Note

The chromatic number and rank of the complements of the Kasami graphs☆

Aidan Roya, Gordon F. Royleb

aDepartment of Combinatorics and Optimization, University of Waterloo, Waterloo, Ont., Canada N2L 3G1
bDepartment of Computer Science and Software Engineering, University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia

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Dedicated to the memory of Dom de Caen

Abstract

We determine the rank and chromatic number of the complements of all Kasami graphs, some of which form an infinite family of counterexamples to the now disproven rank-coloring conjecture.

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1. Introduction

The rank and chromatic number are important parameters of a graph, and significant interest has developed in finding a relation between them. One reason for this is the 1983 conjecture of Van Nuffelen [13] that for any graph X,

\[ \chi(X) \leq \text{rk}(X), \]

where \( \chi \) is the chromatic number and \( \text{rk} \) is the rank of the adjacency matrix over the real numbers. Alon and Seymour [1] found a counterexample to Van Nuffelen’s “rank-coloring” conjecture, which turns out to be the complement of the folded 7-cube. While generalizing to folded \( n \)-cubes does not provide any further examples, de Caen observed that the folded 7-cube is also a Kasami graph. As our results show, the complements of a large subset of the Kasami graphs prove to be counterexamples to Van Nuffelen’s conjecture.

Interest in the rank and chromatic number has also been stimulated by the following problem in communications: suppose two parties A and B want to compute a binary-valued function \( f(x, y) \), where A chooses \( x \) and B chooses \( y \). The deterministic communication complexity (DCC) of \( f \) is the minimum number of bits that must be communicated between A and B to determine the value of \( f \). In 1982, Mehlhorn and Schmidt [9] showed that the DCC of a \( \{0, 1\} \) matrix \( M \) satisfies

\[ \text{DCC}(M) \geq \log_2 \text{rk}(M); \]

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E-mail addresses: aroy@math.uwaterloo.ca (A. Roy), gordon@csse.uwa.edu.au (G.F. Royle).
Lovász and Saks [7] then observed that some power of $\log_2 \text{rk}(M)$ is also an upper bound for $\text{DCC}(M)$ if and only if there is a constant $c$ such that

$$\log_2 \chi(X) \leq (\log_2 \text{rk}(X))^c$$

for every graph $X$.

### 2. Kasami graphs

For the remainder of this note we assume the reader has some knowledge of distance-regular graphs and association schemes. For more information on these topics see [2].

Let $s$ and $t$ be powers of 2 so that $s - 1$ and $t + 1$ are relatively prime and $s > t$. The Kasami graph $G(s, t)$ is the graph with vertex set $F_s \times F_s$ and the following adjacency condition:

$$(a, x) \sim (b, y) \iff a + b = (x + y)^{t+1}.$$  \hfill (1)

This definition of the Kasami graph is due to de Caen and Van Dam [5]. In fact, the Kasami graph is the coset graph of the Kasami code; see Brouwer, Cohen, and Neumaier [2, Section 11.2] for the details of this description. Kasami codes are fairly well-known in coding theory, but their graphs have been much less studied. The graphs are distance-regular, and association schemes arising from the graphs were given by Calderbank and Goethals in [4] and by de Caen and Van Dam in [5]. Kasami graphs are also studied in [11]. Kasami codes are discussed by MacWilliams and Sloane in [8, Section 15.4].

There is a second type of Kasami graph, in which $s = t^2$ instead of having $s - 1$ and $t + 1$ relatively prime; in this case the graph has vertex set $F_t \times F_s$ with the same adjacency condition (1). However, throughout this note we will assume that $\gcd(s - 1, t + 1) = 1$. When this is the case, it can easily be shown that there is a power of 2, say $q = q(s, t)$, such that

$$s = q^{2j+1}, \quad t = q^m \quad \text{and} \quad (m, 2j + 1) = 1.$$  \hfill (2)

We may also assume that $m \leq j$, since replacing $t$ by $s/t$ yields an isomorphic graph.

We investigate the rank and chromatic number of $\overline{G(s, t)}$, the complement of the Kasami graph. The result is the following:

**Theorem 1.** If $s, t, q$ satisfy (2), then

$$\text{rk}(\overline{G(s, t)}) = q^{4j+1} - q^{2j} + 1$$

and

$$\chi(\overline{G(s, t)}) = q^{4j+1}.$$  \hfill (3)

Hence the complements of all Kasami graphs with $s, t$ as in (2) have larger chromatic number than rank.

Using the same methods that are described in this paper, it can be shown that the Kasami graphs with $s = t^2$ have rank $t^3 - t^2 + 1$ and chromatic number $t^2$. Since these graphs are not counterexamples to the rank-coloring conjecture, we omit the details.

