Extensions of p-adic vector measures

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ABSTRACT

For $\mathcal{R}$ being a separating algebra of subsets of a set $X$, $E$ a complete Hausdorff non-Archimedean locally convex space and $m : \mathcal{R} \to E$ a bounded finitely additive measure, it is shown that:

(a) If $m$ is $\sigma$-additive and strongly additive, then $m$ has a unique $\sigma$-additive extension $m''$ on the $\sigma$-algebra $\mathcal{R}''$ generated by $\mathcal{R}$.

(b) If $m$ is strongly additive and $\tau$-additive, then $m$ has a unique $\tau$-additive extension $m'$ on the $\sigma$-algebra $\mathcal{R}^{bc}$ of all $\tau_\mathcal{R}$-Borel sets, where $\tau_\mathcal{R}$ is the topology having $\mathcal{R}$ as a basis.

Also, some other results concerning such measures are given.

I. PRELIMINARIES

Throughout this paper, $\mathbb{K}$ will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over $\mathbb{K}$, we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [12] or [13]). For $E$ a locally

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579
convex space, we will denote by \( cs(E) \) the collection of all continuous seminorms on \( E \). For \( X \) a set, \( f \in \mathbb{K}^X \) and \( A \subset X \), we define

\[
\| f \|_A = \sup\{ |f(x)| : x \in A \} \quad \text{and} \quad \| f \| = \| f \|_X.
\]

Also for \( A \subset X \), \( A^c \) will be its complement in \( X \) and \( \chi_A \) the \( \mathbb{K} \)-valued characteristic function of \( A \). The family of all subsets of \( X \) will be denoted by \( \mathcal{P}(X) \).

Assume next that \( X \) is a non-empty set and \( \mathcal{R} \) a separating algebra of subsets of \( X \), i.e. \( \mathcal{R} \) is a family of subsets of \( X \) such that

1. \( X \in \mathcal{R} \), and, if \( A, B \in \mathcal{R} \), then \( A \cup B, A \cap B, A^c \) are also in \( \mathcal{R} \).
2. If \( x, y \) are distinct elements of \( X \), then there exists a member of \( \mathcal{R} \) which contains \( x \) but not \( y \).

Then \( \mathcal{R} \) is a base for a Hausdorff zero-dimensional topology \( \tau_\mathcal{R} \) on \( X \). For \( E \) a locally convex space, we denote by \( M(\mathcal{R}, E) \) the space of all finitely-additive measures \( m : \mathcal{R} \to E \) such that \( m(\mathcal{R}) \) is a bounded subset of \( E \) (see [10]). For a net \( (V_\delta) \) of subsets of \( X \), we write \( V_\delta \downarrow \emptyset \) if \( (V_\delta) \) is decreasing and \( \bigcap V_\delta = \emptyset \). An element \( m \in M(\mathcal{R}, E) \) is said to be \( \sigma \)-additive if \( m(V_n) \to 0 \) for each sequence \( (V_n) \) in \( \mathcal{R} \) which decreases to the empty set. We denote by \( M_\sigma(\mathcal{R}, E) \) the space of all \( \sigma \)-additive members of \( M(\mathcal{R}, E) \). An \( m \) of \( M(\mathcal{R}, E) \) is said to be \( \tau \)-additive if \( m(V_\delta) \to 0 \) for each net \( (V_\delta) \) in \( \mathcal{R} \) with \( V_\delta \downarrow \emptyset \). We will denote by \( M_\tau(\mathcal{R}, E) \) the space of all \( \tau \)-additive members of \( M(\mathcal{R}, E) \). For \( m \in M(\mathcal{R}, E) \) and \( p \in cs(E) \), we define

\[
m_p : \mathcal{R} \to \mathbb{K}, \quad m_p(A) = \sup\{ p(m(V)) : V \in \mathcal{R}, V \subset A \}
\]

and

\[
\| m \|_p = m_p(X).
\]

We also define

\[
N_{m, p} : X \to \mathbb{K}, \quad N_{m, p}(x) = \inf\{ m_p(V) : x \in V \in \mathcal{R} \}.
\]

Next we will recall the definition of the integral of an \( f \in \mathbb{K}^X \) with respect to some \( m \in M(\mathcal{R}, E) \). Assume that \( E \) is a complete Hausdorff locally convex space. For \( A \subset X \), let \( \mathcal{D}_A \) be the family of all \( \alpha = \{ A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n \} \), where \( \{ A_1, A_2, \ldots, A_n \} \) is an \( \mathcal{R} \)-partition of \( A \) and \( x_k \in A_k \). We make \( \mathcal{D}_A \) into a directed set by defining \( \alpha_1 \geq \alpha_2 \) if the partition of \( A \) in \( \alpha_1 \) is a refinement of the one in \( \alpha_2 \). For \( \alpha = \{ A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n \} \), we define \( \omega_\alpha(f, m) = \sum_{k=1}^n f(x_k)m(A_k) \).

