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A not so simple local multiplier algebra ☆

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To the memory of Gert Kjærgaard Pedersen

Abstract

We construct an AF-algebra A such that its local multiplier algebra $M_{\mathrm{loc}}(A)$ does not agree with $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$, thus answering a question raised by G.K. Pedersen in 1978.
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1. Introduction

For a C^*-algebra A denote by $M(A)$ its multiplier algebra. A closed (two-sided) ideal I of A is called *essential* if it has non-zero intersection with each non-zero ideal of A. Let I, J be closed essential ideals of A such that $J \subseteq I$. Then the restriction mapping induces a *-monomorphism $M(I) \to M(J)$. The direct limit of all $M(I)$ along the downward directed family of all closed essential ideals and with these connecting mappings is the local multiplier algebra $M_{\mathrm{loc}}(A)$ of A, first introduced by Pedersen in [9]. Further properties of $M_{\mathrm{loc}}(A)$ were studied in [2].

Among the questions which were left open in [9] are the following:

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(1) Is every derivation of $M_{\mathrm{loc}}(A)$ inner?
(2) Does the equality $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$ hold?

Pedersen showed that a derivation d on A can be (uniquely) extended to a derivation on $M_{\mathrm{loc}}(A)$, and becomes inner in $M_{\mathrm{loc}}(A)$ provided A is separable. (For a detailed account of his argument, and related questions, see [2, Section 4.2].) Despite some interesting contributions by Somerset [13], question (1) remains open. Note that a positive answer includes the classical results for simple C^*-algebras, for von Neumann algebras and for AW^*-algebras by Sakai, Kadison and Olesen, respectively (compare [10, Section 8.6]).

If the answer to question (2) were positive, to prove (1) it would suffice to show that every derivation on A becomes inner in $M_{\mathrm{loc}}(A)$. This occurs in particular when $M(A)$ is an AW^*-algebra or A is simple; for, in this case, $M_{\mathrm{loc}}(M(A)) = M(A)$ and $M_{\mathrm{loc}}(M(A))$ and $M_{\mathrm{loc}}(A)$ always coincide [2, Section 2.3]. It also occurs when $M_{\mathrm{loc}}(A)$ is an AW^*-algebra or is simple; the former holds for every commutative C^*-algebra [2, 3.1.5] and the latter is indeed possible in non-trivial cases as was shown in [1]. Further evidence for a positive answer is provided by the local Dauns–Hofmann theorem which implies that $Z(M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))) = Z(M_{\mathrm{loc}}(A))$ for every C^*-algebra A [2, 3.2.6]. Somerset showed in [13, Theorem 2.7] that (2) holds for every unital separable C^*-algebra A such that its primitive spectrum $\mathrm{Prim}(A)$ contains a dense G_δ of closed points; hence in particular if $\mathrm{Prim}(A)$ is Hausdorff. Argerami and Farenick recently derived (2) under the assumption that A is separable and contains a minimal essential ideal of compact elements; in this case $M_{\mathrm{loc}}(A)$ coincides with the injective envelope of A and is a type I von Neumann algebra [3].

In general, however, it turns out that the answer to question (2) is negative. In this paper we provide a class of examples to this effect. Our main result is the following.

Theorem 1.1. *There exist unital, primitive AF-algebras A such that $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$.*

The strategy to obtain such AF-algebras follows the ideas in [1], where we gave examples of non-simple AF-algebras A with the property that $M_{\mathrm{loc}}(A)$ is simple. To specify an AF-algebra it is, of course, enough to write down its K-theoretic invariant. It emerges, however, that working with the monoid $V(A)$ of equivalence classes of projections in $M_\infty(A)$ gives us a better control on the order-theoretic properties. Since $V(A)$ is cancellative in this case, this approach is of course equivalent to the usual K-theoretic one; however, it allows for a description of the ideal structure of the multiplier algebras of closed essential ideals of A (which is the decisive step in understanding $M_{\mathrm{loc}}(A)$). By work of Goodearl [7] and Perera [11], for a σ-unital C^*-algebra A of real rank zero and stable rank one, the monoid $V(M(A))$ can be completely described by the monoid of countably generated complemented intervals on $V(A)$. In order to obtain a like description of $V(M_{\mathrm{loc}}(A))$, a localisation procedure is needed, which was carried out in [1, Theorem 2].

Theorem 1.2. *Let A be a unital AF-algebra. Then $M_{\mathrm{loc}}(A)$ has real rank zero and $V(M_{\mathrm{loc}}(A))$ is isomorphic to $\Lambda_{\mathrm{loc}}(V(A), [1_A])$, the monoid of local intervals.*

All the necessary concepts and notation will be introduced in Section 2, where we shall construct a certain countable, abelian monoid M which, endowed with the algebraic order, leads to a localised monoid M' (representing $V(M_{\mathrm{loc}}(A))$) that has a unique minimal order-ideal. As a result, $M_{\mathrm{loc}}(A)$ has a unique minimal closed ideal I so that $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = M_{\mathrm{loc}}(A)$.

However, the tools available in the literature are not sufficient to compute the structure of the projections in the C^*-algebra $M(I)$. The reason is that I is not σ-unital and, moreover, the projections in I can fail to satisfy cancellation. To resolve this problem we use a different technique in Section 3, which is inspired by the geometry of our examples. We construct a *sequence* of projections in $M_{\mathrm{loc}}(A)$ strictly converging in the I-topology to a projection in $M(I) \setminus M_{\mathrm{loc}}(A)$. This allows us to conclude that $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) \neq M_{\mathrm{loc}}(A)$.

Both parts of the construction of our example are fairly technical. Thus, in a brief Section 4, we reflect on the nature of the example and possible further studies on the ideal structure of the local multiplier algebra.

2. Monoids

This section is devoted to the construction of an ordered monoid with very special properties. These will be exploited when it comes to exhibit the structure of the local multiplier algebra associated with the corresponding AF-algebra in the following section.

We begin by fixing our setting. Let X be an infinite compact metrizable space, and let t_0 be a non-isolated point in X. Denote by $C(X)$ the set of all continuous *real-valued* functions on X equipped with pointwise order. Let G be a countable, additive subgroup of $C(X)$ with the following properties:

(i) G is a sublattice of $C(X)$ and $\mathbb{Q}G \subseteq G$;
(ii) G contains a function f_0 such that $0 \leqslant f_0 \leqslant 1$, $f_0(t_0) = 0$ and $f_0(t) > 0$ for all $t \in X \setminus \{t_0\}$;
(iii) for each $f \in G$ there are an open neighbourhood V of t_0 and $\lambda,\ \mu \in \mathbb{Q}$ such that $f = \lambda + \mu f_0$ on V;
(iv) for every closed subset $K \subseteq X$, every open subset V with $K \subseteq V$ and every $\rho \in \mathbb{Q}_+$, there are an open subset U with $K \subseteq U \subseteq V$ and $r \in G$ such that $0 \leqslant r \leqslant \rho$ on X, $r = \rho$ on U and $r = 0$ on ${}^cV = X \setminus V$.

Property (iv) requires G to contain enough "Urysohn functions." It implies in particular that $1 \in G$ (take $K = X$ and $\rho = 1$).

Proposition 2.1. *There exists a countable subgroup G of $C(X)$ with the above properties* (i)–(iv).

Proof. We can take a countable set T of Urysohn functions such that, for every $f \in T$, either $f = 0$ or $f = 1$ on a neighbourhood of t_0 as follows. For each $n \in \mathbb{N}$, take open balls $U_0^{(n)}, U_1^{(n)}, \ldots, U_{k_n}^{(n)}$ of radius $1/n$ and centres $t_i^{(n)}$, for $i = 1, \ldots, k_n$, with $t_0^{(n)} = t_0$, which form a cover of X. Then consider pairs of open subsets U and V such that $\overline{V} \subseteq U$, and such that V and U are finite unions of some of the open balls $U_j^{(m_j)}$, but only those pairs (U, V) such that either $t_0 \in V$ or t_0 belongs to the interior of $X \setminus U$. For each such pair (U, V), choose a Urysohn function $f_{(U,V)}\colon X \to [0, 1]$ such that $f_{(U,V)} = 1$ on V and $f_{(U,V)} = 0$ on $X \setminus U$. The set T is defined as the set of all these functions $f_{(U,V)}$. It is clear that each function in T is either 0 or 1 on a neighbourhood of t_0, and it is not hard to see that the set $\{\mu f \mid f \in T,\ \mu \in \mathbb{Q}_+\}$ contains enough Urysohn functions in the sense of property (iv).