### 3. The rank

We first compute the graph’s rank. The graph $G(s, t)$ is distance-regular (see Theorem 11.2.1 of [2]) with following intersection array:

$$\{q^{2j+1} - 1, q^{2j+1} - q, q^{2j}(q - 1) + 1; 1, q, q^{2j} - 1\}.$$  \hfill (4)

From the array we can compute the eigenvalues of the graph and their multiplicities.
Lemma 1. The graph $G(s, t)$ has eigenvalues
\[ \{q^{2j+1} - 1, q^{j+1} - 1, -1, -(q^{j+1} + 1)\} \]
with respective multiplicities
\[ \left\{ 1, \frac{q^{4j+1} + q^{3j+1} - q^{2j} - q^{j}}{2}, q^{4j+2} - q^{4j+1} + q^{2j} - 1, \frac{q^{4j+1} - q^{3j+1} - q^{2j} + q^{j}}{2} \right\}. \]

Proof. A distance-regular graph of diameter 3 has four distinct eigenvalues, which are the eigenvalues of the $4 \times 4$ matrix
\[
\begin{pmatrix}
  a_0 & b_0 & 0 & 0 \\
  c_1 & a_1 & b_1 & 0 \\
  0 & c_2 & a_2 & b_2 \\
  0 & 0 & c_3 & a_3 \\
\end{pmatrix}.
\]

For the Kasami graph with the above intersection array, this matrix is
\[
\begin{pmatrix}
  0 & q^{2j+1} - 1 & 0 & 0 \\
  1 & q - 2 & q^{2j+1} - q & 0 \\
  0 & q & q^{2j} - q - 2 & q^{2j}(q - 1) + 1 \\
  0 & 0 & q^{2j} - 1 & q^{2j}(q - 1) \\
\end{pmatrix}.
\]

It is straightforward to verify that the characteristic polynomial of this matrix is
\[ (x + 1)(x - (q^{2j+1} - 1))(x - (q^{j+1} - 1))(x + (q^{j+1} + 1)) \]
and therefore the eigenvalues of the Kasami graph are as stated.

The valency $q^{2j+1} - 1$ has multiplicity one, but we need to calculate the multiplicities of the other three eigenvalues. Let eigenvalues $q^{j+1} - 1, -1, -(q^{j+1} + 1)$ have multiplicities $a, b, c$, respectively. The total number of eigenvalues is the number of vertices of $G(s, t)$, so we have
\[ a + b + c = q^{4j+2} - 1. \]

Letting $A$ be the adjacency matrix of the graph, we have that the sum of the eigenvalues is $tr A$, which is equal to 0, so
\[ a(q^{j+1} - 1) - b - c(q^{j+1} + 1) = 1 - q^{2j+1}. \]

Finally, the sum of the squares of the eigenvalues is $tr A^2$, which is equal to the sum of the valencies of the vertices, and so
\[ a(q^{j+1} - 1)^2 + b + c(q^{j+1} + 1)^2 = q^{4j+2}(q^{2j+1} - 1) - (q^{2j+1} - 1)^2. \]

The unique solution to this system of linear equations gives the stated set of multiplicities. □

Each $-1$ eigenvalue in $G(s, t)$ becomes a zero eigenvalue in the graph’s complement (see, for example, [6, Section 8.5.1]). Hence we can compute the rank:
\[ rk(G(s, t)) = q^{4j+2} - (q^{4j+2} - q^{4j+1} + q^{2j} - 1) = q^{4j+1} - q^{2j} + 1. \]

4. The chromatic number

Next we compute the chromatic number. To do this, we establish a simple lower bound using the intersection array, and show that the bound is exact by giving a coloring of the graph.
Lemma 2.
\[ \chi(G(s, t)) \geq q^{4j+1}. \]

\textbf{Proof.} Note that \( G(s, t) \) has \( s^2 \) vertices, and the intersection array has parameter
\[ a_1 = q - 2. \]
If \( G(s, t) \) contains a complete graph, that complete graph has no more than \( q \) vertices, since no more than \( a_1 = q - 2 \) vertices can be adjacent to \( (a, x) \) and \( (b, y) \) where \( (a, x) \sim (b, y) \). Hence the largest independent set in the complement of \( G(s, t) \) has at most \( q \) vertices, and the chromatic number of \( \overline{G}(s, t) \) is at least \( v/q \), where \( v = q^{4j+2} \) is the number of vertices in the graph. \( \square \)

To show that this lower bound is realized, we establish that the vertices of \( G(s, t) \) can be partitioned into sets of size \( q \), each of which forms a complete subgraph. Each of these \( v/q \) complete subgraphs is an empty subgraph in \( \overline{G}(s, t) \), giving a coloring of size \( v/q \).

Lemma 3. Let
\[ K = \{(x^{t+1}, x) \mid x \in F_s, x^t = x \}. \]
Then \( K \) is a complete subgraph in \( G(s, t) \) of size \( q \).

