{\alpha+1}$ is a homeomorphism onto B_α. $L\text{-Ind}\,X_{C,\alpha+1} \leqslant \alpha + 1$ by Lemma 4.1, and $L\text{-Ind}_{b+} X_{C,\alpha+1} \geqslant \alpha + 1$ for each $b \in B_{\alpha+1}$ by Lemma 4.3.

Assume that α is a limit ordinal, and there are $X_{C,\beta} \supset B_\beta$ for $\beta < \alpha$. Let D be the one-point compactification of the topological sum $\bigoplus_{\beta<\alpha} X_{C,\beta}$, and $d_0 \in D$ the unique point in the remainder. In the disjoint sum of C and D, identify d_0 with a point $c_0 \in C$, and call the resulting space Y. Using the fact that $A_\mathfrak{n}$ is Fréchet for every $\mathfrak{n}$, one routinely checks that Y is Fréchet. It follows from Lemma 2.2 that $L\text{-Ind}\,Y = \alpha$. Put $X = C$, $\mathfrak{m} = 2^{\aleph_0} + (\sup\{wX_{C,\beta}\colon \beta < \alpha\})^+$, $X_{C,\alpha} = Z(X, Y)$, and $B_\alpha = \pi_X^{-1}(K(C)) \cap H(\mu)$. $X_{C,\alpha}$ is Fréchet, (a), (c) are satisfied (see Proposition 3.1 and Lemma 4.1), and (b), (d) are evident.

The conditions (c), (d) yield the equality $L\text{-Ind}\,X_{C,\alpha} = \alpha$, and $L\text{-dim}\,X_{C,\alpha} = n$ by Theorem 1.8. □

Remark 4.7. The foregoing construction is essentially the same as the one in the proof of [10, Theorem 5] (see Remarks 3–4 therein), which yields a compact Fréchet space $X_{C,\alpha}$ with $\dim X_{C,\alpha} = n$, $\mathrm{trind}\,X_{C,\alpha} = \mathrm{trInd}\,X_{C,\alpha} = \alpha$, and with components homeomorphic to C. The proofs of Lemmas 4.1 and 4.3 in the present paper are more complex than the proofs of corresponding Lemmas 6 and 7 in [10].

We may add at this place that Lemma 7 in [10] needs one more assumption (necessary but missed out): the space B in that statement should be a non-degenerate continuum (then each component of any non-empty open subspace of B is uncountable).

Proposition 2.1 and Theorem 4.6 yield

Corollary 4.8. *Let L be a non-contractible, compact metric ANR, $1 \leqslant n \in \mathbb{N}$, and let $\alpha \geqslant n$ be an ordinal. If $n = 1$ or the join $L * L$ is non-contractible, then there exists a compact Fréchet space $X_{n,\alpha}$ such that*

(a) *$L\text{-dim}\,X_{n,\alpha} = n$,*
(b) *$L\text{-Ind}\,X_{n,\alpha} = \alpha$, and*
(c) *each component of $X_{n,\alpha}$ is homeomorphic to a cube $[0,1]^m$ for a certain natural number $m = m(L,n)$.* □

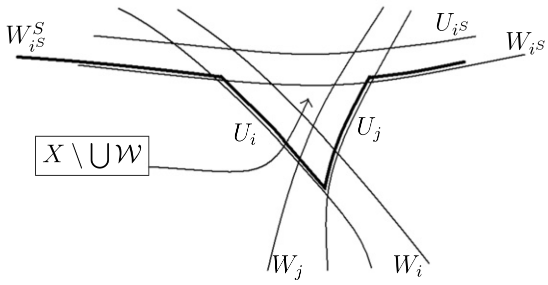

Fig. 1. Check that $\mathcal{W}^S$ and $\mathcal{N}^x$ are $\partial\Delta^k$-tuples (i, j, i^S above are distinct—this is why we need $k \geqslant 2$).

5. Compact spaces with K-dim < K-Ind or K-Ind < $|K|$-Ind, where K is a simplicial complex

This section is devoted to the behaviour of K-Ind under the operation $Z(X,Y)$. We obtain inequalities for K-Ind that resemble those in the preceding section for L-Ind, and we establish conditions in order that $K\text{-Ind}\, Z(X,X) = K\text{-Ind}\, X$ or $K\text{-Ind}\, Z(X,X) = K\text{-Ind}\, X + 1$.