If the limit \( \lim \omega_\alpha(f, m) \) exists in \( E \), we will say that \( f \) is \( m \)-integrable over \( A \) and denote this limit by \( \int_A f \, dm \). For \( A = X \), we write simply \( \int f \, dm \). It is easy to
see that if \( f \) is \( m \)-integrable over \( X \), then it is \( m \)-integrable over every \( A \in \mathcal{R} \) and
\[
\int_A f \, dm = \int \chi_A f \, dm. 
\]
If \( f \) is bounded on \( A \), then
\[
p \left( \int_A f \, dm \right) \leq \| f \|_A \cdot m_p(A).
\]

2. STRONGLY ADDITIVE MEASURES

Throughout the paper, \( \mathcal{R} \) will be a separating algebra of subsets of a set \( X \), \( E \) a complete Hausdorff locally convex space and \( M(\mathcal{R}, E) \) the space of all bounded \( E \)-valued finitely-additive measures on \( \mathcal{R} \). We will denote by \( \tau_\mathcal{R} \) the topology on \( X \) which has \( \mathcal{R} \) as a basis. Every member of \( \mathcal{R} \) is \( \tau_\mathcal{R} \)-clopen, i.e. both closed and open. By \( S(\mathcal{R}) \) we will denote the space of all \( \mathbb{K} \)-valued \( \mathcal{R} \)-simple functions. As in [10], if \( m \in M(\mathcal{R}, E) \), then a subset \( A \) of \( X \) is said to be \( m \)-measurable if the characteristic function \( \chi_A \) is \( m \)-integrable. By [10, Theorem 4.7], \( A \) is \( m \)-measurable iff, for each \( p \in cs(E) \) and each \( \epsilon > 0 \), there exist \( V, W \) in \( \mathcal{R} \) such that \( V \subset A \subset W \) and \( m_p(W \setminus V) < \epsilon \).

Let \( \mathcal{R}_m \) be the family of all \( m \)-measurable sets. The following theorem gives some results contained in [10] which are needed for the paper.

**Theorem 2.1.**

1. \( \mathcal{R}_m \) is an algebra of subsets of \( X \).
2. If \( \hat{m} : \mathcal{R}_m \to E, \hat{m}(A) = \int \chi_A \, dm \), then \( \hat{m} \in M(\mathcal{R}_m, E) \).
3. \( \hat{m} \) is \( \sigma \)-additive iff \( m \) is \( \sigma \)-additive.
4. \( \hat{m} \) is \( \tau \)-additive iff \( m \) is \( \tau \)-additive.
5. For \( p \in cs(E) \), we have \( N_{m,p} = N_{\hat{m},p} \).
6. \( \mathcal{R}_m = \mathcal{R}_{\hat{m}} \).
7. For \( A \in \mathcal{R} \), we have \( m_p(A) = \hat{m}_p(A) \).
8. For \( A \in \mathcal{R}_m \), we have
\[
\hat{m}_p(A) = \inf \{ m_p(W) : W \in \mathcal{R}, A \subset W \}.
\]
9. If \( m \) is \( \sigma \)-additive and \( V_n \downarrow \emptyset \), then \( m_p(V_n) \to 0 \).
10. If \( m \) is \( \tau \)-additive and \( V_\emptyset \downarrow \emptyset \), then \( m_p(V_\emptyset) \to 0 \).
11. If \( m \) is \( \sigma \)-additive and if \( (V_n) \) is a sequence in \( \mathcal{R} \), then for every set \( V \) in \( \mathcal{R} \) contained in \( \bigcup V_n \), we have \( m_p(V) \leq \sup_n m_p(V_n) \).
12. If \( m \) is \( \tau \)-additive and if \( (V_\emptyset) \) is a family in \( \mathcal{R} \), then for every set \( V \) in \( \mathcal{R} \) contained in \( \bigcup V_\emptyset \), we have \( m_p(V) \leq \sup_\emptyset m_p(V_\emptyset) \).
13. An \( f \in \mathbb{K}^X \) is \( m \)-integrable iff, for each \( p \in cs(E) \) and each \( \epsilon > 0 \), there exists an \( \mathcal{R} \)-partition \( \{A_1, \ldots, A_n\} \) of \( X \) such that, for each \( 1 \leq k \leq n \), we have
\[
|f(x) - f(y)| \cdot m_p(A_k) \leq \epsilon \text{ if } x, y \in A_k. \text{ In this case, if } x_k \in A_k, \text{ then}
\]
\[
p \left( \int f \, dm - \sum_{k=1}^n f(x_k)m(A_k) \right) \leq \epsilon.
\]
14. If \( m \) is \( \tau \)-additive, then a subset \( A \) of \( X \) is measurable iff \( A \) is \( \tau_{\mathcal{R}_m} \)-clopen.