Set $G_1 = \mathbb{Q}T + \mathbb{Q}f_0 + \mathbb{Q}1$, the $\mathbb{Q}$-linear span of T, f_0 and 1, and observe that, for $f \in G_1$, there are rational numbers λ and μ such that $f = \lambda + \mu f_0$ on a neighbourhood of t_0.

Let T_1 be the set consisting of all the functions of the form $f \wedge g$ and $f \vee g$, for $f, g \in G_1$. It is then clear that each function in T_1 is of the form $\lambda + \mu f_0$, for some $\lambda, \mu \in \mathbb{Q}$, on a neighbourhood of t_0.

Proceeding inductively, suppose that we have a countable set of functions T_n with the property that, for each $f \in T_n$, there are $\lambda, \mu \in \mathbb{Q}$ such that $f = \lambda + \mu f_0$ on a neighbourhood of t_0. Then define G_{n+1} as the $\mathbb{Q}$-linear span of T_n, and define T_{n+1} as the set of all functions $f \wedge g$, $f \vee g$, for $f, g \in G_{n+1}$. Clearly T_{n+1} enjoys the same property as T_n. Finally, set $G = \bigcup_{n=1}^{\infty} G_n$. Then G satisfies the desired conditions. □

From now on, G will denote a countable subgroup of $C(X)$ satisfying the above properties (i)–(iv). Whenever $t \in X$, we write $V(t)$ to denote *some* open neighbourhood of t and $V[t]$ to denote the punctured neighbourhood $V(t) \setminus \{t\}$. Let

$$M = \{ f \in G \mid f \geqslant 0,\ f(t) > 0 \text{ on } V[t_0] \} \cup \{0\}.$$

Note that M is a countable, additive monoid, closed under multiplication by positive rational numbers. The *algebraic order* in M will be denoted by $\leqslant_M$. We fix a canonical order-unit $u = 1$ in M. The notation $f \ll g$ on Y will be used as a shorthand for $f(t) < g(t)$ for all $t \in Y$, where $f, g \in C(X)$ and $Y \subseteq X$.

We recall that the *algebraic pre-order* on an abelian monoid L is defined by

$$x \leqslant_L y \quad \text{if } y = x + z \text{ for some } z \in L.$$

In the case where L is cancellative, $\leqslant_L$ is a (partial) order. We write $x <_L y$ if $x \leqslant_L y$ and $x \neq y$. We also recall a few order-theoretic concepts that will be used in the following. (For more details, see [1,11].)

An *order-ideal* of $(L, \leqslant_L)$ is a hereditary submonoid; an *order-unit* is an element such that L is the smallest order-ideal containing it; an *interval* is an upward directed hereditary non-empty subset of L. The monoid L is said to be *prime* if each pair of non-zero order-ideals of L has non-zero intersection. Suppose that L is a *Riesz monoid*; that is, whenever $x, y_1, y_2 \in L$ satisfy $x \leqslant_L y_1 + y_2$ there exist $x_1, x_2 \in L$ such that $x = x_1 + x_2$ and $x_i \leqslant_L y_i$ for $i = 1, 2$. Then the sum $E + F$ of two intervals E and F in L is defined by

$$E + F = \{x + y \mid x \in E,\ y \in F\}$$

and is an interval in L. Let D be a fixed interval. The interval E is said to be *complemented* (*with respect to* D) if there are an interval F and some $k \geqslant 1$ such that $E + F = kD$. We denote by $\Lambda(L, D)$ the abelian monoid of all complemented intervals in L.

The following important localisation procedure will be applied to various Riesz monoids in the sequel. Suppose L is a prime Riesz monoid with order-unit v. Let N be an order-ideal in L; then its *canonical interval* D_N is defined by $D_N = \{x \in N \mid x \leqslant_L v\}$. Let N_1, N_2 be order-ideals in L with $N_1 \subseteq N_2$. The restriction mapping

$$\phi_{N_1,N_2} : \Lambda(N_2, D_{N_2}) \to \Lambda(N_1, D_{N_1})$$

defined by $\phi_{N_1,N_2}(E) = E \cap N_1$ is a monoid homomorphism. Whenever $N_1 \subseteq N_2 \subseteq N_3$ is a chain of order-ideals, we have $\phi_{N_1,N_3} = \phi_{N_1,N_2}\phi_{N_2,N_3}$. Therefore we can define the *monoid of*

local intervals $\Lambda_{\mathrm{loc}}(L, v)$ of (L, v) as the direct limit of the family

$$\{\Lambda(N, D_N);\ \phi_{N_1,N_2},\ N_1 \subseteq N_2\},$$

where N runs through the downward directed set of all non-zero order-ideals of L.

This procedure will now be applied to the monoid (M, u).

The proof of our first lemma is exactly the same as the one of the first part of [1, Theorem 3] and hence is omitted.

Lemma 2.2. *The monoid $(M, \leqslant_M)$ is a prime cancellative Riesz monoid.*

For each non-zero $f \in M$, in order to simplify the notation, set

$$\begin{aligned}
N_f &= \{g \in M \mid g \leqslant_M nf \text{ for some } n \geqslant 1\},\\
N_f^* &= N_f \setminus \{0\},\\
D_f &= \{g \in N_f \mid g \leqslant_M u\},\\
L_f &= \{g \in C_b(U_f)_+ \mid \exists z \in N_f^*:\ z \leqslant g \text{ on } U_f\} \cup \{0\},
\end{aligned}$$

where U_f stands for the co-zero set of f. For $f' \leqslant_M f$ we have a canonical map

$$\phi_{f',f} : \Lambda(N_f, D_f) \to \Lambda(N_{f'}, D_{f'})$$

defined by $\phi_{f',f}(E) = E \cap N_{f'}$. (That is, $\phi_{f',f} = \phi_{N_{f'},N_f}$.)

We will denote by M' the prime Riesz monoid $M' = \Lambda_{\mathrm{loc}}(M, u) = \varinjlim \Lambda(N_f, D_f)$; in general, this may not be cancellative. For an interval E in $\Lambda(N_f, D_f)$, where $f \in M \setminus \{0\}$, we denote the class of E in M' by $\overline{E}$. There is a canonical order-unit in M' given by $u' = \overline{[0, u]}$. Let J be the order-ideal of M' generated by $\overline{[0, f_0]}$.

One of the key properties of G, as stated in Proposition 2.1, is that each f in G is of the form $\lambda + \mu f_0$ on $V(t_0)$ for some $\lambda, \mu \in \mathbb{Q}$. This obviously implies that, given $f \in M$ such that $f(t_0) = 0$, there is a rational number μ such that $f = \mu f_0$ on $V(t_0)$.

Proposition 2.3. *The monoid M' has a unique minimal order-ideal J, the order-ideal generated by $\overline{[0, f_0]}$.*

Proof. It suffices to show that, for every non-zero $x \in M'$, we have $\overline{[0, f_0]} \leqslant_{M'} nx$ for some $n \in \mathbb{N}$. Let $E \in \Lambda(N_f, D_f)$ be a representative of x, where $0 \neq f \leqslant_M f_0$. Take a non-zero element g in E. Since $g(t_0) = 0$, there is a rational number $\mu > 0$ such that $g = \mu f_0$ on $V(t_0)$. Thus, on $V[t_0]$, we have $f_0 \ll ng$ for some $n \in \mathbb{N}$. Take $f' \in M \setminus \{0\}$ with $f' \leqslant_M f$ such that $U_{f'} \subseteq V(t_0)$. Observe that

$$[0, f_0] \cap N_{f'} + \big[0, (n\mu - 1) f_0\big] \cap N_{f'} = [0, n\mu f_0] \cap N_{f'} = [0, ng] \cap N_{f'}.$$

On the other hand, $ng \in nE$ so that $nE = [0, ng] + E'$, where E' is the interval in M defined as

$$E' = \{z \in M \mid z + ng \in E\}.$$

It follows that $[0, ng] \cap N_{f'} + E' \cap N_{f'} = nE \cap N_{f'}$, and so

$$[0, f_0] \cap N_{f'} + \big([0, (n\mu - 1) f_0] \cap N_{f'} + E' \cap N_{f'}\big) = n(E \cap N_{f'}),$$

which shows that $\overline{[0, f_0]} \leqslant_{M'} n\overline{E} = nx$, as desired. □

In Section 3 we shall need a functional representation of the monoid M'. Let f be a non-zero element of M such that $f \leqslant_M f_0$. Note that the set of order-ideals N_f with such f is cofinal; so in order to study M', we may restrict attention to those $\Lambda(N_f, D_f)$. Fix such an element f. Then, for every $z \in N_f$, we have $z(t_0) = 0$, so our hypothesis gives that for some $\mu \in \mathbb{Q}^+$ we have $z = \mu f_0$ on $V(t_0)$.