Lemma 5.1. *If $\mathcal{F}$ is a closed K-tuple of $Z(X,Y)$, then there is a set $A \subset A_{\mathrm{m}}$ such that $\mu \in A$, $A_{\mathrm{m}} \setminus A$ is finite, and $\pi_X(\mathcal{F}|\pi_{A_{\mathrm{m}}}^{-1}(A))$ is a closed K-tuple of X.*

Proof. Take any closed K-tuple $\mathcal{F}$ of $Z(X,Y)$. Then, the K-tuple $\pi_X(\mathcal{F}|H(\mu))$ has a K-neighbourhood $\mathcal{U}$ in X. Since $\pi_X(F_i \cap H(\mu)) \subset U_i$ for $i = 0,\dots,k$, we have $\mu \notin A_i = \pi_{A_{\mathrm{m}}}(F_i \setminus \pi_X^{-1}(U_i))$ for each i. Since A_i are closed in A_{m}, they are finite. As easily checked, $A = A_{\mathrm{m}} \setminus \bigcup_{i=0}^k A_i$ has the required properties. □

Lemma 5.2. *Suppose that $\mathcal{U}$ is an open K-tuple of X, and $K\text{-obs}\,\mathcal{U} = \emptyset$. Then there is a K-neighbourhood $\mathcal{V}$ of $\pi_X^{-1}(\mathcal{U})$ in $Z(X,Y)$ with*

$$Z(X,Y) \setminus \bigcup \mathcal{V} = H(\mu) \setminus \pi_X^{-1}\Big(\bigcup \mathcal{U}\Big).$$

If moreover $K = \partial\Delta^k$, where $k \geqslant 2$, and $\mathrm{cl}\,\mathcal{U}$ is a $\partial\Delta^k$-tuple, then $\mathcal{V}$ can be chosen so that $\partial\Delta^k\text{-obs}\,\mathcal{V} = \emptyset$.

Proof. Each $S \in \mathcal{S}_X$ is metrisable, and by Lemma 2.6, the K-tuple $\mathcal{U}|S = (U_0 \cap S, \dots, U_k \cap S)$ has a K-neighbourhood $\mathcal{V}^S$ in S which covers S (a direct proof is easy, too). Let $\alpha \neq \mu$. Then π_X maps $H(\alpha)$ onto $S = \varphi(\alpha)$. The sets $\pi_X^{-1}(V_i^{\varphi(\alpha)}) \cap H(\alpha)$, $i = 0,\dots,k$, form an open K-cover of $H(\alpha)$. Now, the unions

$$V_i = \pi_X^{-1}(U_i) \cup \bigcup_{\alpha \in A_{\mathrm{m}} \setminus \{\mu\}} \big(\pi_X^{-1}\big(V_i^{\varphi(\alpha)}\big) \cap H(\alpha)\big)$$

form the requested K-neighbourhood $\mathcal{V}$ of $\mathcal{F}$.

Assume that $k \geqslant 2$ and $\mathrm{cl}\,\mathcal{U}$ is a $\partial\Delta^k$-tuple. Then there is a $\partial\Delta^k$-neighbourhood $\mathcal{W}$ of $\mathrm{cl}\,\mathcal{U}$. Take an $S \in \mathcal{S}_X$, and let $x^S \in S$ be the limit of S if S is infinite. We choose an index $i^S \in \{0,\dots,k\}$ so that (1) $i^S = 0$ when S is finite or $x^S \notin \bigcup \mathcal{W}$, and (2) $x^S \in W_{i^S}$ when $x^S \in \bigcup \mathcal{W}$. Now, we define a $\partial\Delta^k$-cover $\mathcal{W}^S$ of X by the formulas

$$W_i^S = \begin{cases} W_{i^S} \cup (X \setminus \bigcup \mathcal{U}) & \text{for } i = i^S, \\ U_i & \text{for } i \neq i^S \end{cases}$$

(see Fig. 1; in general, $W_{i^S}^S$ is not open!), and we put $\mathcal{V}^S = \mathcal{W}^S|S$. Since x^S is the unique non-isolated point of an infinite S, it is easily seen that $V_{i^S}^S = S \cap W_{i^S}^S$ is open in S. Hence, $\mathcal{V}^S$ is a $\partial\Delta^k$-neighbourhood of $\mathcal{U}|S$. We define V_i's and $\mathcal{V}$ by the same formula as in the first paragraph of this proof.

There remains to prove that $\mathcal{V}$ has an empty $\partial\Delta^k\text{-obs}\,\mathcal{V}$. If $z \in Z(X,Y) \setminus \bigcup \mathcal{V} \subset H(\mu)$, then $x = \pi_X(z) \in X \setminus \bigcup \mathcal{U}$. There are two cases. (A) When $x \in W_{i^x}$ for some index i^x, we put $N^x = W_{i^x}$. (B) When $x \notin \bigcup \mathcal{W}$, we put $N^x = X \setminus \bigcup \mathrm{cl}\,\mathcal{U}$ and $i^x = 0$. Thus, N^x is an open neighbourhood of x, and the sets