**Definition 2.2.** An element \( m \) of \( M(\mathcal{R}, E) \) is said to be strongly additive if, for each sequence \((A_n)\) of pairwise disjoint members of \( \mathcal{R} \), we have that \( m(A_n) \to 0 \).

It is clear that, if \( \mathcal{R} \) is a \( \sigma \)-algebra and \( m \) \( \sigma \)-additive, then \( m \) is strongly additive.

**Theorem 2.3.** For an \( m \in M(\mathcal{R}, E) \), the following are equivalent:

1. \( m \) is strongly additive.
2. For each decreasing sequence \((A_n)\) of members of \( \mathcal{R} \), the sequence \((m(A_n))\) is convergent in \( E \).
3. For each sequence \((A_n)\) of pairwise disjoint members of \( \mathcal{R} \) and each \( p \in cs(E) \), we have \( m_p(A_n) \to 0 \).
4. For each decreasing net \((V_\delta)\) in \( \mathcal{R} \), the net \((m(V_\delta))\) converges in \( E \).
5. For each decreasing net \((V_\delta)\) in \( \mathcal{R} \), each \( p \in cs(E) \) and each \( \epsilon > 0 \), there exists \( \delta_0 \) such that \( m_p(V_\delta \triangle V_{\delta'}) < \epsilon \) for all \( \delta, \delta' \geq \delta_0 \).
6. For each family \((V_i)_{i \in I}\) of pairwise disjoint members of \( \mathcal{R} \), each \( p \in cs(E) \) and each \( \epsilon > 0 \), there exists \( J \subset I \) finite such that \( m_p(V_i) < \epsilon \) for all \( i \notin J \).
7. Let \((V_i)_{i \in I}\) be a family of pairwise disjoint members of \( \mathcal{R} \). For \( J \subset I \) finite, let \( W_J = \bigcup_{i \in J} V_i \). Then the net \((m(W_J))\) is convergent.

**Proof.**

(1) \( \Rightarrow \) (3). Assume the contrary. Then, there exist \( p \in cs(E) \), \( \epsilon > 0 \) and \( n_1 < n_2 < \cdots \) such that \( m_p(A_{n_k}) > \epsilon \) for all \( k \). For each \( k \), there exists a \( B_k \) contained in \( A_{n_k} \) such that \( p(m(B_k)) > \epsilon \). This contradicts our hypothesis (1).

(3) \( \Rightarrow \) (5). Assume that (5) does not hold. Then, there exist \( p \in cs(E) \) and \( \epsilon > 0 \) such that, for each \( \delta \) there are \( \delta_1, \delta_2 \geq \delta \) with \( m_p(V_{\delta_1} \triangle V_{\delta_2}) > \epsilon \). Thus, for each \( \delta \), there exists \( \delta' \geq \delta \) such that \( m_p(V_\delta \triangle V_{\delta'}) > \epsilon \). Now, there exist \( \delta_1 \leq \delta_2 \leq \cdots \) such that \( m_p(V_{\delta_k} \triangle V_{\delta_{k+1}}) > \epsilon \) for all \( k \). If \( G_n = V_{\delta_n} \setminus V_{\delta_{n+1}} \), then the sets \( G_n \) are pairwise disjoint, which contradicts (3).

(5) \( \Rightarrow \) (4). Let \((V_\delta)\) be a decreasing net in \( \mathcal{R} \) and \( p \in cs(E) \). Then, for all \( \delta, \delta' \), we have \( p(m(V_\delta) - m(V_{\delta'})) \leq m_p(V_\delta \triangle V_{\delta'}) \). This, by our hypothesis, implies that the net \((m(V_\delta))\) is Cauchy and hence convergent.

(4) \( \Rightarrow \) (2). It is trivial.

(2) \( \Rightarrow \) (1). For \((A_n)\) a sequence of pairwise disjoint members of \( \mathcal{R} \), let

\[
B_n = \left( \bigcup_{k=1}^{n} A_k \right)^c.
\]

Then \((B_n)\) is decreasing and so the sequence \((m(B_n))\) is convergent. Thus, given \( p \) in \( cs(E) \) and \( \epsilon > 0 \), there exists \( n_0 \) such that

\[
p(m(B_n \setminus B_{n+1})) = p(m(B_n) - m(B_{n+1})) < \epsilon
\]

for \( n > n_0 \). But \( B_n \setminus B_{n+1} = A_{n+1} \). Thus \( m(A_n) \to 0 \).
(3) \implies (6). Let \((V_i)_{i \in I}\) be a family of pairwise disjoint members of \(\mathcal{R}\) and suppose that, for some \(p \in \text{cs}(E)\) and some \(\varepsilon > 0\), the set \(\{i \in I : m_p(V_i) \geq \varepsilon\}\) is infinite. Hence there are distinct \(i_k, k = 1, 2, \ldots\), such that \(m_p(V_{i_k}) \geq \varepsilon\), which contradicts our hypothesis (3).

