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Discrete piecewise linear functions

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ABSTRACT

The concept of permutograph is introduced and properties of integral functions on permutographs are investigated. The central result characterizes the class of integral functions that are representable as lattice polynomials. This result is used to establish lattice polynomial representations of piecewise linear functions on $\mathbb{R}^d$ and continuous selectors on linear orders.

1. Introduction

We begin with a simple yet instructive example. Let f be a piecewise linear function (PL-function) defined by

$$f(x) = \begin{cases} g_1(x), & \text{for } x \leq -1, \\ g_2(x), & \text{for } -1 \leq x \leq 1, \\ g_3(x), & \text{for } x \geq 1 \end{cases}$$

where

$$g_1(x) = x + 2, \qquad g_2(x) = -x, \qquad g_3(x) = 0.5x - 1.5.$$

The graph of this function is shown in Fig. 1.

It is easy to verify that

$$f = g_1 \wedge (g_2 \vee g_3) = (g_1 \wedge g_2) \vee (g_1 \wedge g_3).$$

We use the notations

$$a \wedge b = \min\{a, b\} \quad \text{and} \quad a \vee b = \max\{a, b\}$$

throughout the article. Thus, the function f can be represented as a lattice polynomial in variables g_1, g_2, and g_3. This is true in general: any continuous PL-function h on a convex domain in $\mathbb{R}^d$ is a lattice polynomial whose variables are linear 'components' of h (Theorem 4.2). In various forms, this result was independently established in [1,20,29] (however, see comments in Section 6, item 1).

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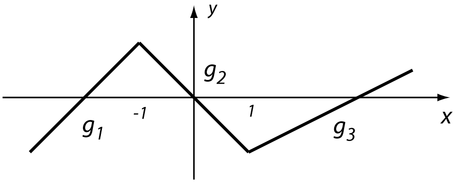

Fig. 1. Graph of a PL-function.

The aim of the article is to show that this result is essentially combinatorial. We introduce a class of functions on permutographs that we call 'discrete piecewise linear (DPL) functions', and show that these functions are representable as lattice polynomials. The discretization of the original problem is achieved by replacing the 'continuity' and 'linearity' properties of PL-functions by the 'separation' and 'linear ordering' properties of integral functions on permutographs.

Permutographs are isometric subgraphs of a weighted Cayley graph of the symmetric group; they are introduced in Section 2. In Section 3, we characterize lattice polynomials on permutographs in terms of the 'separation property' and as 'DPL-functions'. These characterizations are used in Section 4 to establish lattice polynomial representations of PL-functions on convex domains in $\mathbb{R}^d$.

In a different setting, the results of Section 3 are used in Section 5 to obtain a polynomial representation for functions on linear orders. Topological properties of linear orders that are used in Section 5 are introduced in Appendix. Some relevant topics are discussed in Section 6.

2. Permutographs

Let X be a linearly ordered finite set of cardinality $n \geq 1$. We assume that X is the set $\{1, \ldots, n\}$ ordered by the usual relation $<$. A *permutation* (*of order* n) is a bijection $\alpha : X \to X$. We write permutations on the right, that is, $x\alpha$ is the image of x under α, compose them left to right (cf. [7,8]), and use the notation

$$\alpha = (x_1 \cdots x_k \cdots x_n),$$

where $x_k = k\alpha$. For a given permutation $\alpha = (x_1 \cdots x_n)$, the elements of X are linearly ordered by the relation $<_\alpha$ defined by

$$x_i <_\alpha x_j \iff i < j.$$

In other words,

$$x <_\alpha y \iff x\alpha^{-1} < y\alpha^{-1}.$$

We write $x \leq_\alpha y$ if $x <_\alpha y$ or $x = y$. Symbols $>_\alpha$ and $\geq_\alpha$ stand for the respective inverse relations.

A pair $\{x, y\}$ of elements of X is called an *inversion* for a pair of permutations $\{\alpha, \beta\}$ if x and y appear in reverse order in α and β. The *distance* $d(\alpha, \beta)$ between permutations α and β is defined as the number of inversions for the pair $\{\alpha, \beta\}$. This distance equals one half of the cardinality of the symmetric difference of the binary relations $<_\alpha$ and $<_\beta$. We say that a permutation γ *lies between* permutations α and β if

$$d(\alpha, \gamma) + d(\gamma, \beta) = d(\alpha, \beta).$$

It is straightforward to see that γ lies between α and β if and only if

$$(x <_\alpha y \text{ and } x <_\beta y) \Rightarrow x <_\gamma y \quad \text{for all } x, y \in X.$$

The set of all permutations of X forms the *symmetric group* S_n with the operation of composition and the identity element $\epsilon = (1 \cdots n)$.

A partition $\pi = (X_1, \ldots, X_m)$ of the ordered set $(X, <)$ into a family of nonempty subsets is said to be an *ordered partition* if

$$(x \in X_i, y \in X_j, i < j) \Rightarrow x < y.$$

The ordered partition $(\{1\}, \ldots, \{n\})$ is said to be *trivial*.