$$V_i' = \begin{cases} V_i \cup \pi_X^{-1}(N^x) & \text{for } i = i^x, \\ V_i & \text{for } i \neq i^x \end{cases}$$

are open in $Z(X,Y)$. We are to show that their intersection is empty. In order to check that $H(\mu)\cap\bigcap_{i=0}^k V_i'=\emptyset$, observe that $\pi_X(H(\mu)\cap V_i')$ is either $U_{i^x}\cup N^x$ for $i=i^x$ or U_i for $i\neq i^x$. These $k+1$ subsets of X do not intersect in both cases (A) and (B), and we are done. When $\alpha\neq\mu$ and $S=\varphi(\alpha)$, we have $H(\alpha)\cap V_i'=H(\alpha)\cap\pi_X^{-1}(N_i^x)$, where

$$N_i^x=\begin{Bmatrix} W_{i^x}^S\cup N^x & \text{for } i=i^x\\ W_i^S & \text{for } i\neq i^x\end{Bmatrix}=\begin{cases} U_i\cup N^x & \text{if } i=i^x\neq i^S,\\ W_{i^S}\cup(X\setminus\bigcup\mathcal{U}) & \text{if } i=i^S,\\ U_i & \text{for } i\notin\{i^S,i^x\}.\end{cases}$$

One checks that $\mathcal{N}^x=(N_0^x,\dots,N_k^x)$ is a $\partial\Delta^k$-tuple in both cases (A) and (B), and hence $H(\alpha)\cap\bigcap_{i=0}^k V_i'=\emptyset$. Now, we infer that the sets V_i' form a $\partial\Delta^k$-neighbourhood of $\mathcal{V}$. Finally, $z\notin\partial\Delta^k\text{-obs}\,\mathcal{V}$ because $z\in V_{i^x}'\subset\bigcup_{i=0}^k V_i'$. □

As $Z(X,Y)$ contains homeomorphic copies of both X and Y, we immediately obtain the inequality $\max\{K\text{-Ind}\,X, K\text{-Ind}\,Y\}\leqslant K\text{-Ind}\,Z(X,Y)$. The following theorem contains upper bounds of $K\text{-Ind}\,Z(X,Y)$.

Theorem 5.3. *Suppose that X and Y are non-empty compact spaces. Then*

$$K\text{-Ind}\,Z(X,Y)\leqslant\max\{K\text{-Ind}\,X+1, K\text{-Ind}\,Y\}.$$

If moreover $K\text{-str}\,X=0$, then

$$K\text{-Ind}\,Z(X,Y)=\max\{K\text{-Ind}\,X, K\text{-Ind}\,Y\}.$$

If $k\geqslant 2$ and $\partial\Delta^k\text{-Ind}\,Y<\partial\Delta^k\text{-Ind}\,X+1=\partial\Delta^k\text{-Ind}\,Z(X,Y)$ then

$$\partial\Delta^k\text{-str}\,Z(X,Y)=0.$$

Proof. Take a closed K-tuple $\mathcal{F}$ of $Z=Z(X,Y)$. Lemma 5.1 yields a set $A\subset A_{\mathrm{m}}$ such that $\mu\in A$, $A_{\mathrm{m}}\setminus A$ is finite, and $\pi_X(\mathcal{F}|\pi_{A_{\mathrm{m}}}^{-1}(A))$ is a closed K-tuple of X. Then there is a K-neighbourhood $\mathcal{U}$ of $\pi_X(\mathcal{F}|\pi_{A_{\mathrm{m}}}^{-1}(A))$ such that $\mathrm{cl}\,\mathcal{U}$ is a K-tuple. Clearly $K\text{-obs}\,\mathcal{U}=\emptyset$, and writing $P=X\setminus\bigcup\mathcal{U}$, we obtain $K\text{-Ind}\,P\leqslant K\text{-Ind}\,X$. As $A_{\mathrm{m}}\setminus A$ is finite, we can think that $\pi_{A_{\mathrm{m}}}^{-1}(A)$ is a $Z(X,Y)$. Hence by Lemma 5.2, $\pi_X^{-1}(\mathcal{U})|\pi_{A_{\mathrm{m}}}^{-1}(A)$ has a K-neighbourhood $\mathcal{V}$ in $\pi_{A_{\mathrm{m}}}^{-1}(A)$ with the corresponding K-partition $Q=H(\mu)\setminus\pi_X^{-1}(\bigcup\mathcal{U})$. Thus, Q is a K-partition in $\pi_{A_{\mathrm{m}}}^{-1}(A)$ for $\mathcal{F}|\pi_{A_{\mathrm{m}}}^{-1}(A)$. As $\pi_X|Q$ is a homeomorphism onto P, we have $K\text{-Ind}\,Q\leqslant K\text{-Ind}\,X$. On the other hand, $H(\alpha)$ is homeomorphic to $\varphi(\alpha)\times Y$ for $\alpha\neq\mu$, and $K\text{-Ind}\,H(\alpha)=K\text{-Ind}\,Y$ by Theorem 2.9(a). For each $\alpha\notin A$, in $H(\alpha)=\pi_{A_{\mathrm{m}}}^{-1}(\alpha)$ there is a K-neighbourhood $\mathcal{W}^\alpha$ of $\mathcal{F}|\pi_{A_{\mathrm{m}}}^{-1}(\alpha)$ such that $R^\alpha=\pi_{A_{\mathrm{m}}}^{-1}(\alpha)\setminus\bigcup\mathcal{W}^\alpha$ has $K\text{-Ind}\,R^\alpha<K\text{-Ind}\,Y$. Since $A_{\mathrm{m}}\setminus A$ is finite, the union