For $h \in C_b(U_f)_+$ we set

$$I_f(h) = \big\{g \in N_f \mid \exists z \in N_f\colon\ g <_M z,\ z \ll h \text{ on } V[t_0],\ z \leqslant h \text{ on } U_f\big\} \cup \{0\}.$$

The following description of $I_f(h)$ will be used subsequently several times without specific reference.

Lemma 2.4. *For every $h \in L_f \setminus \{0\}$,*

$$\begin{aligned} I_f(h) &= \big\{g \in N_f \mid g \leqslant h \text{ and } g \ll g' \ll h \text{ on } V[t_0] \text{ for some } g' \in N_f\big\} \\ &= \{g \in N_f \mid g <_M z \leqslant h \text{ for some } z \in N_f\}. \end{aligned}$$

Moreover, $I_f(h)$ is an interval in N_f.

Proof. Put $\tilde{I}_f(h) = \{g \in N_f \mid g \leqslant h \text{ and } g \ll g' \ll h \text{ on } V[t_0] \text{ for some } g' \in N_f\}$. It is evident that $I_f(h) \subseteq \tilde{I}_f(h)$.

To show the reverse inclusion, assume that $g \in N_f$ is such that $g \leqslant h$ and $g \ll g' \ll h$ on $V[t_0]$ for some $g' \in N_f$. Take an open neighbourhood V of t_0 with $\overline{V} \subseteq V(t_0)$. Let $r \in M$ be such that $r = \rho \gg g'$ on V, $r = 0$ on ${}^cV(t_0)$ and $0 \leqslant r \leqslant \rho$ for some $\rho \in \mathbb{Q}$. Then $g' \wedge r = g'$ on V, $g' \wedge r \ll h$ on $V[t_0]$ and $g' \wedge r = 0$ on ${}^cV(t_0)$. Let $z = (g' \wedge r) \vee g \in N_f$. Then $g <_M z$, $z \ll h$ on $V[t_0]$ and $z \leqslant h$ on U_f; thus $g \in I_f(h)$.

Put $I'_f(h) = \{g \in N_f \mid g <_M z \leqslant h \text{ for some } z \in N_f\}$; clearly $I_f(h) \subseteq I'_f(h)$. On the other hand, if $g \in I'_f(h)$ and $z \in N_f$ satisfies $z \leqslant h$ on U_f and $g <_M z$ then we take $r \in M$ such that $r = \rho \gg z - g$ on V, an open neighbourhood of t_0 with $\overline{V} \subseteq V(t_0)$ and $g \ll z$ on $V[t_0]$, $r = 0$ on ${}^cV(t_0)$ and $0 \leqslant r \leqslant \rho$. Upon replacing z by $\frac{1}{2}(z - g) \wedge r + g \in N_f$ we find that $g \in I_f(h)$.

Clearly $I_f(h)$ is a non-empty hereditary subset of N_f. To show that it is upward directed take $g_1, g_2 \in I_f(h)$. There are $\mu_1, \mu_2 \in \mathbb{Q}$ such that $g_i = \mu_i f_0$ on $V(t_0)$, $i = 1, 2$. We may assume that $\mu_1 \geqslant \mu_2$. There exists $g' \in N_f$ such that $g_1 <_M g'$, $g' \ll h$ on $V'[t_0]$, where $V'(t_0) \subseteq V(t_0)$, and $g' \leqslant h$ on U_f. Take an Urysohn function $r \in M$ as above, where $r = \rho \gg g' - g_1$ on V with $\overline{V} \subseteq V'(t_0)$ and $r = 0$ on ${}^cV'(t_0)$. Set $g'' = \frac{1}{2}(g' - g_1) \wedge r + g_1 \vee g_2$. Then $g_1 \leqslant_M g''$, $g_2 \leqslant_M g''$ and for $z = ((g' - g_1) \wedge r) + g_1 \vee g_2 \in N_f$ we have $g'' <_M z$, $z \ll h$ on $V'[t_0]$ and $z \leqslant h$ on U_f. Thus $g'' \in I_f(h)$. □

Under the standing assumption that $f \in M \setminus \{0\}$ with $f \leqslant_M f_0$ is given, we will now define a monoid homomorphism $\tau_f : \Lambda(N_f, D_f) \to L_f$. For $E \in \Lambda(N_f, D_f)$ let $\tau_f(E) = \sup E$ be

the pointwise supremum over all functions in E, restricted to U_f. Then τ_f has the following properties.

(1) $\tau_f(D_f) = 1$ on U_f;

This follows easily from the existence of sufficiently many Urysohn functions in G.

(2) $\tau_f(E_1 + E_2) = \tau_f(E_1) + \tau_f(E_2)$ for all $E_1, E_2 \in \Lambda(N_f, D_f)$;

This is straightforward. As a consequence of (1) and (2), $\tau_f(E)$ is a continuous function on U_f and τ_f is a monoid homomorphism.

(3) $h = \sup I_f(h)$ for each $h \in L_f$.

It follows from the definition of $I_f(h)$ that $h \geqslant \sup I_f(h)$. In order to show the converse inequality, suppose that $h \neq 0$, let $\varepsilon > 0$ and take $z \in N_f^*$ with $z \leqslant h$. Let $t \in U_f$ be such that $h(t) > 0$. Let V, V_0 be disjoint open neighbourhoods of t and t_0, respectively, with the property that $\overline{V} \subseteq U_f$, $h(s) - \varepsilon < \rho < h(s)$ for all $s \in V$ and some $\rho \in \mathbb{Q}_+$ and $z \gg 0$ on $V_0 \setminus \{t_0\}$. There is $r \in M$ such that $0 \leqslant r \leqslant \rho$, $r = \rho$ on W and $r = 0$ on ${}^cV$, where W is some open neighbourhood of t with $\overline{W} \subseteq V$. There is $r' \in M$ such that $0 \leqslant r' \leqslant 1$, $r' = 1$ on W' and $r = 0$ on ${}^cV_0$, where W' is some open neighbourhood of t_0 with $\overline{W'} \subseteq V_0$. Then $g = \frac{1}{4}(z \wedge r') \vee r \in N_f$. In fact, $g \in I_f(h)$ since, for $g' = \frac{1}{2}(z \wedge r') \in N_f$, we have $g \ll g' \ll h$ on $W' \setminus \{t_0\}$. As $g(t) = r(t) = \rho > h(t) - \varepsilon$ it follows that $h = \sup I_f(h)$.

Let $f' \in M \setminus \{0\}$, $f' \leqslant_M f$. Let $R: L_f \to L_{f'}$ denote the restriction mapping. (Note that $h_{|U_{f'}} \in L_{f'}$ if $h \in L_f$. For $h = 0$ this is evident, so assume that h is non-zero. By definition, there is $z \in N_f^*$ with $z \leqslant h$ on U_f wherefore $z \wedge f' \in N_{f'}^*$ and $z \wedge f' \leqslant h$ on $U_{f'}$.)

We obtain the following commutative diagram.

$$\begin{array}{ccc} \Lambda(N_f, D_f) & \xrightarrow{\tau_f} & L_f \\ {\scriptstyle \phi_{f',f}}\downarrow & & \downarrow{\scriptstyle R} \\ \Lambda(N_{f'}, D_{f'}) & \xrightarrow[\tau_{f'}]{} & L_{f'} \end{array} \tag{2.1}$$

To verify the commutativity of the diagram take $E \in \Lambda(N_f, D_f)$ and note at first that

$$\tau_{f'} \circ \phi_{f',f}(E) = \sup(E \cap N_{f'}) \leqslant \sup E_{|U_{f'}}.$$

For the reverse inequality, let $h = \sup E$ and let $t \in U_{f'}$ such that $h(t) > 0$. Let $\varepsilon > 0$. By property (3) above, there is $g' \in I_{f'}(h)$ such that $h(t) \geqslant g'(t) > h(t) - \varepsilon$. By definition of h, there is $z \in E$ such that $z(t) \geqslant g'(t)$. Put $g = g' \wedge z \in E \cap N_{f'} = \phi_{f',f}(E)$. As $g(t) = g'(t)$ it follows that $\sup \phi_{f',f}(E) \geqslant h_{|U_{f'}}$.

Set $L = \varinjlim L_f$, where the morphisms $L_f \to L_{f'}$ for $f' \leqslant_M f$ are given by restriction. The commutativity of the above diagram enables us to define a monoid homomorphism

$$\tau : M' \to L$$

via the maps $\tau_f : \Lambda_f(N_f, D_f) \to L_f$. In contrast to [1, Theorem 3], τ may be not injective in general and hence M' need not be cancellative.