(6) \implies (7). Let \((V_i)_{i \in I}\) be a family of pairwise disjoint members of \(\mathcal{R}\). For \(J \subset I\) finite, let \(W_J = \bigcup_{i \in J} V_i\). Let \(J_0\) be a finite subset of \(I\) such that \(m_p(V_i) < \varepsilon\) for all \(i \notin J_0\). If now \(J\) is any finite subset of \(I\) containing \(J_0\), then
\[
p(m(W_J) - m(W_{J_0})) = p\left(\bigcup_{i \in J \setminus J_0} V_i\right) \leq \max_{i \in J \setminus J_0} m_p(V_i) < \varepsilon.
\]
Hence the net \((m(W_J))\) is Cauchy and therefore convergent.

(7) \implies (1). It follows easily. \(\Box\)

**Definition 2.4.** A family \(H\) of members of \(M(\mathcal{R}, E)\), is said to be uniformly strongly additive iff, for each sequence \((A_n)\), of pairwise disjoint members of \(\mathcal{R}\), we have that \(m(A_n) \to 0\) uniformly for \(m \in H\).

Using arguments analogous to the ones used in the proof of Theorem 2.3, we get the following theorem.

**Theorem 2.5.** For a subset \(H\) of \(M(\mathcal{R}, E)\), the following are equivalent:

1. \(H\) is uniformly strongly additive.
2. For each sequence \((A_n)\), of pairwise disjoint members of \(\mathcal{R}\), and each \(p \in \text{cs}(E)\), we have that
   \[
   \lim_{n \to \infty} m_p(A_n) = 0
   \]
   uniformly for \(m \in H\).
3. If \((A_n)\) is a decreasing sequence of members of \(\mathcal{R}\), then, for each \(p \in \text{cs}(E)\) and each \(\varepsilon > 0\), there exists \(n_0\) such that \(m_p(A_n \setminus A_k) \leq \varepsilon\) for all \(k > n \geq n_0\).

**Theorem 2.6.** Let \(H \subset M_\sigma(\mathcal{R}, E)\) be uniformly strongly additive and let \((A_n)\) be a sequence in \(\mathcal{R}\) such that \(A_n \downarrow \emptyset\). Then, for each \(p \in \text{cs}(E)\), \(m_p(A_n) \to 0\) uniformly for \(m \in H\).

**Proof.** Given \(p \in \text{cs}(E)\) and \(\varepsilon > 0\), there exists \(n_0\) such that \(m_p(A_n \setminus A_k) < \varepsilon\) for all \(k > n \geq n_0\) and each \(m \in H\). Let now \(n \geq n_0\). For \(k > n\), we have \(A_n = (A_n \setminus A_k) \cup A_k\) and so
\[
m_p(A_n) = \max\{m_p(A_n \setminus A_k), m_p(A_k)\} \leq \max\{\varepsilon, m_p(A_k)\}.
\]
Since \(m_p(A_k) \to 0\) when \(k \to \infty\), it follows that \(m_p(A_n) \leq \varepsilon\) for all \(n \geq n_0\) and all \(m \in H\). This completes the proof. \(\Box\)
Theorem 2.7 (Nikodym boundedness theorem). Assume that $\mathcal{R}$ is a $\sigma$-algebra and let $H$ be a subset of $M(\mathcal{R}, E)$ consisting of strongly additive measures. If, for each $A \in \mathcal{R}$, the set $H(A) = \{m(A) : m \in H\}$ is bounded in $E$, then the set $H(\mathcal{R}) = \{m(A) : A \in \mathcal{R}, m \in H\}$ is bounded, equivalently $\sup_{m \in H} \|m\|_p < \infty$ for each $p \in cs(E)$.

**Proof.** Assume the contrary. Then, there exist a $p \in cs(E)$ and a sequence $(m_n)$ in $H$ such that $\sup_n \|m_n\|_p = \infty$.

**Claim I.** If $G \in \mathcal{R}$ is such that $\sup_n (m_n)_p(G) = \infty$, then, for each $\alpha > 0$, there exist an $n$ and an $\mathcal{R}$-partition $\{A, B\}$ of $G$ such that $p(m_n(A)) = p(m_n(B)) > \alpha$.