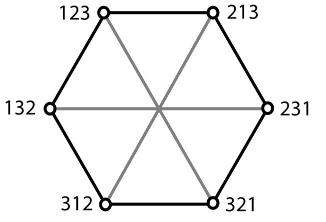

Fig. 2. Big permutograph on S_3. The weights of edges are not shown.

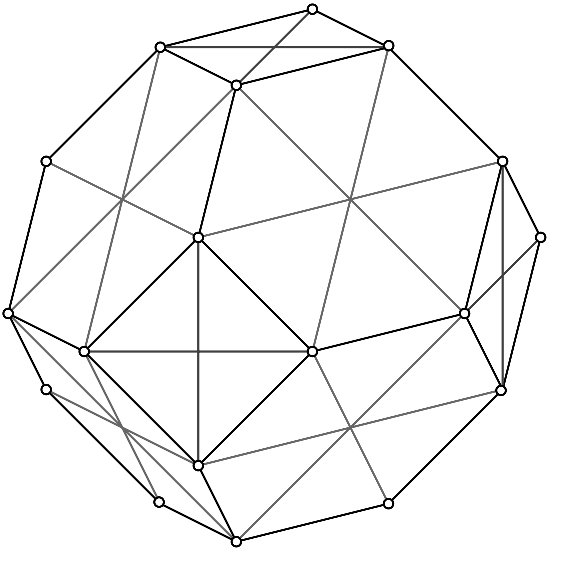

Fig. 3. Big permutograph on S_4. The graph is drawn on the permutohedron Π_3. Only 'visible' edges of the graph are shown. The weights of edges are not shown.

For a given nontrivial ordered partition $\pi = (X_1, \ldots, X_m)$, the permutation τ_π reverses the order of elements in every set X_i. For instance,

$$\tau_\pi = (1\,432\,5\,6\,87)$$

for the ordered partition $\pi = (\{1\}, \{2, 3, 4\}, \{5\}, \{6\}, \{7, 8\})$. Two permutations α and β are π-*adjacent* if $\alpha\beta^{-1} = \tau_\pi$ for a nontrivial ordered partition π. They are *adjacent* if they are π-adjacent for some π. The adjacency relation on S_n is symmetric and irreflexive. It defines a Cayley graph [3] on S_n that we denote by Υ_n. By definition, the *weight* of an edge $\alpha\beta$ is the distance $d(\alpha, \beta)$ between permutations α and β. We call the weighted graph (Υ_n, d) the *big permutograph on* S_n. This graph is k-regular for $k = 2^{n-1} - 1$. By definition, a *permutograph on* S_n (cf. [26]) is an isometric weighted subgraph of the big permutograph.

Example 2.1. The graph Υ_3 is shown in Fig. 2. It is the complete bipartite graph $K_{3,3}$.

Example 2.2. The graph of the permutohedron Π_{n-1} is a permutograph on S_n. (See [2,6,26,30]; the term "permutohedron" was coined by Guilbaud and Rosenstiehl [11] in 1963.) This graph is a spanning graph of Υ_n. The edges of the big permutograph on S_n link the opposite vertices of the faces of the permutohedron (see Fig. 3).

3. Functions on permutations

Let S be a subset of S_n. We denote $\mathcal{F}(S)$ the set of all functions from S to X. The ordering $<$ of the set X induces a partial order on $\mathcal{F}(S)$:

$$F < G \Longleftrightarrow F(\alpha) <_\alpha G(\alpha) \quad \text{for all } F, G \in \mathcal{F}(S) \text{ and } \alpha \in S.$$

The poset $\mathcal{F}(S)$ is a complete distributive lattice with meet and join operations defined pointwise by

$$(F \wedge G)(\alpha) = F(\alpha) \wedge G(\alpha) \quad \text{and} \quad (F \vee G)(\alpha) = F(\alpha) \vee G(\alpha), \tag{3.1}$$

respectively. In the right-hand sides of the equations in (3.1), the meet and join operations are defined with respect to the linear ordering $\leq_\alpha$. We use this convention throughout the article.

For a given $k \in X$, we define

$$G_k(\alpha) = k \quad \text{for all } \alpha \in S, \tag{3.2}$$

the *constant function* on the set S.

Example 3.1. The kth *order statistic* M_k on S is defined by

$$M_k(\alpha) = x_k \quad \text{for } \alpha = (x_1 \cdots x_n) \in S.$$

Let $\mathcal{X}_k$ be a family of subsets of X defined by

$$\mathcal{X}_k = \{Y \subseteq X : |X \setminus Y| = k - 1\}.$$

We have the following formula for the order statistic M_k (cf. [17]):

$$M_k = \bigvee_{Y \in \mathcal{X}_k} \bigwedge_{j \in Y} G_j. \tag{3.3}$$

Indeed, it suffices to note that $x_1 <_\alpha \cdots <_\alpha x_n$ for $\alpha = (x_1 \cdots x_n)$, so the maximum in (3.3) is attained at $Y = \{x_k, \cdots, x_n\}$.