In order to describe the image of τ we use a similar approach as in [1]. For each non-zero $f \in M$, we say that the function $h \in L_f$ has the property (C) provided that, for each $t \in U_f \setminus \{t_0\}$, there exists $z_t \in N_f$ such that $z_t \leqslant h$ on U_f, $h = z_t$ on $V(t)$ and $z_t \ll h$ on $V[t_0]$. The subset C of L consists of those elements $y \in L$ such that there is a representative $h \in L_f$ of y with property (C), together with $y = 0$. Evidently C is a submonoid of L.

Lemma 2.5. *Let $x, y \in C$. If $x <_L y$ then $y - x \in C$.*

Proof. Without restricting the generality we can assume that both x and y are non-zero. By hypothesis, there is $x' \in L \setminus \{0\}$ such that $y = x + x'$. Let $f \in M \setminus \{0\}$ and $g, h \in C_b(U_f)_+$ be such that g, h and $h - g$ are representatives of x, y and x', respectively, and both g and h have property (C). We need to show that $h - g$ has property (C) as well.

Fix $t \in U_f \setminus \{t_0\}$. By assumption, there are $z = z_t \in N_f^*$, $w = w_t \in N_f^*$ and $v \in N_f^*$ with the following properties:

(i) $z \leqslant g$, $w \leqslant h$ and $v \leqslant h - g$ on U_f;
(ii) $z = g$, $w = h$ on $V(t)$;
(iii) $z \ll g$, $w \ll h$ and $v \ll h - g$ on $V[t_0]$.

Note that $V(t) \cap V(t_0) = \emptyset$.

Since $g \leqslant h$ on U_f, it follows from (ii) that $z \leqslant w$ on $V(t)$. Take $r \in G$, $r \geqslant 0$ such that $r \geqslant w - z$ on $V'(t) \subseteq V(t)$ and $r = 0$ on ${}^cV(t)$. Then

$$(w - z) \wedge r \in G, \qquad (w - z) \wedge r = w - z \quad \text{on } V'(t),$$
$$(w - z) \wedge r \leqslant w - z \quad \text{on } U_f, \qquad (w - z) \wedge r \leqslant 0 \quad \text{on } {}^cV(t).$$

Put $d = 0 \vee ((w - z) \wedge r) \in G$; then $0 \leqslant d$, $d \leqslant w - z$ on $V(t)$, $d = w - z$ on $V'(t)$ and $d = 0$ on ${}^cV(t)$. Let $e = d \vee v \in G$; in fact, $e \in N_f$. Then $e \leqslant h - g$ on U_f, $e \ll h - g$ on $V[t_0]$ and $e = h - g$ on $V'(t)$. Thus $h - g$ has property (C) and so $y - x \in C$. □

We next show that the functions with property (C) enjoy an even stronger property.

Lemma 2.6. *Let $h \in L_f$ have property* (C)*, where $f \in M \setminus \{0\}$, $f \leqslant_M f_0$. For a compact subset K of X such that $K \subseteq U_f$, there is $z \in I_f(h)$ such that $z = h$ on K. Moreover, if $v \in I_f(h)$, then we can choose z such that, in addition, $v \leqslant_M z$.*

Proof. Since K is compact, there exist $t_1, \ldots, t_n \in K$ such that, for some $z_j = z_{t_j} \in N_f$, $z_j \leqslant h$ on U_f, $z_j = h$ on $V(t_j)$, $z_j \ll h$ on $V[t_0]$ for all $1 \leqslant j \leqslant n$ and $K \subseteq V$, where $V = \bigcup_{j=1}^n V(t_j)$. Let U_j be open neighbourhoods of t_0 such that $0 \ll z_j \ll h$ on $U_j \setminus \{t_0\}$ and put $U = \bigcap_{j=1}^n U_j$. Note that $U_j \cap V(t_j) = \emptyset$ so that $U \cap V = \emptyset$.

Letting $z' = z_1 \vee \cdots \vee z_n \in N_f$ we have $z' \leqslant h$ on U_f, $z' = h$ on K and $z' \ll h$ on $U \setminus \{t_0\}$. Take $r \in M$ such that $r \geqslant h$ on some open neighbourhood U' of t_0 contained in U and $r = 0$ on ${}^cU$. Take $r' \in M$ such that $r' \geqslant h$ on some open set $V' \subseteq V$ containing K and $r' = 0$ on ${}^cV$. Letting $z = \frac{1}{2}(z' \wedge r) + z' \wedge r' \in N_f$ we have $z \leqslant h$ on U_f, $z = h$ on K and $z \ll z' \ll h$ on $U' \setminus \{t_0\}$. Thus $z \in I_f(h)$.

Now assume that $v \in I_f(h)$. Since both z and v belong to the interval $I_f(h)$, there is $z'' \in I_f(h)$ such that $z \leqslant_M z''$ and $v \leqslant_M z''$, and obviously $z'' = h$ on K. □

The image of τ can now be identified.

Proposition 2.7. *The monoid homomorphism τ maps M' onto C.*

Proof. We first show $\tau(M') \subseteq C$. Let $E \in \Lambda(N_f, D_f) \setminus \{0\}$, where $f \in M \setminus \{0\}$, $f \leqslant_M f_0$. We have to verify that $h = \tau_f(E) = \sup E$ has property (C).

Let $t \in U_f$ and take $b \in N_f$ such that $b = 1$ on $V(t)$ and $b \leqslant_M 1$; then $b \in D_f$. Let $E' \in \Lambda(N_f, D_f)$ and $k \geqslant 1$ be such that $E + E' = kD_f$. Take $z \in E$ and $z' \in E'$ with $z + z' = kb$. Letting $h' = \sup E'$ it follows that $h + h' = k$, since τ_f is a monoid homomorphism. Consequently

$$k = h + h' \geqslant z + z' = kb = k \quad \text{on } V(t)$$

and so $z = h$ on $V(t)$. Clearly $z \leqslant h$ on U_f.

Let $r, r' \in M$ be such that $r \geqslant h$ on $V'(t) \subseteq V(t)$, $r = 0$ on ${}^cV(t)$ and $r' \geqslant h$ on $V'(t_0) \subseteq V(t_0)$, $r' = 0$ on ${}^cV(t_0)$ for some $V(t_0)$ such that $V(t_0) \cap V(t) = \emptyset$. (This can be achieved by possibly shrinking the neighbourhood $V(t)$.) Letting $z_t = \frac{1}{2}(z \wedge r') + z \wedge r$ we obtain $z_t \in N_f$ such that $z_t \leqslant h$ on U_f, $z_t = h$ on $V'(t)$ and $z_t \ll h$ on $V'[t_0]$. Therefore, h has property (C) and thus $\tau(\overline{E}) \in C$.

In order to establish the reverse inclusion, $C \subseteq \tau(M')$, note that for each non-zero $y \in C$ there is $n \geqslant 1$ such that $y <_L n1$, whence $n1 - y \in C$ by Lemma 2.5. There exists a representative $h \in L_f$ of y, for some non-zero $f \in M$ with $f \leqslant_M f_0$, such that both h and $n1 - h$ have property (C). Moreover, we can assume that $n1 - h \gg \varepsilon$ on U_f for some $\varepsilon > 0$. We will show that $I_f(h) + I_f(n1 - h) = nD_f$ which, together with property (3) above, entails that $h = \sup I_f(h) = \tau_f(I_f(h))$ and thus $y \in \tau(M')$.

To this end take $0 \neq g \in nD_f$. Put $\delta = \varepsilon/3$ and let $K_1 = \{t \in X \mid g(t) \leqslant \delta/3\}$ and $K_2 = \{t \in X \mid g(t) \geqslant \delta/2\}$. For each rational number ρ with $0 < n1 - \rho < \delta$ we get, by using the assumption on the richness in Urysohn functions, an element $a \in G$ such that $0 \leqslant a \leqslant \rho$, $a = \rho$ on K_2 and $a = 0$ on K_1. Let $g' = g \wedge (n1 - a) \in N_f$. Note that $g' = g$ on K_1, $g' \leqslant g$ and $g' \ll \delta$ on U_f. Set $K_3 = \{t \in X \mid g(t) \geqslant \delta/3\}$. Then $g - g'$ is supported on K_3. Take a non-zero function $v \in N_f$ supported on a compact neighbourhood K_4 of t_0 such that $K_4 \cap K_3 = \emptyset$, and with $v \ll g'$ on $K_4 \setminus \{t_0\}$. Put $g'' = g - g' + v$. Note that $g' + v \ll 2\delta \ll n1 - h$ on U_f. Set $h' = n1 - h - g'$; then $v \in I_f(h')$ because $v + \delta \ll h'$ on U_f, and h' has property (C) by the arguments in Lemma 2.5.