Indeed, there exist an $n$ and $A \in \mathcal{R}, A \subset G$, such that

$$p(m_n(A)) > \max \left\{ \alpha, \sup_k p(m_k(G)) \right\} = \max \{\alpha, p(m_n(G))\}.$$

If $B = G \setminus A$, then

$$p(m_n(A)) > p(m_n(G)) = p(m_n(A) + m_n(B)).$$

Thus $p(m_n(A)) = p(m_n(B)) > \alpha$. Let now $n_1$ be a positive integer and $\{A_1, B_1\}$ an $\mathcal{R}$-partition of $X$ such that $p(m_{n_1}(A_1)) = p(m_{n_1}(B_1)) > 1$. The $\sup_n (m_n)_p(A_1)$, $\sup_n (m_n)_p(B_1)$ cannot both be finite. If the former is infinite, take $G_1 = A_1$ and $F_1 = B_1$, otherwise take $G_1 = B_1$ and $F_1 = A_1$. Let $n_2 > n_1$ and $\{A_2, B_2\}$ an $\mathcal{R}$-partition of $G_1$ such that

$$p(m_{n_2}(A_2)) = p(m_{n_2}(B_2)) > \max \left\{ 2, \sup_k p(m_k(F_1)) \right\} = \max \{ 2, p(m_{n_2}(F_1)) \}.$$

One of the $\sup_n (m_n)_p(A_2)$, $\sup_n (m_n)_p(B_2)$ must be infinite. If the former is infinite, take $G_2 = A_2$ and $F_2 = B_2$, otherwise take $G_2 = B_2$ and $F_2 = A_2$. We continue using the same argument and get by induction a sequence $(F_k)$, of pairwise disjoint members of $\mathcal{R}$, and $n_1 < n_2 < \cdots$ such that

$$p(m_{n_k}(F_k)) > \max \left\{ k, \max_{1 \leq j < k} p(m_{n_k}(F_j)) \right\}.$$

Let $\mu_k = m_{n_k}$.

**Claim II.** For each $m \in H$ and each infinite subset $\Omega$ of $\mathbb{N}$, there exists an infinite subset $Z$ of $\Omega$ such that $m_p(\bigcup_{n \in Z} F_n) < 1$.

Indeed, there exists an infinite partition $\Omega_1, \Omega_2, \ldots$ of $\Omega$ into infinite sets. The sets $D_k = \bigcup_{n \in \Omega_k} F_n, k \in \mathbb{N}$, are pairwise disjoint members of $\mathcal{R}$. Since $m$ is strongly additive, there exists a $k$ such that $m_p(D_k) < 1$. 584
Let now \( r_1 = 1 \), \( W_{1,j} = F_j \) for \( j \in \mathbb{N} \). By the preceding claim, there exists a subsequence \((W_{2,j})\) of \((W_{1,j})\), with \( W_{2,1} = F_{r_2} \) and \( r_2 > r_1 \), such that

\[
(\mu_{r_1})_p \left( \bigcup_j W_{2,j} \right) < 1.
\]

Next, there exists a subsequence \((W_{3,j})\) of \((W_{2,j})\), with \( W_{3,1} = F_{r_3} \), and \( r_3 > r_2 \), such that

\[
(\mu_{r_2})_p \left( \bigcup_j W_{3,j} \right) < 1.
\]

We continue the same argument using induction. Let \( W = \bigcup_{i=1}^{\infty} F_{r_i} \). For each \( j \) we have

\[
\mu_{r_j}(W) = \mu_{r_j}(F_{r_j}) + \mu_{r_j} \left( \bigcup_{k < j} F_{r_k} \right) + \mu_{r_j} \left( \bigcup_{k > j} F_{r_k} \right).
\]

Now \( p[\mu_{r_j}(F_{r_j})] > r_j \geq 1 \), \( (\mu_{r_j})_p(\bigcup_{k > j} F_{r_k}) < 1 \) and

\[
p \left( \mu_{r_j} \left( \bigcup_{k < j} F_{r_k} \right) \right) < p(\mu_{r_j}(F_{r_j})).
\]

Thus \( p(\mu_{r_j}(W)) = p(\mu_{r_j}(F_{r_j})) > r_j \) and so \( \sup_j p(\mu_{r_j}(W)) = \infty \), a contradiction. This completes the proof. \( \square \)

**Theorem 2.8.** If \( m \in M(\mathcal{R}, E) \) is strongly additive, then \( \tilde{m} \in M(\mathcal{R}_m, E) \) is also strongly additive.

**Proof.** Let \((A_n)\) be a sequence, of pairwise disjoint members of \( \mathcal{R}_m \), and let \( p \in cs(E) \) and \( \varepsilon > 0 \). For each \( n \), there exist \( V_n, W_n \) in \( \mathcal{R} \) such that \( V_n \subset A_n \subset W_n \) and \( m_p(W_n \setminus V_n) < \varepsilon \). As \( m \) is strongly additive, there exists \( n_o \) such that \( m_p(V_n) < \varepsilon \) for each \( n \geq n_o \). If now \( n \geq n_o \), then \( A_n \subset V_n \cup (W_n \setminus V_n) \) and so

\[
\tilde{m}_p(A_n) \leq \max\{m_p(W_n \setminus V_n), m_p(V_n)\} < \varepsilon.
\]

Hence \( \tilde{m} \) is strongly additive. \( \square \)

### 3. Absolute Continuity

**Definition 3.1.** An element \( m \) of \( M(\mathcal{R}, E) \) is said to be absolutely continuous with respect to some \( \mu \in M(\mathcal{R}) \), and we write \( m \ll \mu \), if

\[
\lim_{|\mu|(A) \to 0} m_p(A) = 0
\]

for each \( p \in cs(E) \). Equivalently, for each \( p \in cs(E) \) and each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( m_p(A) < \varepsilon \) for each \( A \in \mathcal{R} \) with \( |\mu|(A) < \delta \).
Theorem 3.2. Let $\mu \in M_\sigma(\mathcal{R})$ and $m \in M_\sigma(\mathcal{R}, E)$. If $\mathcal{R}$ is a $\sigma$-algebra, then $m \ll \mu$ iff $m(A) = 0$ for each $A \in \mathcal{R}$ with $|\mu|(A) = 0$.