$$R=\left(H(\mu)\setminus\pi_X^{-1}\left(\bigcup\mathcal{U}\right)\right)\cup\bigcup_{\alpha\in A_{\mathrm{m}}\setminus A}R^\alpha \tag{‡}$$

is a K-partition for $\mathcal{F}$, and $K\text{-Ind}\,R<\max\{K\text{-Ind}\,X+1, K\text{-Ind}\,Y\}$. We have shown the first inequality of the theorem's assertion.

In the case when $K\text{-str}\,X=0$, only a slight modification of the above proof is needed. Indeed, we do not need the K-tuple $\mathrm{cl}\,\mathcal{U}$, but instead, $\pi_X(\mathcal{F}|\pi_{A_{\mathrm{m}}}^{-1}(A))$ has a K-neighbourhood $\mathcal{U}$ such that $K\text{-obs}\,\mathcal{U}=\emptyset$ and the corresponding K-partition P satisfies the inequality $K\text{-Ind}\,P<K\text{-Ind}\,X$. At the end, we obtain $K\text{-Ind}\,R<\max\{K\text{-Ind}\,X, K\text{-Ind}\,Y\}$ and $K\text{-Ind}\,Z\leqslant\max\{K\text{-Ind}\,X, K\text{-Ind}\,Y\}$.

If $k\geqslant 2$, $K=\partial\Delta^k$, and $\partial\Delta^k\text{-Ind}\,Y<\partial\Delta^k\text{-Ind}\,X+1=\partial\Delta^k\text{-Ind}\,Z$, then we make another modification. We take $\mathcal{U}$ with the $\partial\Delta^k$-tuple $\mathrm{cl}\,\mathcal{U}$, and Lemma 5.2 yields $\mathcal{V}$ with $\partial\Delta^k\text{-obs}\,\mathcal{V}=\emptyset$. As $\partial\Delta^k\text{-Ind}\,Y<\partial\Delta^k\text{-Ind}\,X+1$, for $\alpha\notin A$ we can take any $\partial\Delta^k$-neighbourhood $\mathcal{W}^\alpha$ in $\pi_{A_{\mathrm{m}}}^{-1}(\alpha)$ of $\mathcal{F}|\pi_{A_{\mathrm{m}}}^{-1}(\alpha)$ with $\mathrm{cl}\,\mathcal{W}^\alpha$ being a $\partial\Delta^k$-tuple, in addition. Then $\partial\Delta^k\text{-obs}\,\mathcal{W}^\alpha=\emptyset$ and $\partial\Delta^k\text{-Ind}\,R<\partial\Delta^k\text{-Ind}\,X+1$. R is the corresponding $\partial\Delta^k$-partition of the open $\partial\Delta^k$-tuple which consists of the sets $V_i\cup\bigcup_{\alpha\in A_{\mathrm{m}}\setminus A}W_i^\alpha$ for $i=0,\dots,k$, and which does not have $\partial\Delta^k$-obstruction points. This completes the proof of the equality $\partial\Delta^k\text{-str}\,Z=0$. □

Proposition 2.8 and Theorem 5.3 yield

Corollary 5.4. *Let $k\geqslant 2$ and $n\geqslant 1$. If X is a compact metric space such that* $\dim X=kn$, *then*

$$\partial\Delta^k\text{-Ind}\,Z(X,Y)=\max\{n,\partial\Delta^k\text{-Ind}\,Y\}$$

for every non-empty compact space Y. □

Corollary 5.5. *Let $k \geqslant 2$ and $n \geqslant 1$. If C is a metric continuum with* $\dim C = kn$, *then $X_C = Z(C, C)$ is a compact Fréchet space such that*

(a) $\partial\Delta^k$-dim $X_C = \partial\Delta^k$-Ind $X_C = n$,
(b) $|\partial\Delta^k|$-Ind $X_C = n + 1$, *and*
(c) *each component of X_C is homeomorphic to C.*

Proof. It follows from Proposition 3.1 that X_C is a compact Fréchet space that satisfies the statement (c).