\textbf{Proof.} Let \((x^{t+1}, x)\) and \((y^{t+1}, y)\) be elements of \( K \), so that \( x^t = x \) and \( y^t = y \). Then
\[ (x + y)^{t+1} = (x^t + y^t)(x + y) = x^{t+1} + y^{t+1} + (x'y + y'x) = x^{t+1} + y^{t+1}. \]
But from the definition of the graph,
\[ (x^{t+1}, x) \sim (y^{t+1}, y) \iff x^{t+1} + y^{t+1} = (x + y)^{t+1}. \]
Thus every pair of vertices \((x^{t+1}, x)\) and \((y^{t+1}, y)\) is adjacent in \( K \).
To establish the size of \( K \), we must find the number of distinct elements of \( F_s \) that satisfy \( x^t = x \). From the proof of Lemma 2 we know that a complete subgraph in \( G(s, t) \) has no more than \( q \) vertices, so if \( q \) distinct elements in \( K \) exist, then \( |K| = q \).
Note that \( s \) is a power of \( q \), so \( q - 1 \) divides \( s - 1 \), and therefore there is an element \( z \) in \( F_s^s \) with order \( q - 1 \). Moreover, each element \( x \) in \( \langle z \rangle \) satisfies \( x^{q-1} = 1 \). Now \( t \) is also a power of \( q \), which means \( t - 1 \) is also a multiple of \( q - 1 \). Thus each \( x \) in \( \langle z \rangle \) satisfies \( x^{t-1} = 1 \) and \( x^t = x \). With 0 then, we have at least \( q \) distinct elements of \( F_s \) satisfying \( x^t = x \). This establishes the size of \( K \). \( \square \)

To partition the entire graph \( G(s, t) \) into complete subgraphs, we show that \( K \) forms a subgroup of the group of vertices and consider its cosets.

Lemma 4. \( K \) is an additive subgroup of \( F_s \times F_s \).

\textbf{Proof.} The identity \((0, 0)\) is clearly in \( K \), so it suffices to show that \( K \) is closed under addition. Let \((x^{t+1}, x)\) and \((y^{t+1}, y)\) be elements of \( K \). Then \( x + y \) is an element of \( F_s \) such that
\[ (x + y)^t = x^t + y^t = x + y. \]
Moreover, from the previous lemma,
\[ (x + y)^{t+1} = x^{t+1} + y^{t+1}. \]
Therefore \((x^{t+1}, x) + (y^{t+1}, y) = ((x + y)^{t+1}, x + y)\) is also an element of \( K \), and \( K \) is closed under addition. \( \square \)
Now consider a coset \((c, z) + K\) in \(G(s, t)\). Since any two vertices \((a, x)\) and \((b, y)\) in \(K\) satisfy \((a + b) = (x + y)^t + 1\), we see that two corresponding vertices \((a + c, x + z)\) and \((b + c, y + z)\) in \((c, z) + K\) satisfy \((a + c + b + c) = (x + z + y + z)^t + 1\). Thus every coset of \(K\) is also a complete subgraph of \(G(s, t)\).

The cosets of \(K\) form a partition of the vertices of \(G(s, t)\), which is the partition we needed to color the complement with \(v/q\) colors. Thus the chromatic number of \(\overline{G(s, t)}\) is \(q^{4^j + 1}\).

5. Remarks

Alon and Seymour noted in their paper that they could not show \(\chi(X)\) is not bounded by a linear function of \(\text{rk}(X)\); the Kasami graphs are also bounded linearly. However, Razborov [10] has found a sequence of graphs with a superlinear gap between rank and chromatic number.

In the case \(q = 2\) the Kasami graphs have a generalization, where the function \(x \mapsto x^{t+1}\) is replaced by an almost bent function (see Van Dam and Fon-Der-Flaass [12]). For \(a\) and \(x\) in \(F_s\), let \(a^T x\) denote the usual inner product when \(F_s\) is treated as a vector space over \(F_2\). Then a function \(f: F_s \to F_s\) is almost bent if

\[
\sum_{x \in F_s} (-1)^{a^T x} (-1)^{b^T f(x)} \in \{0, \pm 2^{(s+1)/2}\}
\]

for all pairs \((a, b) \neq (0, 0)\) in \(F_s \times F_s\). For any almost bent \(f\) with \(f(0) = 0\), the graph on vertices \(F_s \times F_s\) such that \((a, x) \sim (b, y) \iff a + b = f(x + y)\)

is distance-regular with the same parameters as the Kasami graph. It follows that the rank is the same as the Kasami graph. Since the clique number is 2 and the graph has a perfect matching (see for example [3, Corollary 4.2]), in fact the chromatic number is also the same as the Kasami graph. Therefore the gap between the rank and chromatic number also holds for this generalization.

It is interesting to note that the graph parameters do not have any reference to \(t\) in them, so \(G(s, t)\) and \(G(s, t')\) have the same parameters but might not be isomorphic.

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References