Proof. The condition is clearly necessary. Conversely, suppose that the condition is satisfied but $m$ is not $\mu$-absolutely continuous. Then there exist $p \in cs(E)$ and $\epsilon > 0$ and a sequence $(A_n)$ in $\mathcal{R}$, with $|\mu|(A_n) < 1/n$, such that $m_p(A_n) > \epsilon$ for all $n$. Let $G_n = \bigcup_{k \geq n} A_k$, $G = \bigcap G_n$. Then

$$|\mu|(G) \leq |\mu|(G_n) = \sup_{k \geq n} |\mu|(A_k) < 1/n \to 0.$$ By our hypothesis, $m_p(G) = 0$. Let $G_0 = X$ and $B_n = G_{n-1} \setminus G_n$ for each $n \in \mathbb{N}$. The sequence $(B_n)$ consists of pairwise disjoint members of $\mathcal{R}$. Moreover $G_n \setminus G = \bigcup_{k > n} B_k$ and so $m_p(G_n \setminus G) \to 0$. Also, $A_n \subseteq G_n = G \cup (G_n \setminus G)$ and hence

$$m_p(A_n) \leq \max\{m_p(G), m_p(G_n \setminus G)\} = m_p(G_n \setminus G) \to 0$$
as $n \to \infty$. This completes the proof. \(\square\)

Theorem 3.3. Let $m \in M(\mathcal{R}, E)$ and $\mu \in M(\mathcal{R})$ be such that $m \ll \mu$. Then:

1. $\mathcal{R}_\mu \subseteq \mathcal{R}_m$.
2. If $m_1 = \tilde{m}|_{\mathcal{R}_\mu}$, then $m_1 \ll \tilde{\mu}$.

Proof. 1. Assume that $A \in \mathcal{R}_\mu$ and let $p \in cs(E)$ and $\epsilon > 0$. Since $m \ll \mu$, there exists $\delta > 0$ such that $m_p(B) < \epsilon$ if $|\mu|(B) < \delta$. As $A \in \mathcal{R}_\mu$, there are $V, W \in \mathcal{R}$, with $V \subseteq A \subseteq W$, such that $|\mu|(W \setminus V) < \delta$. But then $m_p(W \setminus V) < \epsilon$, which proves that $A$ is in $\mathcal{R}_m$.

2. Let $p \in cs(E)$ and $\epsilon > 0$. There exists $\delta > 0$ such that $m_p(B) < \epsilon$ if $|\mu|(B) < \delta$. Let now $A \in \mathcal{R}_\mu$ with $|\mu|(A) < \delta$. We will show that $m_p(A) < \epsilon$. In fact, there are $V, W \in \mathcal{R}$, with $V \subseteq A \subseteq W$, such that $|\mu|(W \setminus V) < \delta$. But then,

$$m_p(W \setminus V) = m_p(W \setminus V) < \epsilon.$$ Also

$$|\mu|(V) = |\tilde{\mu}|(V) \leq |\tilde{\mu}|(A) < \delta$$

and hence $m_p(V) = m_p(V) < \epsilon$. Since $A \subseteq V \cup (W \setminus V)$, we have that

$$m_p(A) \leq \max\{m_p(V), m_p(W \setminus V)\} < \epsilon.$$ Hence the result follows. \(\square\)

Definition 3.4. Let $\mu \in M(\mathcal{R})$. A collection $H \subset M(\mathcal{R}, E)$ is said to be uniformly absolutely continuous with respect to $\mu$ if, for each $p \in cs(E)$, we have

$$\lim_{|\mu|(A) \to 0} \sup_{m \in H} m_p(A) = 0.$$

586
Theorem 3.5. Let $H$ be a uniformly strongly additive subset of $M(\mathcal{R}, E)$ and let $\mu \in M(\mathcal{R})$ be such that $m \ll \mu$ for each $m \in H$. Then $H$ is uniformly absolutely continuous with respect to $\mu$.