Applying Lemma 2.6 to h and h', respectively, we get $z_1 \in I_f(h)$ and $z_2 \in I_f(h')$ such that $z_1 = h$ on K_3, $z_2 = h'$ on K_3 and $v \leqslant_M z_2$. We claim that $g'' \leqslant_M z_1 + z_2$. Indeed, on K_3, we have $z_1 = h$ and $z_2 = h' = n1 - h - g'$, so that $z_1 + z_2 = n1 - g' \geqslant g - g' = g''$. On ${}^cK_3$, we have $g'' = v$, because $g - g'$ is supported on K_3 and $v \leqslant_M z_2 <_M z_1 + z_2$. It follows that $g'' \leqslant_M z_1 + z_2$.

Since M has the Riesz property, we get $g'' = g_1 + g_3$ with $g_1, g_3 \in N_f$ and $g_1 \leqslant_M z_1$ and $g_3 \leqslant_M z_2$. Set $g_2 = g_3 + g' - v \in N_f$, and observe that

$$g_2 = g_3 + g' - v \leqslant_M z_2 + g' - v \leqslant z_2 + g' \in I_f(n1 - h)$$

so that $g = g'' + g' - v = g_1 + g_2$ with $g_1 \in I_f(h)$ and $g_2 \in I_f(n1 - h)$, as desired. □

Remark 2.8. The proof of Proposition 2.7 in fact shows that, for every $h \in L_f$ with property (C), the interval $I_f(h)$ is complemented.

3. Multiplier algebras

In this section we will use the properties of the monoids studied above to construct a C^*-algebra A with the property that $M_{\mathrm{loc}}(A) \neq M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$. Throughout let (M, u) be a fixed monoid as considered in Section 2. For every C^*-algebra B of real rank zero, see, e.g., [5], there is an isomorphism between its lattice of closed ideals and the lattice of order-ideals of the monoid $V(B)$ ([6], [14, Theorem 2.3], [11, Theorem 2.1]).

Let A be the prime, unital AF-algebra such that $(V(A), [1_A]) = (M, u)$. By the Blackadar–Handelman theorem [4, Theorem 6.9.1], the trace simplex of A is precisely $M_1^+(X)$, the simplex of probability measures on X. By Lin's theorem [8, Corollary 3.7], $M(I)$ has real rank zero for every closed ideal I of A. For an element $f \in M$, $f \leqslant_M u$ denote by $I_f = \overline{ApA}$ the closed ideal generated by a projection $p \in A$ such that $[p] = f$. The order-ideal $V(I_f)$ is precisely $N_f = \{g \in M \mid g \leqslant_M nf \text{ for some } n \in \mathbb{N}\}$. For a projection $q \in M(I_f)$ define

$$\operatorname{supp}(q) = \big\{t \in U_f \mid \tau_f(D)(t) \neq 0\big\} \subseteq U_f,$$

where D is the interval in $\Lambda(N_f, D_f)$ corresponding to $[q]$ via the canonical isomorphism between $V(M(I_f))$ and $\Lambda(N_f, D_f)$ [11, Theorem 2.4] and $\tau_f : \Lambda(N_f, D_f) \to L_f$ is the canonical map defined in Section 2.

Consider a fundamental sequence $(I_n)_{n\geqslant 0}$ of closed ideals of A (compare Section 4). The closed ideals I_n are assumed to be of the form I_{f_n}, where $f_n \in M$ and $f_n \leqslant_M f_{n-1}$ for $n \geqslant 1$, and f_0 is the distinguished function in M. The fact that $(I_n)_{n\geqslant 0}$ is a fundamental sequence means that, for each open neighbourhood $V(t_0)$, there is some n_0 such that $U_{f_n} \subseteq V(t_0)$ for all $n \geqslant n_0$. Given such a sequence $(I_n)_{n\geqslant 0}$ we have $M_{\mathrm{loc}}(A) = \varinjlim M(I_n)$, with canonical maps $\varphi_{m,n}: M(I_n) \to M(I_m)$, $n \leqslant m$ and $\varphi_n: M(I_n) \to M_{\mathrm{loc}}(A)$, $n \geqslant 0$. Since the φ_n's are isometric embeddings, we will subsequently suppress them when no ambiguity can arise.

Let I be the closed ideal of $B = M_{\mathrm{loc}}(A)$ generated by p_0, where $p_0 \in A$ is a projection such that $[p_0] = f_0$. Then $V(I) = J$ is the unique minimal order-ideal of M', cf. Proposition 2.3; thus I is the unique minimal closed ideal of B (use Theorem 1.2) and so $M_{\mathrm{loc}}(B) = M(I)$. Our aim is to construct a sequence of projections $(p_n)_{n\in\mathbb{N}}$ in B such that $(p_n)_{n\in\mathbb{N}}$ is strictly convergent in $M(I)$ to a projection $p \in M(I)$, but $p \notin B$. This will ensure that $B \neq M_{\mathrm{loc}}(B)$.

Put $B_n = M(I_n) \subseteq B$, $n \geqslant 0$, and let I_n' denote the closed ideal of B_n generated by I_0. Then $I_0 = I_0' \subseteq I_1' \subseteq \cdots \subseteq I_n' \subseteq \cdots$ and $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n \subseteq \cdots$, and

$$B = \overline{\bigcup_{n=1}^{\infty} B_n} \quad \text{and} \quad I = \overline{\bigcup_{n=1}^{\infty} I_n'}. \tag{3.1}$$

Moreover, note that $B_n = M(I_n')$ for all $n \geqslant 0$ since I_n is an ideal in I_n' and hence $B_n \subseteq M(I_n') \subseteq M(I_n) = B_n$.

The following easy observation enables us to manoeuvre between different multiplier algebras.

Lemma 3.1. *Let (x_n) be a sequence in the ideal I_0 converging in the strict I-topology to $x \in M(I)$. Then $x \in M(I_0)$.*

Proof. Clearly (x_n) converges in the strict I_0-topology too, with limit $y \in M(I_0)$, say. Since $(x-y)I_0 = 0$, it suffices to show that $cI_0 \neq 0$ whenever $0 \neq c \in M(I)$.

Let c be a non-zero element in $M(I)$. There are a closed essential ideal I' of A such that $I' \subseteq I_0$ and $z \in M(I') \cap I$ such that $\|cz\| = 1$. Given $0 < \varepsilon < 1/2$, there are another closed essential ideal $I'' \subseteq I'$ and $z' \in M(I'')$, $\|z'\| = 1$ such that $\|cz - z'\| < \varepsilon$. Let $a \in I''$ with $\|a\| = 1$ be such that $\|z'a\| > 1 - \varepsilon$. Then

$$\|cza\| \geqslant \|z'a\| - \|cza - z'a\| > 1 - 2\varepsilon > 0.$$

Hence $c(za) \neq 0$, and $za \in M(I')I'' \subseteq I'' \subseteq I_0$ as desired. □

The next two results are at the core of our construction.

Lemma 3.2. *Let $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ be increasing sequences of projections in B such that, for each n, $p_n, q_n \in B_n$ and $p_n + q_n \in B_0$. Suppose that* $\operatorname{dist}(p_n, B_{n-1}) \geqslant \delta$ *for all $n \geqslant 1$ and some $\delta > 0$. If $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ converge strictly in $M(I)$ to projections p and q, respectively, such that $p + q = 1$ then $p \notin B$ and $q \notin B$.*

Proof. Suppose that $p \in B$. There is a projection $p' \in B_n$ for some $n \in \mathbb{N}$ such that $\|p - p'\| < \delta$. Note that

$$(p_{n+1} + q_{n+1})p = p_{n+1}$$

and $(p_{n+1} + q_{n+1})p' \in B_0B_n = B_n$. Since

$$\big\|p_{n+1} - (p_{n+1} + q_{n+1})p'\big\| = \big\|(p_{n+1} + q_{n+1})(p - p')\big\| \leqslant \|p - p'\| < \delta,$$

we obtain that $\operatorname{dist}(p_{n+1}, B_n) < \delta$ contradicting our hypothesis. Therefore $p \notin B$ and so $q = 1 - p \notin B$. □

Proposition 3.3. *With the same notation and caveats as above, take compact neighbourhoods K_n, $n \geqslant 0$, of t_0 such that $K_n \subseteq K_{n-1} \subseteq U_{f_{n-1}} \cup \{t_0\}$ for all $n \geqslant 1$. Let $(h_n)_{n\geqslant 0}$ be an increasing approximate identity consisting of projections for I_0. Let $(h''_n)_{n\geqslant 0}$ be a sequence of non-zero projections in I_0 such that $h''_n \leqslant h_n - h_{n-1}$ and* $\operatorname{supp}(h''_n) \subseteq K_n$ *for all n (so that, in particular, $h''_n \in I_n$). Set $h'_n = h''_0 + h''_1 + \cdots + h''_n$. Then $(h'_n)_{n\geqslant 0}$ converges in the strict topology of $M(I)$.*

Proof. By identity (3.1), the fact that I_0 is an AF-algebra and since $B_nI_0B_n$ is dense in I'_n, every element in I can be approximated by an element of the form $\sum_{i=1}^{k} y_ie_iz_i$, for some projections $e_i \in I_0$ and $y_i, z_i \in M(I_n)$. It suffices to consider the case $k = 1$, hence assume that $x = yez$ with $e \in I_0$ a projection and $y, z \in M(I_n)$. Given $\varepsilon > 0$, we have to find n_0 such that, for $m > \ell \geqslant n_0$, we have $\|(h'_m - h'_\ell)x\| < \varepsilon$.