The right-hand side of the equation in (3.3) is a lattice polynomial written in its disjunctive normal form. Since $\mathcal{F}(S)$ is a distributive lattice, any lattice polynomial in variables G_j's can be written in the disjunctive normal form [5]. This fact motivates the following definition.

Definition 3.1. Let $\{K_i\}_{i \in I}$ be a family of subsets of the set $X = \{1, \ldots, n\}$. A *polynomial* on S is a function $F : S \to X$ defined by

$$F(\alpha) = \bigvee_{i \in I} \bigwedge_{j \in K_i} G_j(\alpha). \tag{3.4}$$

Note that we may assume that the family $\{K_i\}_{i \in I}$ is an antichain in the lattice 2^X of all subsets of X.

In this section, we give two characterizations of polynomial functions on permutations.

Definition 3.2. Let S be a nonempty subset of S_n. A function $F : S \to X$ satisfies the *separation property* (*S-property*) if, for any $\alpha, \beta \in S$, there is $u \in X$ such that

$$u \leq_\alpha F(\alpha) \quad \text{and} \quad u \geq_\beta F(\beta).$$

Theorem 3.1. *A function $F : S \to X$ satisfies the S-property if and only if it is a polynomial.*

Proof (*Necessity.*). Suppose that F satisfies the S-property. For $\gamma \in S$, let $K_\gamma = \{v \in X \mid v \geq_\gamma F(\gamma)\}$. By the S-property, there is $u \in X$ such that $u \leq_\alpha F(\alpha)$ and $u \geq_\gamma F(\gamma)$, so $u \in K_\gamma$. Since $u \leq_\alpha F(\alpha)$, we have

$$\bigwedge_{j \in K_\gamma} G_j(\alpha) \leq_\alpha F(\alpha) \quad \text{for every } \gamma \in S.$$

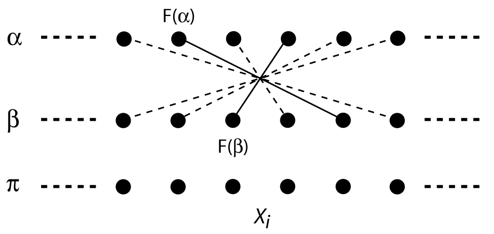

Fig. 4. Values of a DPL-function on two adjacent permutations.

Clearly,

$$\bigwedge_{j\in K_\alpha} G_j(\alpha) = F(\alpha).$$

Thus,

$$\bigvee_{\gamma\in S}\bigwedge_{j\in K_\gamma} G_j(\alpha) = F(\alpha),$$

that is, F is a polynomial.

(Sufficiency.) Suppose that F is a polynomial in the form (3.4). Let $\alpha, \beta \in S$ and suppose that $u <_\beta F(\beta)$ for any $u \leq_\alpha F(\alpha)$. By (3.4), for every i there is $j_i \in K_i$ such that $j_i \leq_\alpha F(\alpha)$ (recall that G_j's are constant functions). By our assumption, $j_i <_\beta F(\beta)$. Since $j_i \in K_i$, we have

$$\bigwedge_{j\in K_i} G_j(\beta) <_\beta F(\beta).$$

Therefore,

$$\bigvee_{i\in J}\bigwedge_{j\in K_i} G_j(\beta) <_\beta F(\beta),$$

contradicting (3.4). It follows that F satisfies the S-property. □

Definition 3.3. A function $F : S \to X$ is said to be a *DPL-function* if, for any two π-adjacent permutations $\alpha, \beta \in S$ with $\pi = (X_1, \ldots, X_m)$, we have

$$F(\alpha) \in X_i\alpha \quad \text{and} \quad F(\beta) \in X_i\beta \quad \text{for some } 1 \leq i \leq m. \tag{3.5}$$

See Fig. 4.

Theorem 3.2. *Let S be the vertex set of a permutograph on S_n. A function F on S is a DPL-function if and only if it satisfies the S-property.*

Proof (*Necessity.*)**.** Let F be a DPL-function on S. We need to show that for given $\alpha, \beta \in S$ there is $u \in X$ such that

$$u \leq_\alpha F(\alpha) \quad \text{and} \quad u \geq_\beta F(\beta).$$

If $F(\beta) \leq_\beta F(\alpha)$, we may choose $u = F(\alpha)$. Thus, in what follows, we assume that

$$F(\alpha) <_\beta F(\beta). \tag{3.6}$$

The proof is by induction on the length k of a shortest $\alpha\beta$-path in S.

For $k = 1$, the permutations α and β are adjacent and the result follows immediately from the definition of a DPL-function: just let u be the maximum element in $X_i\beta$ (see (3.5) and recall that elements of $X_i\alpha$ and $X_i\beta$ are in reverse order).