Proof. Assume the contrary. Then, there exist $p \in cs(E)$ and $\epsilon > 0$ such that, for each $\delta > 0$, there are $m \in H$ and $A \in \mathcal{R}$ such that $|\mu|(A) < \delta$ and $m_p(A) > \epsilon$. Let $A_1 \in \mathcal{R}$, $m_1 \in H$, $\delta_1 = 1$ be such that $|\mu|(A_1) < \delta_1$ and $(m_1)_p(A_1) > \epsilon$. Since $m_1 \ll \mu$, there exists $\delta_2 > 0$ such that, if $|\mu|(A) < \delta_2$, then $(m_1)_p(A) < \epsilon$. There exist $A_2 \in \mathcal{R}$ and $m_2 \in H$ such that $|\mu|(A_2) < \delta_2$ and $(m_2)_p(A_2) > \epsilon$. Next there exists $\delta_3 > 0$ such that, if $|\mu|(A) < \delta_3$, then $(m_1)_p(A) < \epsilon$ and $(m_2)_p(A) < \epsilon$. Let $m_3 \in H$ and $A_3 \in \mathcal{R}$ be such that $|\mu|(A_3) < \delta_3$ and $(m_3)_p(A_3) > \epsilon$. Inductively, we get a sequence $(m_n)_n$ in $H$ and a sequence $(A_n)_n$ in $\mathcal{R}$ such that $(m_n)_p(A_n) > \epsilon$ and $(m_k)_p(A_n) < \epsilon$ if $k < n$.

Claim. There are $n_0 = 1 < n_1 < n_2 < \cdots < n_k$ such that, for $G_o = A_1$, $G_1 = G_o \setminus A_1, \ldots, G_k = G_{k-1} \setminus A_k$, we have

1. $(m_{n_j})_p(G_{j-1} \cap A_{n_j}) > \epsilon$, for $j = 1, 2, \ldots, k$.
2. $(m_n)_p(G_k \cap A_n) \leq \epsilon$ for every $n > n_k$.

In fact, if $(m_n)_p(G_o \cap A_n) \leq \epsilon$ for every $n > 1$, take $k = 0$, $n_0 = 1$. Otherwise, choose $n_1 > n_0 = 1$ such that $(m_{n_1})_p(G_o \cap A_{n_1}) > \epsilon$ and let $G_1 = G_o \setminus A_{n_1}$. If $(m_n)_p(G_1 \cap A_n) \leq \epsilon$ for all $n > n_1$, take $k = 1$. Otherwise, choose $n_2 > n_1$ such that $(m_{n_2})_p(G_1 \cap A_{n_2}) > \epsilon$ and let $G_2 = G_1 \setminus A_{n_2}$. If this process does not eventually terminate, we find by induction $n_0 = 1 < n_1 < n_2 < \cdots$ such that, for $G_o = A_1$ and $G_k = G_{k-1} \setminus A_{n_k}$, for $k \geq 1$, we have $(m_{n_k})_p(G_{k-1} \cap A_{n_k}) > \epsilon$ for all $k \geq 1$. Let $D_k = G_{k-1} \setminus G_k$, $k \geq 1$. The sets $D_k$ are pairwise disjoint. Moreover, $D_k = G_{k-1} \cap A_{n_k}$ and so $(m_{n_k})_p(D_k) > \epsilon$, for all $k$, which contradicts the fact that $H$ is uniformly strongly additive. Hence the claim holds. Let now $n_0 = 1 < n_1 < n_2 < \cdots < n_k$ be as in the claim. Since

$$A_1 = \left[ \bigcup_{j=1}^k A_1 \cap A_{n_j} \right] \cup G_k$$

and $(m_1)_p(A_1) > \epsilon$ while $(m_1)_p(A_1 \cap A_{n_j}) \leq (m_1)_p(A_{n_j}) < \epsilon$ for $j \leq k$, it follows that $(m_1)_p(G_k) > \epsilon$. Let $F_1 = G_k \subset A_1$ and $r_1 = n_k$. For $n > r_1$, let $B_n = A_n \setminus F_1$. Let $n > r_1$. Then $A_n = (A_n \cap F_1) \cup B_n$. Since $(m_n)_p(A_n \cap F_1) \leq \epsilon$ and $(m_n)_p(A_n) > \epsilon$, it follows that $(m_n)_p(B_n) > \epsilon$. Also, for $r_1 < n < N$, we have $(m_n)_p(B_n) \leq (m_n)_p(A_n) < \epsilon$. Now, we can apply the same argument as above, replacing $(A_n)$ by $(B_n)_n_{n>r_1}$ and $(m_n)$ by $(m_n)_{n>r_1}$. We will then get an $r_2 > r_1$ and $F_2 \subset B_1$ such that $(m_{r_2+1})_p(F_2) > \epsilon$ and $(m_n)_p(F_2 \cap B_n) \leq \epsilon$ for all $n > r_2$. For $n > r_2$, let $Z_n = B_n \setminus F_2$. Since $B_n = (B_n \cap F_2) \cup Z_n$ and since $(m_n)_p(B_n \cap F_2) \leq \epsilon$, while $(m_n)_p(B_n) > \epsilon$, we get that $(m_n)_p(Z_n) > \epsilon$. Also, for $r_2 < n < N$, we have $(m_n)_p(Z_n) \leq (m_n)_p(B_n) < \epsilon$. Thus we may repeat the same argument for the sequences $(Z_n)_{n>r_2}$ and $(m_n)_{n>r_2}$. Inductively, we get a sequence $(F_k)$, of pairwise
disjoint members of $\mathcal{R}$, and a sequence $(m'_k)$ in $H$ such that $(m'_k)_{p(F_k)} > \epsilon$. This contradicts our hypothesis that $H$ is uniformly strongly additive. Hence the result follows. \qed