For $m \geqslant n \geqslant 1$, we have $h'_m - h'_{n-1} = \sum_{i=n}^{m} h''_i \in I_n$. The sequence $(h'_m - h'_{n-1})_{m\geqslant n}$ converges in the strict I_0-topology to an element $q \in M(I_0)$. This follows because $h''_i \leqslant h_i - h_{i-1}$ and $(h_i)_{i\geqslant 0}$ is an approximate identity for I_0. Note that $\operatorname{supp}(q) \subseteq K_n$ by hypothesis.

We note that $qy \in M(I_n)$ and that $M(I_n)$ is a C^*-algebra of real rank zero (as I_n is an AF-algebra). Hence, given $0 < \eta < 1$, there is a projection $p \in M(I_n)qy$ such that $\|qy - qyp\| < \eta$.

Then p is equivalent to a subprojection of q and thus $\operatorname{supp}(p) \subseteq K_n$. It follows from Lemma 3.4 below that $pe = pe'$ for some projection $e' \in I_n$, so that $pe \in I_n$ and thus

$$qypez \in M(I_n) I_n M(I_n) \subseteq I_n.$$

From

$$\|qyez - qypez\| < \eta \|z\|$$

we find $\operatorname{dist}(qyez, I_n) < \eta\|z\|$. Since this holds for all $0 < \eta < 1$, it follows that $qyez \in I_n$.

For all $m \geqslant n$ we have $(h'_m - h'_{n-1})(yez) = (h'_m - h'_{n-1})q(yez)$ and, since $qyez \in I_n$ and $(h'_k)_{k\geqslant 0}$ is a Cauchy sequence in the strict topology of $M(I_0)$, there is some $n_0 \geqslant n$ such that, for all $m > \ell \geqslant n_0$,

$$\big\|(h'_m - h'_\ell)yez\big\| = \big\|(h'_m - h'_\ell)qyez\big\| < \varepsilon,$$

as desired. □

The following somewhat technical lemma completes the proof of Proposition 3.3.

Lemma 3.4. *Let $f \in M$ be such that $f \leqslant_M f_0$. Let p be a non-zero projection in $M(I_f)$ such that the closure in X of* $\operatorname{supp}(p)$ *is contained in $U_f \cup \{t_0\}$. Then, given a projection e in I_0, there is a projection e' in I_f such that $pe = pe'$. In particular, $pI_0 \subseteq I_f$.*

Proof. Let K denote the closure in X of $\operatorname{supp}(p)$. By hypothesis, $K \subseteq U_f \cup \{t_0\}$.

Set $P = G \cap C(X)_+$; then P is a countable dimension monoid. Let N be the order-ideal of P consisting of those functions $g \in P$ such that $g = 0$ on K. Put $S = M + N \subseteq P$ and observe that S is the disjoint union of N and $M \setminus \{0\}$. (Note that the sum of an element in N and a non-zero element in M is an element in M, because an element in P is in M if and only if it is strictly positive on $V[t_0]$.)

Since S is a dimension monoid, there is an (up to isomorphism unique) unital AF-algebra D such that $(V(D), [1_D]) \cong (S, u)$. We have a monoid homomorphism $\lambda : (M, u) \to (S, u)$ and therefore there is a unital *-homomorphism $\psi : A \to D$ with the property $V(\psi) = \lambda$. Let I' be the closed ideal of D such that $V(I') = N$.

Since $p \in M(I_f)$, p is the limit with respect to the strict topology of I_f of an increasing sequence (h_n) of projections, which forms an approximate identity for $pI_f p$. Let I denote the closed ideal of A generated by (h_n), that is, $I = \overline{\bigcup_{n=1}^{\infty} Ah_nA}$. Then $I \subseteq I_f$ and clearly p is also the strict limit of (h_n) with respect to I. Observe that $\operatorname{supp}(h_n) \subseteq K$ for all n, so that $\lambda(V(I))$ is an order-ideal of S. Let I'' be the closed ideal of D corresponding to $\lambda(V(I))$, that is, the closed ideal generated by all the projections q in D such that $[q] \in \lambda(V(I))$. The map $\psi_{|I} : I \to I''$ induces an isomorphism of monoids $V(\psi_{|I}) : V(I) \to V(I'')$ sending the canonical interval in $V(I)$ onto the canonical interval in $V(I'')$. Since both I and I'' are AF-algebras, it follows from Elliott's theorem that $\psi_{|I}$ is an isomorphism from I onto I''. Since A is prime and I is a non-zero ideal of A, we conclude that the map $\psi : A \to D$ is injective. Therefore we can identify A with a C^*-subalgebra of D via ψ. Note that, under this identification, I is a closed ideal of D such that $II' = 0$. Thus $I' \subseteq I^\perp$, where we denote by $I^\perp$ the orthogonal ideal of I in D, that is, the set of all elements in D that annihilate I.

Now we need a suitable decomposition for the projection $e \in I_0$. Let f' be a fixed non-zero element of $V(I)$, so that $U_{f'} \subseteq K$. There is a compact neighbourhood K_1 of t_0 such that $K_1 \subseteq U_{f'} \cup \{t_0\}$. Let W be an open neighbourhood of t_0 such that $W \subseteq K_1$ and $[e] = \beta f_0$ on W for some $\beta \in \mathbb{Q}_+ \setminus \{0\}$. We can select a compact subset K_2 of X and an open subset U of X such that $K \subseteq U \subseteq K_2 \subseteq U_f \cup \{t_0\}$, so that we have the following situation:

$$t_0 \in W \subseteq K_1 \subseteq U_{f'} \cup \{t_0\} \subseteq K \subseteq U \subseteq K_2 \subseteq U_f \cup \{t_0\}.$$

Using suitable Urysohn functions, we will establish an orthogonal decomposition $e = e_1 + e_2$, where e_1, e_2 are projections in A with $e_1 \in I_f$ and $\operatorname{supp}([e_2]) \subseteq K_1 \cup K'$, where K' is a compact subset of X such that $K \cap K' = \emptyset$.

Put $g = [e] \in M$. Take $\rho \in \mathbb{Q}$ such that $\rho \gg g$ on X. Then there is $r_1 \in M$ and an open subset V of X with $K \subseteq V \subseteq U$ such that $r_1 = \rho$ on V and $r_1 = 0$ on $X \setminus U$. Set $g_1' = g \wedge r_1$ and note that $g_1' \in M$ and $\operatorname{supp}(g_1') \subseteq K_2 \subseteq U_f \cup \{t_0\}$. It follows that there is a positive integer k such that $g_1' \leqslant_M kf$ and hence $g_1' \in N_f$.

Take a rational number α such that $0 < \alpha < \beta$, so that $\alpha f_0 \ll g$ on $W \setminus \{t_0\}$. In addition take $\rho' \in \mathbb{Q}$ such that $\alpha f_0 \ll \rho'$ on W. There is a Urysohn function $r_2 \in M$ such that $r_2 = \rho'$ on an open neighbourhood $W' \subseteq W$ of t_0, and $r_2 = 0$ on $X \setminus W$. Set $g_2' = \alpha f_0 \wedge r_2 \in M$ and note that $g_2' \leqslant_M g_1'$ and $\operatorname{supp}(g_2') \subseteq K_1$. It follows that there is $\ell \in \mathbb{N}$ such that $g_2' \leqslant_M \ell f'$, so that $g_2' \in V(I)$. Now consider the element $g_2 = g_2' + (g - g_1')$. Note that $g_2 \in P$ and $g_2 \gg 0$ on $W' \setminus \{t_0\}$ so that $g_2 \in M$. Finally we set $g_1 = g_1' - g_2' \in M$. Then $g = g_1 + g_2$ with $g_1, g_2 \in M$, and we have $g_1 \in N_f$ and $\operatorname{supp}(g_2) \subseteq K_1 \cup K'$, where $K' = X \setminus V \subseteq X \setminus K$. There is a corresponding orthogonal decomposition $e = e_1 + e_2$, with $e_1 \in I_f$ and $[e_2] = g_2$.