For the inductive step, suppose that

$$\alpha = \alpha_0, \quad \gamma = \alpha_1, \ldots, \alpha_k = \beta$$

is a shortest $\alpha\beta$-path of length $k > 1$ in S. By the induction hypothesis, there is $u \in X$ such that

$$u \geq_\beta F(\beta) \quad \text{and} \quad u \leq_\gamma F(\gamma).$$

Since $\alpha = (x_1, \ldots, x_n)$ and $\gamma = (y_1, \ldots, y_n)$ are π-adjacent for some ordered partition $\pi = (X_1, \ldots, X_m)$, we have $F(\alpha) \in X_i\alpha$ and $F(\gamma) \in X_i\gamma$ for some $1 \leq i \leq m$, that is,

$$x_p \leq_\alpha F(\alpha) \leq_\alpha x_q \quad \text{and} \quad y_p \leq_\gamma F(\gamma) \leq_\gamma y_q,$$

where p and q are the minimum and maximum elements of X_i, respectively.

If $u <_\gamma y_p$, then $u <_\alpha x_p$, because π is an ordered partition. Therefore, $u <_\alpha F(\alpha)$, since $x_p \leq_\alpha F(\alpha)$.

Otherwise, $u \in X_i\gamma$, implying $u \in X_i\alpha$, since α and γ are π-adjacent. Since γ lies between α and β, and $F(\alpha) \in X_i\gamma$, any element of $X_i\alpha$ which is greater than $F(\alpha)$ in the linear ordering $(X_i, <_\alpha)$ must be less than $F(\alpha)$ in the linear ordering $(X, <_\beta)$. Hence, if $u >_\alpha F(\alpha)$, then we must have $F(\beta) \leq_\beta u <_\beta F(\alpha)$, in contradiction with our assumption in (3.6). It follows that $u \leq_\alpha F(\alpha)$.

(Sufficiency.) We assume that F satisfies the S-property. Let α and β be two π-adjacent permutations in S with $\pi = (X_1, \ldots, X_m)$ and suppose that $F(\alpha) \in X_i\alpha$ and $F(\beta) \in X_j\beta$ with $i < j$. Clearly, $u \geq_\beta F(\beta)$ implies $u >_\alpha F(\alpha)$, contradicting the S-property. Hence, $i \geq j$. By symmetry, $j = i$. Therefore, F is a DPL-function. $\square$

The following theorem summarizes the results of Theorems 3.1 and 3.2.

Theorem 3.3. *Let S be a permutograph on S_n and F be a function on S with values in X. The following statements are equivalent:*

(i) *F is a DPL-function.*
(ii) *F satisfies the S-property.*
(iii) *F is a polynomial.*

4. Piecewise linear functions on $\mathbb{R}^d$

A *closed domain* in $\mathbb{R}^d$ is the closure of a nonempty open set in $\mathbb{R}^d$. In this section, D is a convex closed domain in $\mathbb{R}^d$ and $\{g_i(x)\}_{1\leq i\leq n}$ is a family of distinct (affine) linear functions on D. We assume that

$$\mathrm{int}(D) \cap \ker(g_i - g_j) \neq \varnothing$$

for at least one pair of distinct functions g_i and g_j, where $\mathrm{int}(D)$ stands for the interior of D.

Let $\mathcal{H}$ be the arrangement of all distinct hyperplanes in $\mathbb{R}^d$ that are solutions of the equations in the form $g_i(x) = g_j(x)$ and have nonempty intersections with $\mathrm{int}(D)$. We denote by $\mathcal{R}$ the family of nonempty intersections of the regions of $\mathcal{H}$ with $\mathrm{int}(D)$ and use the same name 'region' for elements of $\mathcal{R}$. The *region graph* $\mathcal{G}$ of the arrangement $\mathcal{H}$ has $\mathcal{R}$ as the set of vertices; the edges of the graph are pairs of adjacent regions.

It is easy to see that the functions $g_1, \ldots, g_n$ are linearly ordered over any region in $\mathcal{R}$, that is, for a given $R \in \mathcal{R}$ there is a permutation $(i_1 \cdots i_n)$ such that

$$g_{i_1}(x) < \cdots < g_{i_n}(x) \quad \text{for all } x \in R.$$

This correspondence defines a mapping $\varphi : \mathcal{R} \to S_n$. We treat the graph $\mathcal{G}$ as a weighted graph: the weight of an edge PQ is the distance between permutations $\varphi(P)$ and $\varphi(Q)$.

Theorem 4.1. *The mapping φ defines an isometric embedding of the weighted region graph $\mathcal{G}$ into the big permutograph Υ_n. Thus, the image $\varphi(\mathcal{G})$ of the region graph is a permutograph.*

Proof. First, we show that φ is a one-to-one function. Let P and Q be two distinct regions in $\mathcal{R}$. Let H be a hyperplane in $\mathcal{H}$ separating P and Q. The hyperplane H is the solution set of some equation $g_i = g_j$ with $i \neq j$. Therefore, the functions g_i and g_j are in reverse order over regions P and Q. It follows that the permutations $\varphi(P)$ and $\varphi(Q)$ are distinct.