All four of the dimensions $\partial\Delta^k$-dim, $|\partial\Delta^k|$-dim, $\partial\Delta^k$-Ind, and $|\partial\Delta^k|$-Ind of C are equal to n by Theorems 1.7 and 1.9. Now, the statements 1.7, 3.2, and 5.4 imply (a). The statement (b) results from 4.4 and 4.5. □

Since any simplicial complex K is a triangulation of the polyhedron $|K|$, we may restate Fedorchuk's question [8, Question 3.1] as follows: *Are the dimensions K-Ind and $|K|$-Ind equal for arbitrary normal spaces?* The foregoing corollary shows that the answer is no. In the simplest case—for $k = 2$, $n = 1$, and $[0, 1]^2$—we obtain $\partial\Delta^2$-Ind $Z([0, 1]^2, [0, 1]^2) = 1 < 2 = |\partial\Delta^2|$-Ind $Z([0, 1]^2, [0, 1]^2)$.

The two above corollaries show that if we take a kn-dimensional compact metric space, then one-time use of the operation $Z(X, Y)$ does not allow us to obtain a space with $\partial\Delta^k$-dim $< \partial\Delta^k$-Ind. We could try to iterate the operation. However, *we even do not know whether $\partial\Delta^2$-str $Z([0, 1]^2, [0, 1]^2)$ is 1 or it is 0.* Let us write $T = Z([0, 1]^2, [0, 1]^2)$. The values of $\partial\Delta^2$-Ind $Z(T, T)$ and $\partial\Delta^2$-Ind $Z(T, [0, 1]^2)$ remain unknown. On the other hand, $\partial\Delta^2$-Ind $Z([0, 1]^2, T) = 1$.

To show that the operation $Z(X, Y)$ sometimes raises the dimension K-Ind by one, we need the following.

Lemma 5.6. *Suppose that X is a compact Fréchet space with $\partial\Delta^k$-Ind $X = \alpha$ and $\partial\Delta^k$-str $X = 1$. Let $\mathcal{F}$ be a $\partial\Delta^k$-tuple in X, where $k \geqslant 1$. Assume that if $\mathcal{U}$ is a $\partial\Delta^k$-neighbourhood of $\mathcal{F}$, and the corresponding $\partial\Delta^k$-partition $P = X \setminus \bigcup\mathcal{U}$ has $\partial\Delta^k$-Ind $P < \alpha$, then $\partial\Delta^k$-obs$\mathcal{U} \neq \emptyset$. Write $\mathcal{G} = \pi_X^{-1}(\mathcal{F})|H(\mu)$. If Q is a $\partial\Delta^k$-partition in $Z(X, Y)$ for $\mathcal{G}$, then one of the following conditions is satisfied*:

(a) $\partial\Delta^k$-Ind$(Q \cap H(\mu)) = \alpha$;
(b) *there is an $\alpha \neq \mu$ such that $\varphi(\alpha) \in \mathcal{S}_X$ is infinite and $\pi_{A_\mathfrak{m}}^{-1}(\alpha) \cap \pi_X^{-1}(x^{\varphi(\alpha)}) \subset Q$, where $x^{\varphi(\alpha)}$ is the accumulation point of $\varphi(\alpha)$ (and the intersection of the point-inverses is homeomorphic to Y).*

Proof. Let $\mathcal{V}$ be any $\partial\Delta^k$-neighbourhood of $\mathcal{G}$ in $Z = Z(X, Y)$, and Q the corresponding $\partial\Delta^k$-partition. Since $\pi_X|H(\mu)$ is a homeomorphism onto X, assume that $\partial\Delta^k$-Ind$(Q \cap H(\mu)) < \alpha$. Hence, $\mathcal{U} = \pi_X(\mathcal{V}|H(\mu))$ has $\emptyset \neq \partial\Delta^k$-obs$\mathcal{U}$. By Lemma 1.10, there is a common element $x_0 \in \mathrm{cl}(\bigcap_{0 \leqslant j \leqslant k,\, j \neq i} U_j)$ for $i = 0, \ldots, k$. Moreover $x_0 \notin \bigcup\mathcal{U}$ because $\partial\Delta^k$-obs$\mathcal{U}$ is disjoint from $\bigcup\mathcal{U}$. As X is Fréchet, for each i there is an infinite sequence $S_i \subset \bigcap_{0 \leqslant j \leqslant k,\, j \neq i} U_j$ that converges to x_0. It follows from Lemma 3.3 that there is a set $A \subset A_\mathfrak{m}$ with $\mathrm{card}(A_\mathfrak{m} \setminus A) < \mathfrak{m}$ and $\pi_{A_\mathfrak{m}}^{-1}(A) \cap \pi_X^{-1}(U_i) \subset V_i$ for each i. Let $S = \{x_0\} \cup \bigcup_{i=0}^k S_i$. Now, we can find an $\alpha \in A \setminus \{\mu\}$ with $\varphi(\alpha) = S$ (because $\mathrm{card}\,\varphi^{-1}(S) = \mathfrak{m}$). $H(\alpha) = \pi_1(\{\alpha\} \times S \times Y)$ is homeomorphic to $S \times Y$. Fix an index i for a while, and note that