The element $g_2 \in M$ decomposes in $S = M + N$ as $g_2 = g_2' + (g - g_1')$, where $g_2' \in N_{f'}$ and $g - g_1' \in N$, because $g - g_1'$ vanishes on K. This implies an orthogonal decomposition $e_2 = e_2' + e_2''$ of e_2 in D such that $[e_2'] = g_2'$ and $[e_2''] = g - g_1'$ in $V(D) = S$. Since $f' \in V(I)$, we have $N_{f'} \subseteq V(I)$, and we know that the closed ideal of D generated by its order-ideal $V(I)$ is precisely I, so that $e_2' \in I$. On the other hand, $e_2'' \in I'$ because $[e_2''] \in N = V(I')$. Therefore we can write

$$e = e_1 + e_2 = (e_1 + e_2') + e_2'',$$

where $e_1 + e_2' \in I_f$ and $e_2'' \in I'$. Set $e' = e_1 + e_2' \in I_f$ and $e'' = e_2'' \in I'$.

Note that $I \oplus I^\perp$ is an essential closed ideal of D and we have a canonical inclusion $\iota : D \to M(I \oplus I^\perp) = M(I) \oplus M(I^\perp)$. The sequence $(h_n, 0)$ converges in the strict topology of $I \oplus I^\perp$ to $(p, 0) \in M(I) \oplus M(I^\perp)$. We have the following commutative diagram:

$$\begin{array}{ccc} D & \xrightarrow{\iota} & M(I) \oplus M(I^\perp) \\ {\scriptstyle\psi}\uparrow & & \downarrow{\scriptstyle\pi_1} \\ A & \longrightarrow & M(I) \end{array}$$

where $\pi_1 : M(I) \oplus M(I^\perp) \to M(I)$ is the projection onto the first component.

Using the above decomposition $e = e' + e''$ and $I' \subseteq I^\perp$, we find that the image of e in $M(I) \oplus M(I^\perp)$ is $(e'_{|I}, e_{|I^\perp})$, where, for a closed ideal J of D, we denote by $x_{|J}$ the image of x

in $M(J)$ under the canonical restriction map $D \to M(J)$. It follows that

$$pe = \pi_1\big((p,0)(e'_{|I}, e_{|I^\perp})\big) = \pi_1\big((pe'_{|I}, 0)\big) = pe' \in M(I).$$

Since $p \in M(I_f)$ and $e' \in I_f$ we get $pe = pe' \in I_f$, as claimed. □

Although not strictly necessary, we shall assume for the remainder of this paper that $X = [0,1]$, $t_0 = 0$ and $f_0(t) = t$, $t \in X$. This allows us to construct certain discontinuous functions without undue notational complications.

In the following lemma we denote by $\varphi_{f_1,f_2} : M(I_{f_2}) \to M(I_{f_1})$, $f_1 \leqslant_M f_2$ the canonical restriction map.

Lemma 3.5. *Let A be the AF-algebra such that $(V(A), [1_A]) = (M, u)$. Let f be a non-zero element in M with $f \leqslant_M f_0$, and let p be a non-zero projection in I_0. Then there exist $f' \leqslant_M f$ and a projection $q \in M(I_{f'})$ with $q \leqslant \varphi_{f',f_0}(p)$ such that q is not equivalent in $M(I_{f'})$ to any projection in $\varphi_{f',f}(M(I_f))$.*

Proof. We can assume that $f = \lambda f_0$ and $h = 2\lambda' f_0$ for some positive rational numbers λ and λ', where $h = [p]$. Let $f' \in M$ be a non-zero function such that $f' \leqslant_M f$ and $U_{f'} \subsetneq U_f = (0,1]$. Let ρ be the left-most positive number t such that $f'(t) = 0$. Let $0 < \mu \in \mathbb{Q}$ be such that $\mu < \lambda'$ and choose a strictly increasing sequence of positive rational numbers $(\mu_n)_{n\geqslant 1}$ converging to ρ. Define a function g on $U_{f'}$ as follows. On $U_{f'} \setminus (0,\rho)$ we set $g = 0$. In the interval $(0,\mu_1]$, we set $g(t) = \mu t$, so that $g(\mu_1) = \mu\mu_1 \in \mathbb{Q}$. In the interval $[\mu_1,\mu_2]$, define g to be the restriction to $[\mu_1,\mu_2]$ of a Urysohn function $r_1 \in G$ such that $r_1 = \mu_1\mu$ on $[0,\mu_1]$, $r_1 = 0$ on $[\mu_2,1]$ and $0 \leqslant r_1 \leqslant \mu_1\mu$. Observe that $g = (\mu f_0) \wedge r_1 \in M$ on $[0,\mu_2]$. On $[\mu_2,\mu_3]$, we define g as the restriction to $[\mu_2,\mu_3]$ of a Urysohn function $r_2 \in G$ such that $r_2 = 0$ on $[0,\mu_2]$, $r_2 = \mu_1\mu$ on $[\mu_3,1]$ and $0 \leqslant r_2 \leqslant \mu_1\mu$. Note that $g = ((\mu f_0) \wedge r_1) \vee r_2 \in M$ on $[0,\mu_3]$. We continue in this way, obtaining a continuous function g on $(0,\rho)$ which cannot be extended to a continuous function at ρ. Observe that, by the above arguments, the function g is locally in M, that is, for each $t \in U_{f'}$ there is a function $z_t \in M$ such that $g = z_t$ on an open neighbourhood of t. It follows easily that g has property (C), and by construction $g \ll h_1$ on $U_{f'}$, where $h_1 = \frac{1}{2}h$.

Let $E_1 = I_{f'}(g)$ and $E_2 = I_{f'}(h_1 - g)$. By Lemmas 2.4 and 2.5 together with Remark 2.8, $E_1, E_2 \in \Lambda(N_{f'}, D_{f'})$ and $\tau_{f'}(E_1) = g$ and $\tau_{f'}(E_2) = h_1 - g$. It follows that $E = E_1 + E_2 \in \Lambda(N_{f'}, D_{f'})$ and $\tau_{f'}(E) = \tau_{f'}(E_1) + \tau_{f'}(E_2) = h_1$. We claim that $E + ([0,h_1] \cap N_{f'}) = [0,h] \cap N_{f'}$. Let $z \in N_{f'}$ be such that $z \leqslant_M h = 2h_1$. By the Riesz property of M, we can write $z = z_1 + z_2$ with $z_1 \leqslant_M h_1$ and $z_2 \leqslant_M h_1$. Since $h_1 = \lambda' f_0$ with $\lambda' > 0$, and $f'(\rho) = 0$, we conclude that $z_1 <_M h_1$ and $z_2 <_M h_1$. Take $z'_1 \in M$ such that its support is contained in a closed interval of the form $[0,\beta]$, with $\beta < \rho$, and $z'_1 <_M z_1$ and $z_1 - z'_1 <_M h_1 - z_2$. (This is possible because there is $\varepsilon > 0$ such that $z_1 \ll \varepsilon$ and $h_1 - z_2 \gg \varepsilon$ on $[\rho - \beta, 1]$ for some $\beta < \rho$.) Since E contains all functions in $[0,h_1] \cap N_{f'}$ whose support is contained in a closed interval $[0,\beta]$ with $\beta < \rho$ (because of the special construction of g), it follows that $z'_1 \in E$. On the other hand, $z_2 + (z_1 - z'_1) <_M z_2 + (h_1 - z_2) = h_1$ and so

$$z = z_1 + z_2 = z'_1 + \big(z_2 + (z_1 - z'_1)\big) \in E + \big([0,h_1] \cap N_{f'}\big).$$

This shows that $E + ([0,h_1] \cap N_{f'}) \supseteq [0,h] \cap N_{f'}$ and the reverse inclusion is obvious.