Now, we show that the permutations corresponding to two adjacent regions $P, Q \in \mathcal{R}$ are adjacent vertices in the graph Υ_n. Let $H \in \mathcal{H}$ be the hyperplane separating P and Q and $F \subseteq H$ be the relative interior of the common facet of P and Q. It is clear that two components g_i and g_j are in the reverse order over P and Q if and only if $\ker(g_i - g_j) = H$. It follows that the values $g_i(x)$ of the components over F define the same ordered partition π of the set $X = \{1, \ldots, n\}$ for every $x \in F$. For this partition π, the permutations $\varphi(P)$ and $\varphi(Q)$ are π-adjacent.

To complete the proof, we need to prove that the image of $\mathcal{G}$ under φ is an isometric subgraph of Υ_n. Let P and Q be two distinct regions in $\mathcal{R}$. A simple topological argument (cf. [21]; convexity of the domain is essential in this argument) shows that there are points $p \in P$ and $q \in Q$ such that the line segment $[p, q]$ does not intersect cells of $\mathcal{H}$ of dimension less than $d - 1$. The regions with nonempty intersections with $[p, q]$ form a path $R_0 = P, R_1, \ldots, R_m = Q$ in $\mathcal{G}$. Since functions g_i's are linear, the number of inversions for the pair $\{\varphi(P), \varphi(Q)\}$ equals the total number of inversions corresponding to the pairs of adjacent regions in the path. Since the number of inversions for two adjacent regions is the weight of the edge joining these regions, the length of the path $\varphi(R_0), \ldots, \varphi(R_m)$ equals the distance between $\varphi(P)$ and $\varphi(Q)$, that is, φ is an isometric embedding. $\square$

Definition 4.1. Let D be a convex closed domain in $\mathbb{R}^d$. A function $f : D \to \mathbb{R}$ is said to be a *PL-function* if there is a finite family $\mathcal{D}$ of closed domains such that $D = \cup \mathcal{D}$ and f is (affine) linear on every domain in $\mathcal{D}$. A linear function g such that $g|_R = f|_R$ for some domain $R \in \mathcal{D}$ is said to be a *component* of f.

Note that in applied articles (see, for instance, [25,29] and references therein) the domain D is a polyhedron in $\mathbb{R}^d$.

Clearly, a PL-function on D is continuous. Let f be a PL-function on a given convex closed domain $D \subseteq \mathbb{R}^d$ and $\{g_1, \ldots, g_n\}$ be the set of components of f.

Let us define a function $F(\alpha)$ on the set of vertices of the permutograph $\varphi(\mathcal{G})$ as follows:

$$F(\alpha) = G_i(\alpha) \Longleftrightarrow f(x) = g_i(x) \quad \text{for } x \in \varphi^{-1}(\alpha).$$

Since f is a continuous function, the function F is a DPL-function on $\varphi(\mathcal{G})$. By Theorem 3.3, there is a family $\{K_i\}_{i \in I}$ such that

$$F(\alpha) = \bigvee_{i \in I} \bigwedge_{j \in K_i} G_j(\alpha).$$

The functions g_i's are ordered over a given region R as functions G_i's are ordered with respect to the relation $<_\alpha$ for $\alpha = \varphi(R)$. Therefore, we have the following theorem (Theorem 2.1 in [20]).

Theorem 4.2. *Let f be a PL-function on a convex closed domain D in $\mathbb{R}^d$ and $\{g_1, \ldots, g_n\}$ be the set of components of f. There is a family $\{K_i\}_{i \in I}$ of subsets of the set $X = \{1, \ldots, n\}$ such that*

$$f(x) = \bigvee_{i \in I} \bigwedge_{j \in K_i} g_j(x), \quad \text{for } x \in D. \tag{4.1}$$

The converse is also true: Let $\{g_1, \ldots, g_n\}$ be a family of affine linear functions on D. Then, a function in the form (4.1) is a PL-function.

The following simple corollary is of importance in some applications (see [1,16,27,28]):

Corollary 4.1. *A PL-function is representable as a difference of two concave (equivalently, convex) PL-functions.*

Proof. Let f be a PL-function in the form (4.1) and let $h_i(x) = \bigwedge_{j \in K_i} g_j(x)$. Note that h_i's are concave functions. Since

$$h_i = \sum_k h_k - \sum_{k \neq i} h_k,$$

we have

$$f = \bigvee_{i\in I} h_i = \sum_k h_k - \bigwedge_{i\in I} \sum_{k\neq i} h_k.$$

Since sums and minimums of concave functions are concave, we have the desired representation. □

Clearly, a PL-function f on D is a 'selector' of its components g_j's, that is, for any $x \in D$ there is i such that $f(x) = g_i(x)$ (cf. Section 5). Conversely, let f be a continuous selector of a family of linear functions $\{g_1, \ldots, g_n\}$ and let R be a region of the arrangement $\mathcal{H}$ defined by this family over D. The functions g_i's are linearly ordered over R. Since f is a continuous function and R is connected, we must have $f = g_i$ over R for some index i. It follows that a continuous selector f is a PL-function on D and therefore admits a polynomial representation (4.1). This case is of interest in the "nonsmooth critical point theory" [1].