$$\pi_1\big(\{\alpha\} \times S_i \times Y\big) = \pi_{A_\mathfrak{m}}^{-1}(\alpha) \cap \pi_X^{-1}(S_i) \subset \bigcap_{0 \leqslant j \leqslant k,\, j \neq i} V_j.$$

We claim that *no point of $\pi_1(\{\alpha\} \times \{x_0\} \times Y)$ belongs to V_i*. Indeed, S_i converges to x_0. If we had $\pi_1(\alpha, x_0, y) \in V_i$, then there would exist a point $x \in S_i$ such that $\pi_1(\alpha, x, y) \in V_i$. In consequence, the intersection $\bigcap_{j=0}^k V_j$ would be non-empty, and $\mathcal{V}$ would not be a $\partial\Delta^k$-tuple. Therefore, $\pi_1(\{\alpha\} \times \{x_0\} \times Y) = \pi_{A_\mathfrak{m}}^{-1}(\alpha) \cap \pi_X^{-1}(x_0)$ does not meet V_i for any i, and is contained in Q. We can write $x^{\varphi(\alpha)} = x_0$. □

As a consequence of Theorem 5.3 and the foregoing lemma we obtain

Theorem 5.7. *Let $k \geqslant 1$. Suppose that X and Y are non-empty compact spaces. If X is a Fréchet space, $\partial\Delta^k$-str $X = 1$, and $\partial\Delta^k$-Ind $X = \partial\Delta^k$-Ind Y, then*

$$\partial\Delta^k\text{-Ind}\, Z(X, Y) = \partial\Delta^k\text{-Ind}\, X + 1. \quad \square$$

Lemma 5.6 and Theorem 5.7 hold for each simplicial complex K (a similar proof with a more complicated description of the set K-obs$\mathcal{U}$ for arbitrary K).

The following corollary results from the statements 2.7, 5.3, and 5.7.

Corollary 5.8. *Let $k \geqslant 2$ and $n \geqslant 1$. If X is a compact metric space such that* $\dim X = k(n + 1) - 1$, *then*

$$\partial\Delta^k\text{-Ind}\, Z(X, Y) = n + 1 \quad \textit{and} \quad \partial\Delta^k\text{-str}\, Z(X, Y) = 0$$

for every compact space Y with $\partial\Delta^k$-Ind $Y = n$. □

Corollary 5.9. *Let $k, n \geqslant 1$. If C is a metric continuum with $\dim C = k(n+1) - 1$, then $X_C = Z(C, C)$ is a Fréchet compact space such that*

(a) $\partial\Delta^k\text{-dim}\, X_C = n$,
(b) $\partial\Delta^k\text{-Ind}\, X_C = |\partial\Delta^k|\text{-Ind}\, X_C = n+1$,
(c) $\partial\Delta^k\text{-str}\, X_C = 0$ *whenever* $k \geqslant 2$,
(d) *every component of X_C is homeomorphic to C.*

Proof. By Proposition 2.7, we obtain $\partial\Delta^k\text{-Ind}\, C = n$ and $\partial\Delta^k\text{-str}\, C = 1$. The statement (a) results from Theorem 1.7 and Lemma 3.2, and (b) is a corollary to the statements 5.7, 1.7, and 4.1. Corollary 5.8 implies (c), and the application of Proposition 3.1 completes the proof. □

Remark 5.10. (a) In the above Corollary 5.9, the metrisable components P of $Z(C, C)$ have $\partial\Delta^k\text{-Ind}\, P = |\partial\Delta^k|\text{-Ind}\, P = n < n+1 = \partial\Delta^k\text{-Ind}\, Z(C, C) = |\partial\Delta^k|\text{-Ind}\, Z(C, C)$. Thus, $\partial\Delta^k$-Ind and $|\partial\Delta^k|$-Ind analogues of Theorem 1.8 do not hold. This is no surprise because there is not such an analogue for the large inductive dimension Ind (Chatyrko [3]; see also Krzempek [10]).