Since

$$E_1 + \big(E_2 + \big([0,h_1] \cap N_{f'}\big)\big) = [0,h] \cap N_{f'},$$

we find that there is a projection $q \in M(I_{f'})$ such that $q \leqslant \varphi_{f',f_0}(p)$ and the interval in $\Lambda(N_{f'}, D_{f'})$ corresponding to $[q]$ is E_1. Since

$$\tau_{f'}(E_1) = \tau_{f'}\big(I_{f'}(g)\big) = g,$$

and g cannot be extended to a continuous function on U_f, we infer from the commutative diagram (2.1) that q is not equivalent in $M(I_{f'})$ to a projection in $\varphi_{f',f}(M(I_f))$. □

With the same notation as in Proposition 3.3, let p be the strict limit of the sequence (h'_n). By Lemma 3.1, $p \in M(I_0) = B_0$. We will now put all the above ingredients together to obtain sequences of projections satisfying the requirements in Lemma 3.2.

Proposition 3.6. *With the same notation and caveats as above, set $p'_n = p - h'_{n-1} \in B_0$ for $n \geqslant 1$ and $p'_0 = p$. Then there are $\delta > 0$, a sequence (f'_n) of elements of M with $f'_0 = f_0$ and $f'_n \leqslant_M f'_{n-1} \leqslant_M f_{n-1}$ for all $n \geqslant 1$ as well as orthogonal decompositions $1 = p'_n + p_n + q_n$ such that $p_n, q_n \in M(I_{f'_n})$ for all $n \geqslant 0$ and $\operatorname{dist}(p_n, M(I_{f'_{n-1}})) \geqslant \delta$ for all $n \geqslant 1$.*

It follows that the sequences (p_n) and (q_n) converge in the strict I-topology to $e \in M(I)$ and $1 - e$, respectively.

Proof. Let $\delta > 0$ be such that, for all C^*-algebras C, D with $C \subseteq D$ and for all projections $e' \in D$, $\operatorname{dist}(e', C) < \delta$ implies that e' is equivalent to a projection in C; cf. [12, Lemma 6.3.1 and Proposition 2.2.4].

The sequences (p_n) and (q_n) are constructed inductively. To start with we set $p_0 = 1 - p$, $q_0 = 0$. Then $p_0, q_0 \in B_0$ and $1 = p'_0 + p_0 + q_0$. Suppose that, for $n \geqslant 0$, we have an orthogonal decomposition $1 = p'_n + p_n + q_n$ satisfying the stated conditions. Note that $p'_n = p'_{n+1} + h''_n$.

By Lemma 3.5, there exist a non-zero $f'_{n+1} \in M$ with $f'_{n+1} \leqslant_M f'_n$ and $f'_{n+1} \leqslant_M f_{n+1}$, and a projection $q'_{n+1} \in M(I_{f'_{n+1}})$ such that $q'_{n+1} \leqslant \varphi_{f'_{n+1},f_0}(h''_n)$ in $M(I_{f'_{n+1}})$ and q'_{n+1} is not equivalent in $M(I_{f'_{n+1}})$ to a projection in $\varphi_{f'_{n+1},f'_n}(M(I_{f'_n}))$. The latter condition implies that $\operatorname{dist}(q'_{n+1}, M(I_{f'_n})) \geqslant \delta$.

Identify all the projections constructed so far with their images in $M(I_{f'_{n+1}})$ under the canonical inclusions; then $q'_{n+1} \leqslant h''_n$. Set $p_{n+1} = p_n + q'_{n+1}$ and $q_{n+1} = q_n + (h''_n - q'_{n+1})$. Then

$$\operatorname{dist}\big(p_{n+1}, M(I_{f'_n})\big) = \operatorname{dist}\big(q'_{n+1}, M(I_{f'_n})\big) \geqslant \delta;$$

moreover,

$$\begin{aligned} 1 &= p'_n + p_n + q_n = p'_{n+1} + h''_n + p_n + q_n \\ &= p'_{n+1} + q'_{n+1} + (h''_n - q'_{n+1}) + p_n + q_n \\ &= p'_{n+1} + p_{n+1} + q_{n+1}. \end{aligned}$$

This concludes the inductive construction.

Now we consider all the projections as projections in $M(I)$ and all the algebras $M(I_{f_n'})$ as C^*-subalgebras of $M(I)$. It is a simple matter to show that (p_n) converges in the strict I-topology. Indeed, fix $a \in I$ and $\varepsilon > 0$. Then there is n_0 such that $\|p'_{n_0}a\| = \|(p - h'_{n_0-1})a\| < \varepsilon$. For $m > m' \geqslant n_0$, we have $p_m - p_{m'} \leqslant p'_{n_0}$ and so $\|(p_m - p_{m'})a\| < \varepsilon$. Similarly, the sequence (q_n) is strictly convergent. Let e be the strict limit of (p_n). Since $p_n + q_n = 1 - p'_n$ converges strictly to 1, it follows that (q_n) converges strictly to $1 - e$. □

We are ready to complete the proof of our main result, Theorem 1.1. By Proposition 3.6, we can construct projections p, q in $M(I)$ such that $p + q = 1$ and p and q are strict limits of sequences (p_n) and (q_n), respectively, satisfying the conditions stated in Lemma 3.2 with respect to the C^*-subalgebras $B_n = M(I_{f_n'})$. Since $(I_{f_n'})$ is a fundamental sequence of closed ideals of A with $I_{f_0'} = I_0$, it follows from Lemma 3.2 that $p, q \notin B$. Therefore

$$M_{\mathrm{loc}}\big(M_{\mathrm{loc}}(A)\big) = M_{\mathrm{loc}}(B) = M(I) \neq B = M_{\mathrm{loc}}(A).$$

4. Local multiplier algebras

In this section we add a few remarks on a systematic approach to understanding the ideal structure of $M_{\mathrm{loc}}(A)$. Let A be a separable prime C^*-algebra. Then A is primitive [10, 4.3.6] and hence $0 \in \mathrm{Prim}(A)$. Since $\mathrm{Prim}(A)$ is second countable [10, 4.3.4], we can find a countable basis (U_n) of open neighbourhoods of 0, with $U_{n+1} \subseteq U_n$ for all n. These open sets correspond to a cofinal countable family (I_n) of non-zero closed ideals of A such that $I_{n+1} \subseteq I_n$ for all n. We call such a sequence (I_n) a *fundamental sequence of ideals* of A. Obviously, we have $M_{\mathrm{loc}}(A) = \varinjlim M(I_n)$ for such a fundamental sequence of ideals.

The sequence (I_n) determines a fundamental sequence (J_n) in $M_{\mathrm{loc}}(A)$, so that 0 has a countable basis of neighbourhoods in $\mathrm{Prim}(M_{\mathrm{loc}}(A))$. Indeed, define J_n as the closed ideal of $M_{\mathrm{loc}}(A)$ generated by I_n. Given a non-zero closed ideal J of $M_{\mathrm{loc}}(A)$ we obtain that $J \cap A$ is a non-zero ideal of A [2, 2.3.2]; hence there is m such that $I_m \subseteq J$, and so $J_m \subseteq J$. It follows that $M_{\mathrm{loc}}(M_{\mathrm{loc}}(A)) = \varinjlim M(J_n)$. In general, the iterated local multiplier algebra $M_{\mathrm{loc}}^{(k)}(A)$ can be computed by taking the direct limit $\varinjlim M(I_n^{(k-1)})$, where $I_n^{(k-1)}$ is the closed ideal of $M_{\mathrm{loc}}^{(k-1)}(A)$ generated by I_n.

We can distinguish three different types of behaviour. The first one corresponds to the case where all ideals J_n are equal to $M_{\mathrm{loc}}(A)$, that is, $M_{\mathrm{loc}}(A)$ is a simple C^*-algebra. Of course, this happens when A is simple and unital, but it can also occur when all the ideals I_n are different; examples of this behaviour were constructed in [1]. A second possibility is that 0 is an isolated point in $\mathrm{Prim}(M_{\mathrm{loc}}(A))$ which has more than one point. This is the case if and only if the sequence (J_n) stabilises and $J_n \neq M_{\mathrm{loc}}(A)$ for large n. In this case we have

$$M_{\mathrm{loc}}\big(M_{\mathrm{loc}}(A)\big) = M(J_{n_0}),$$

where $J_{n_0} = J_n$ for all $n \geqslant n_0$, and so $M_{\mathrm{loc}}^{(k)}(A) = M_{\mathrm{loc}}(M_{\mathrm{loc}}(A))$ for all $k \geqslant 2$. Our examples in the present paper are of this type.

The third kind of prime C^*-algebras consists of those such that the family (J_n) is strictly decreasing. Although we will not go into the details of the construction, it is possible to give explicit examples of AF-algebras in this class using the methods developed in the present paper. However it seems technically challenging to analyze the possible lack of stabilisation of the increasing chain $(M_{\mathrm{loc}}^{(k)}(A))_{k\in\mathbb{N}}$.

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