5. Selectors and invariant functions

Let X be an arbitrary set and D be a subset of X^d. Let $\{g_1, \ldots, g_n\}$ be a family of functions on D with values in X. A function $f : D \to X$ is said to be a *selector of the functions g_i's* if for any $x \in D$ there is i such that $f(x) = g_i(x)$. A *coordinate selector* (cf. [15]) is a selector of the coordinate functions $g_i(x) = x_i$ for $1 \leq i \leq d$.

In the rest of this section, we assume that X is a linearly ordered set endowed with interval topology (see Appendix for notations and relevant results).

Suppose that X is a connected space and let f be a continuous coordinate selector on X^d. For a given permutation $\alpha \in S_d$, the sets

$$A_i = \{x \in \mathcal{O}_\alpha : f(x) = x_i\}, \quad 1 \leq i \leq d,$$

are closed disjoint sets and the chamber $\mathcal{O}_\alpha$ is their finite union. Since $\mathcal{O}_\alpha$ is a connected set (Theorem A.1), we must have $f(x) = x_k$ on $\mathcal{O}_\alpha$ for some $1 \leq k \leq d$. We define a function F on the vertices of the permutohedron Π_{d-1} by letting $F(\alpha) = k$ if $f(x) = x_k$ on $\mathcal{O}_\alpha$. By Theorem A.2(2), F is a DPL-function on the permutograph Π_{d-1}. By Theorems 3.3 and A.2(1), we have the following result:

Theorem 5.1. *Let X be a connected linear order. A function $f : X^d \to X$ is a continuous coordinate selector if and only if it is a lattice polynomial in variables $x_1, \ldots, x_d$.*

Note that the result of this theorem does not hold for disconnected linear orders. Indeed, let $X = U \cup V$ where U and V are nonempty disjoint open sets and let us define $f : X^d \to X$ by

$$f(x_1, \ldots, x_d) = \begin{cases} x_1, & \text{if } x_1 \in U, \\ x_2, & \text{if } x_1 \in V. \end{cases}$$

The function f is a continuous selector which is not representable as a lattice polynomial.

Let X be a linear order and let f be a lattice polynomial in variables $x_1, \ldots, x_d$, that is,

$$f(x) = \bigvee_{i\in I} \bigwedge_{j\in K_i} x_j, \quad \text{for } x = (x_1, \ldots, x_d) \in X^d, \tag{5.1}$$

where $\{K_i\}_{i\in I}$ is a family of subsets of the set $\{1, \ldots, d\}$. It is clear that $f(x\psi) = f(x)\psi$ for any automorphism $\psi \in \mathcal{A}(X)$ (cf. Appendix). We show below that the converse is true for a special class of linear orders.

In the rest of this section, X is a doubly homogeneous linear order, that is, there is a doubly transitive ℓ-permutation group G acting on X. By Theorem A.3, G is m-transitive for all $m \geq 2$.

A function $f : X^d \to X$ is said to be *invariant* (under actions from G) if

$$f(x_1\psi, \ldots, x_d\psi) = f(x_1, \ldots, x_d)\psi$$

for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\psi \in G$.

Theorem 5.2. *An invariant function $f : X^d \to X$ is a coordinate selector.*

Proof. Let f be an invariant function on X^d. If $y = f(x_1, \dots, x_d)$, then we have

$$y = f(x_1, \dots, x_d) = f(x_1\psi, \dots, x_d\psi) = f(x_1, \dots, x_d)\psi = y\psi,$$

for any automorphism $\psi \in G$ that fixes elements $x_1, \dots, x_d$. Suppose that $y \neq x_i$ for all $1 \le i \le d$. Since X is $(d+1)$-homogeneous, there is an automorphism in G that fixes elements $x_1, \dots, x_d$ and such that $y\psi \neq y$, a contradiction. Therefore, $f(x_1, \dots, x_d) \in \{x_1, \dots, x_d\}$, that is, f is a coordinate selector. $\square$

Note that the linear order X is not necessarily connected, so we cannot simply apply the result of Theorem 5.1 to show that an invariant function is a polynomial. However, this result holds as the following argument demonstrates.

Since X is d-homogeneous and the coordinates appear in the same order for all points in a chamber $\mathcal{O}_\alpha$, we must have (by Theorem 5.2) $f(x) = x_k$ on $\mathcal{O}_\alpha$ for some $1 \le k \le d$. As before, we define a function F on the vertices of the permutohedron Π_{d-1} by letting $F(\alpha) = k$ if $f(x) = x_k$ on $\mathcal{O}_\alpha$. By Theorem A.2(2), F is a DPL-function on the permutograph Π_{d-1}. By Theorems 3.3 and A.2(1), we have the following result [18]:

Theorem 5.3. *Let X be a doubly homogeneous linear order. A continuous function $f : X^d \to X$ is invariant if and only if it is a lattice polynomial in variables $x_1, \dots, x_d$.*