(b) Spaces similar to $Z(C, C)$ in Corollary 5.9 are constructed by Chatyrko [3] for $k = n = 1$ and $C = [0, 1]$. The spaces have $\dim = 1$, $\text{ind} = \text{Ind} = 2$, and each of their components is either a singleton or a subspace homeomorphic to $[0, 1]$. Also for $k = 1$ and each integer $n > 1$, similar spaces have been expected in [3, Remark 5.1]. We believe that if X is a compact metric space with $\dim X = k(n+1) - 1$, where $k, n \geqslant 1$, then $Z(X, X)$ contains compact subspaces $Q \subset P$ such that $\partial\Delta^k\text{-Ind}\, Q = |\partial\Delta^k|\text{-Ind}\, Q = n$, $\partial\Delta^k\text{-Ind}\, P = |\partial\Delta^k|\text{-Ind}\, P = n+1$, and $P \setminus Q$ is a discrete space of cardinality $\mathfrak{c}$ (cf. [3], a construction for $k = n = 1$ and Ind).

(c) Suppose that X is a compact metric space with $\dim X = k(n+1) - 1$. Then $|\partial\Delta^k|\text{-dim}\, X = n$, and X is the union of pairwise disjoint subspaces $X_0, \dots, X_n$ with $|\partial\Delta^k|\text{-dim}\, X_i = 0$ for $i = 0, \dots, n$ (Fedorchuk [6, Corollary 5.16]). Consider $Z(X, X)$ and its compact subspaces

$$Z_i = H(\mu) \cup \bigcup \{ H(\alpha)\colon\ \varphi(\alpha) \text{ is finite or its unique accumulation point is in } X_i \}$$

for $i = 0, \dots, n$. Evidently $Z(X, X) = Z_0 \cup \dots \cup Z_n$. We shall sketch a proof of the equalities $\partial\Delta^k\text{-Ind}\, Z_i = |\partial\Delta^k|\text{-Ind}\, Z_i = n$ for $i = 0, \dots, k$. Therefore, the space $Z(X, X)$ with $\partial\Delta^k\text{-Ind}\, Z(X, X) = |\partial\Delta^k|\text{-Ind}\, Z(X, X) = n+1$ is the union of $n+1$ closed subspaces Z_i with $\partial\Delta^k\text{-Ind}\, Z_i = |\partial\Delta^k|\text{-Ind}\, Z_i = n$. This is similar to the properties of several well-known spaces (for instance, Lokucievskiĭ's Example 2.2.14 in [5], Chatyrko's spaces in [3], Charalambous and Chatyrko's examples for the dimension Ind_0 in [1]).

We have $n = \partial\Delta^k\text{-Ind}\, X \leqslant \partial\Delta^k\text{-Ind}\, Z_i \leqslant |\partial\Delta^k|\text{-Ind}\, Z_i$. The dimension $M\text{-Ind}_0$ modulo a simplicial complex M [respectively: modulo an ANR M] is defined similarly as M-Ind—in order that $M\text{-Ind}_0\, X \leqslant \alpha$ we stipulate that the M-partition P in the statement 1.3(b) [respectively: 1.4(b′)] is a zero set with $M\text{-Ind}_0\, P < \alpha$ (see [2, p. 670]). It is easily shown by transfinite induction that $M\text{-Ind} \leqslant M\text{-Ind}_0$, and Theorem 1 in [2] may be summarised as follows: $K\text{-Ind}_0 = |K|\text{-Ind}_0$ *for any simplicial complex K and all normal spaces.* Thus, we have $n \leqslant |\partial\Delta^k|\text{-Ind}\, Z_i \leqslant |\partial\Delta^k|\text{-Ind}_0\, Z_i = \partial\Delta^k\text{-Ind}_0\, Z_i$. It is sufficient to show that $\partial\Delta^k\text{-Ind}_0\, Z_i \leqslant n$.

We need the following claim: *For each closed $\partial\Delta^k$-tuple $\mathcal{F}$ of X, there exists a $\partial\Delta^k$-partition P disjoint from X_i.* Indeed, Lemma 6 in [2] directly yields a map $f : \bigcup \mathcal{F} \to |\partial\Delta^k|$ with $F_j \subset f^{-1}(K_j)$ for $j = 0, \dots, k$ (see the definition of K_j's before Lemma 2.3 herein). By Fedorchuk [8, Proposition 2.7], there is a $\partial\Delta^k$-partition P for f disjoint from X_i, and hence, f has an extension $f' : X \setminus P \to |\partial\Delta^k|$. Since the sets $K'_j = \{x \in |\partial\Delta^k|\colon\ x_j > 0\}$ form a $\partial\Delta^k$-neighbourhood $\mathcal{K}'$ of $\mathcal{K}$, we can take the pre-image $\partial\Delta^k$-tuple $f'^{-1}(\mathcal{K}')$. Thus, P is a $\partial\Delta^k$-partition for $\mathcal{F}$. Using the above claim, remembering that each closed subset of X is a zero subset, and modifying the proof of Theorem 5.3, one can show that each closed $\partial\Delta^k$-tuple of Z_i has a metrisable zero $\partial\Delta^k$-partition P in Z_i with $\partial\Delta^k\text{-Ind}\, P = \partial\Delta^k\text{-Ind}_0\, P < n$. This means that $\partial\Delta^k\text{-Ind}_0\, Z_i \leqslant n$.