We define the kth *order statistic* (cf. Example 3.1) $x^{(k)}$ on X^d by arranging a d-tuple $(x_1, \dots, x_d)$ in the increasing order:

$$x^{(1)} \le \cdots \le x^{(k)} \le \cdots \le x^{(d)}.$$

Clearly, an order statistic is a symmetric, continuous, and invariant function on X^d. The converse is also true [17]:

Theorem 5.4. *Let f be a symmetric continuous invariant function on X^d. Then, f is an order statistic.*

Proof. For a given sequence $a_1 < \cdots < a_d$, we have $f(a_1, \dots, a_d) = a_k$ for some $1 \le k \le d$, by Theorem 5.2. Suppose that x_i's are distinct elements of X. Since f is d-homogeneous, there is $\psi \in G$ such that $a_i\psi = x^{(i)}$ for all $1 \le i \le d$. Since f is symmetric and invariant, we have

$$\begin{aligned} f(x_1, \dots, x_d) &= f(x^{(1)}, \dots, x^{(d)}) = f(a_1\psi, \dots, a_2\psi) \\ &= f(a_1, \dots, a_d)\psi = a_k\psi = x^{(k)}. \end{aligned}$$

Therefore, f is the kth order statistics over chambers in X^d. The result follows from Theorem A.2(1), since f is a continuous function. $\square$

As in Example 3.1, we have the following lattice polynomial representation for the kth order statistics:

$$x^{(k)} = \bigvee_{Y \in X_k} \bigwedge_{j \in Y} x_j,$$

where X_k is the family of $(d-k+1)$-element subsets of $\{1, \dots, d\}$. Clearly, $x^{(1)} = \min\{x_1, \dots, x_d\}$ and $x^{(d)} = \max\{x_1, \dots, x_d\}$.

6. Concluding remarks

1. The statement that a continuous PL-function admits a representation as a max–min composition of its linear components appears to be intuitively clear. Apparently, this result was first stated in [29] and repeated in [25]. As it happens in applied areas, these publications lack precise definitions and assumptions. For instance, the result of Theorem 4.2 does not hold for non-convex domains (see the next remark), but this condition is not used in the 'proofs' found in [25]. In a different context, this result appears as Corollary 2.1 in [1], but again the proof is unsatisfactory. In its present form, the result was formulated and proven independently in [20]. A multidimensional analog of Theorem 4.2 is also found there.

2. The convexity of the domain D in Theorem 4.2 is an essential assumption. Consider, for instance, the domain

$$D = \{(x, y) \in \mathbb{R}^2 : y \leq |x|\}$$

in $\mathbb{R}^2$ and define a PL-function f on D by

$$f(x, y) = \begin{cases} y, & \text{if } x \wedge y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that f is not representable as a lattice polynomial in terms of functions $g_1(x, y) = 0$ and $g_2(x, y) = y$.

3. The result of Theorem 4.2 does not hold in general for piecewise polynomial functions. For instance, the function

$$h(x) = \begin{cases} 0, & \text{if } x < 0, \\ x^2, & \text{if } x \geq 0, \end{cases}$$

cannot be expressed by means of minimum and maximum operations on the zero function and x^2 (polynomial 'components' of h). On the other hand, we have

$$h(x) = ((x^3 + x) \vee 0) \wedge x^2,$$

that is, h is definable by means of the operations $\wedge$ and $\vee$ in the polynomial ring $\mathbb{R}[x]$. The "Pierce–Birkhoff conjecture" states that any continuous piecewise polynomial function on $\mathbb{R}^d$ can be obtained from the polynomial ring $\mathbb{R}[x_1, \ldots, x_d]$ by iterating the operations $\wedge$ and $\vee$. (The problem is still open; see [4,12,14,13]).

4. It is not difficult to show that selectors in the form (5.1) are in one-to-one correspondence with nonempty antichains of subsets of the set $\{1, \ldots, d\}$. Thus, the total number of continuous selectors on X^d is the Dedekind number (entry A007153 in [23]).

   A more involved problem is counting functions on X^d that can be expressed as lattice polynomials using the operations $\wedge$ and $\vee$, in which every variable appears exactly once. They are known as 'read-once expressions' [9] and of importance in the PL Morse theory [15]. It can be shown that the number $M(d)$ of distinct read-once functions on X^d equals twice the number of total partitions of d and satisfies the recurrence relation

$$M(n) = (n + 1)M(n - 1) + \sum_{k=2}^{n-2} \binom{n-1}{k} M(k)M(n - k),$$

with initial conditions $M(0) = 1$, $M(1) = 1$, and $M(2) = 2$ (cf. [24] and entry A000311 in [23]).

5. Infinite lattice polynomials on some normed spaces are instances of the Choquet integral; they are used for the representation of invariant functionals on those spaces [22]. These representations are of interest in the "aggregation problem" [20]. For an application in analysis, the reader is referred to [19].

Appendix. Interval topology

Let $(X, <)$ be a linear order with $|X| > 2$. We write $x \leq y$ if $x < y$ or $x = y$ in X. An *open ray* in X is a subset in the form

$$(\leftarrow, a) = \{x \in X : x < a\} \quad \text{or} \quad (a, \rightarrow) = \{z \in Z : z > a\}, \quad a \in X.$$

An *open interval* in X is either an open ray or a subset in the form

$$(a, b) = \{x \in X : a < x < b\}, \quad \text{for } a < b \text{ in } X.$$

Closed intervals $[a, b]$, $(\leftarrow, a]$, and $[a, \rightarrow)$ are defined similarly. A *gap* in X is an empty open interval (a, b). The family of open intervals is a base for the *interval topology* (order topology) on X.

Let $\alpha = (i_1 \cdots i_d)$ be a permutation of order d. A *chamber* $\mathcal{O}_\alpha$ is a subset of X^d defined by

$$\mathcal{O}_\alpha = \{(x_1, \ldots, x_d) : x_{i_1} < \cdots < x_{i_d}\}.$$

Two chambers $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ are *adjacent* if

$$\alpha = (i_1 \cdots i_k i_{k+1} \cdots i_d) \quad \text{and} \quad \beta = (i_1 \cdots i_{k+1} i_k \cdots i_d).$$

Note that for $X = \mathbb{R}$, the chambers in $\mathbb{R}^d$ are regions of the braid arrangement in $\mathbb{R}^d$.

Theorem A.1. *Let X be a linear order which is connected in its interval topology. Then, the chambers of X^d are connected sets.*

Proof. It suffices to show that the chamber $\mathcal{O}_\epsilon = \{x \in X^d : x_1 < \cdots < x_d\}$ is a connected set. Indeed, any other chamber is an image of $\mathcal{O}_\epsilon$ under a homeomorphism of X^d onto itself defined by a permutation of coordinates.

For any two points $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ in X^d such that $x_i = y_i$ for all $i \neq k$, the 'line segment'

$$\{(z_1, \ldots, z_d) : z_i = x_i \ (i \neq k), \ x_k \wedge y_k \leq z_k \leq x_k \vee y_k\}$$

is connected. It is not difficult to see that for any two points in the chamber $\mathcal{O}_\epsilon$ there is a sequence of points in $\mathcal{O}_\epsilon$ such that consecutive points differ in exactly one coordinate. The union of corresponding 'line segments' is a connected set (a 'path') containing the two points. It follows that the chamber $\mathcal{O}_\epsilon$ is a connected set. □

The next theorem puts forth some properties of chambers established in [17] and [18].

Theorem A.2. *Let X be a linear order without gaps.*

(1) *The chambers in X^d are open sets and their union is dense in X^d.*
(2) *Let $\mathcal{O}_{(i_1 \cdots i_k i_{k+1} \cdots i_d)}$ and $\mathcal{O}_{(i_1 \cdots i_{k+1} i_k \cdots i_d)}$ be two adjacent chambers in X^d and $f : X^d \to X$ be a continuous function such that*

$$f(x) = x_p \quad \text{for } x \in \mathcal{O}_{(i_1 \ldots i_k i_{k+1} \ldots i_d)} \quad \text{and} \quad f(x) = x_q \quad \text{for } x \in \mathcal{O}_{(i_1 \ldots i_{k+1} i_k \ldots i_d)}.$$

Then, one of the following holds:

(i) $p = q$,
(ii) $p = i_k$, $q = i_{k+1}$,
(iii) $p = i_{k+1}$, $q = i_k$.

Clearly, the results of Theorem A.2 hold for connected linear orders. Another class of linear orders without gaps consists of doubly homogeneous linear orders. For details about ordered permutation groups, the reader is referred to [10].

Let X be a linear order and $\mathcal{A}(X)$ be the group of automorphisms (order-preserving permutations) of X. This group inherits the pointwise order from X, that is, $\alpha < \beta$ if and only if $x\alpha < x\beta$ for all $x \in X$. This order makes $\mathcal{A}(X)$ a lattice-ordered permutation group (ℓ-permutation group), that is, $\mathcal{A}(X)$ is a lattice and the order is preserved by multiplication on both sides. The meet and join operations are also defined pointwise. A subgroup G of $\mathcal{A}(X)$ which is also a sublattice is called an ℓ-permutation group.

A subgroup $G \subseteq \mathcal{A}(X)$ is said to be m-transitive if for all

$$x_1 \leq \cdots \leq x_m \quad \text{and} \quad y_1 \leq \cdots \leq y_m$$

in X, there exists $\alpha \in G$ such that $x_i\alpha = y_i$ for all $1 \leq i \leq m$. If G is m-transitive, we say that X is m-homogeneous. It is clear, that a 2-homogeneous (doubly homogeneous) linear order does not have gaps.

The result of the following theorem is Lemma 1.10.1 in [10].

Theorem A.3. *If $G \subseteq \mathcal{A}(X)$ is a doubly transitive ℓ-permutation group, then it is m-transitive for all $m \geq 2$.*

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