(d) Let $T = Z(C, C)$ be the space in Corollary 5.9. If $k \geqslant 2$, then $\partial\Delta^k\text{-str}\, T = 0$, and we obtain $\partial\Delta^k\text{-Ind}\, Z(T, T) = n+1$ by Theorem 5.7. In the proof of Theorem 4.6 we iterate the operation $Z(X, Y)$. In the case of $\partial\Delta^k$-Ind for $k \geqslant 2$, we do not know whether $\partial\Delta^k\text{-str}\, Z(T, T) = 0$ or $\partial\Delta^k\text{-str}\, Z(T, T) = 1$. In consequence, *for $k \geqslant 2$ we do not know if the operation $Z(X, Y)$ allows us to construct compact spaces X with metrisable components and $\partial\Delta^k\text{-Ind}\, X > \partial\Delta^k\text{-dim}\, X + 1$.*

6. Conclusion and open problems

The theories of inductive dimensions investigate problems which involve partitioning a given space in some admissible ways. The following two questions arise. (1) What closed subsets are *sufficient or large enough* to partition the space in all considered circumstances/ways? (2) How large closed subsets are *necessary* to partition the space? In the case of L-Ind and K-Ind of $Z(X, Y)$, it is sufficient to consider L-partitions and K-partitions which are finite disjoint unions described by formulas (†) and (‡) on pp. 3019 and 3023. An answer to the latter question is stated by the alternatives (a) or (b) of Lemmas 4.2 and 5.6.

In Sections 4 and 5 we have drawn up two maps of the $Z(X, Y)$ spaces' land. The difference between the maps has enabled us to detect compact Fréchet spaces with $\partial\Delta^k\text{-Ind} < |\partial\Delta^k|\text{-Ind}$ (Corollary 5.5). We have found a quite exhaustive solution to the problem stated in the Introduction in the case of L-Ind, where L is a compact metric ANR: for arbitrarily large ordinals $\alpha \geqslant n$, we have constructed compact Fréchet spaces with L-dim $= n$, L-Ind $= \alpha$, and all components metrisable (see Corollary 4.8 for necessary obstructions). In the case of K-Ind, where K is a finite simplicial complex, we have succeeded only for $K = \partial\Delta^k$ and $\alpha = n + 1$ (Corollary 5.9). Crucial properties of $\partial\Delta^k$-Ind and $\partial\Delta^k$-str may be summarised as follows (Propositions 2.7, 2.8, and Theorems 5.3, 5.7).

Theorem 6.1. *Let $k, n \geqslant 1$ be natural numbers. Suppose that X and Y are non-empty compact spaces with $\partial\Delta^k\text{-Ind}\, X = \partial\Delta^k\text{-Ind}\, Y$. Then the following implications hold.*

X is metrisable and $\dim X = kn$, *where* $k \geqslant 2$	*X is metrisable and* $\dim X = k(n+1) - 1$
$\Downarrow$	$\Downarrow$
$\partial\Delta^k\text{-Ind}\, X = n$ *and* $\partial\Delta^k\text{-str}\, X = 0$	*X is a Fréchet space,* $\partial\Delta^k\text{-Ind}\, X = n$, *and* $\partial\Delta^k\text{-str}\, X = 1$
$\Downarrow$	$\Downarrow$
$\partial\Delta^k\text{-Ind}\, Z(X, Y) = n$	$\partial\Delta^k\text{-str}\, Z(X, Y) = 0$ *unless* $k = 1$, *and* $\partial\Delta^k\text{-Ind}\, Z(X, Y) = n + 1$. □

The specific question we are not able to answer is

Question 6.2. Is it true that $\partial\Delta^2\text{-str}\, Z([0,1]^2, [0,1]^2) = 1$?

An answer in the affirmative would give us hopes for finding a proof of the equality $\partial\Delta^2\text{-str}\, Z(T, T) = 1$, where $T = Z([0,1]^3, [0,1]^3)$. Having such a proof, we could apply Theorem 5.7 to $X = Y = Z(T, T)$, and state a positive answer to

Question 6.3. Do there exist a simplicial complex K and a compact space X such that the underlying polyhedron $|K|$ is connected, $K\text{-Ind}\, X > K\text{-dim}\, X + 1$, and each component of X is metrisable?

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