4. EXTENSIONS OF $\sigma$-ADDITIVE MEASURES

In this section we will examine the problem of extending an $m \in M(\mathcal{R}, E)$ to a $\sigma$-additive measure defined on a $\sigma$-algebra containing $\mathcal{R}$. In order for such an extension to exist, it is clearly necessary that $m$ is $\sigma$-additive and strongly additive. We will show that these two conditions are also sufficient.

**Note.** Throughout the rest of this section, $m$ will be a strongly additive member of $M_\sigma(\mathcal{R}, E)$.

For $p \in cs(E)$, we define

$$\hat{m}_p : P(X) \to \mathbb{R}, \quad \hat{m}_p(A) = \inf \sup_n m_p(V_n),$$

where the infimum is taken over the collection of all sequences $(V_n)$ in $\mathcal{R}$ which cover $A$. It is easy to show that $\hat{m}_p(A \cup B) = \max\{\hat{m}_p(A), \hat{m}_p(B)\}$.

**Lemma 4.1.** $m_p(A) = \hat{m}_p(A)$ for all $A \in \mathcal{R}$.

**Proof.** Clearly $\hat{m}_p(A) \leq m_p(A)$. On the other hand, if $(V_n)$ is a sequence in $\mathcal{R}$ covering $A$, then $m_p(A) \leq \sup_n m_p(V_n)$ since $m$ is $\sigma$-additive. This implies that $\hat{m}_p(A) \geq m_p(A)$.

Let now

$$\hat{d}_p : P(X) \times P(X) \to \mathbb{R}, \quad \hat{d}_p(A, B) = \hat{m}_p(A \Delta B),$$

where $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Then $\hat{d}_p$ is an ultrapseudometric on $P(X)$. Let $\mathcal{U}_m^\sigma$ be the uniformity induced by the pseudometrics $\hat{d}_p$, $p \in cs(E)$. For the map

$$m : \mathcal{R} \to E,$$

we have $p(m(A) - m(B)) \leq \hat{d}_p(A, B)$. Thus $m$ is $\mathcal{U}_m^\sigma$-uniformly continuous and hence there exists a unique uniformly continuous extension

$$m^{\sigma} : \hat{\mathcal{R}}_m \to E,$$

where $\hat{\mathcal{R}}_m$ is the closure of $\mathcal{R}$ in $P(X)$ with respect to the topology induced by $\mathcal{U}_m^\sigma$. \qed

**Lemma 4.2.** $\hat{\mathcal{R}}_m$ is a separating algebra of subsets of $X$ and $m^{\sigma} \in M(\hat{\mathcal{R}}_m, E)$. 

588
Proof. Let $A, B \in \hat{R}_m$, $p \in cs(E)$ and $\epsilon > 0$. There are $V_1, V_2$ in $\mathcal{R}$ such that $\hat{m}_p(A \Delta V_1) < \epsilon$, $\hat{m}_p(B \Delta V_2) < \epsilon$. If $V = V_1 \cup V_2$ and $W = V_1 \cap V_2$, then

$$(A \cup B) \Delta V \subset (A \Delta V_1) \cup (B \Delta V_2), \quad (A \cap B) \Delta W \subset (A \Delta V_1) \cup (B \Delta V_2)$$

and $A^c \Delta V_1^c = A \Delta V_1$. Hence

$$\hat{m}_p((A \cup B) \Delta V) < \epsilon, \quad \hat{m}_p((A \cap B) \Delta W) < \epsilon, \quad \hat{m}_p(A^c \Delta V_1^c) < \epsilon,$$

which proves that the sets $A \cup B$, $A \cap B$ and $A^c$ are in $\hat{R}_m$. Also $A \setminus B = A \cap B^c \in \hat{R}_m$ and so $\hat{R}_m$ is an algebra. Since

$$m'^\alpha(\hat{R}_m) \subset m(\mathcal{R}).$$

it follows that $m'^\alpha$ is bounded. Finally we need to show that $m'^\alpha$ is finitely additive. To this end, we consider the set

$$\Delta = \{(p, n): p \in cs(E), n \in \mathbb{N}\}$$

and make $\Delta$ into a directed set by defining $(p_1, n_1) \triangleright (p_2, n_2)$ iff $p_1 \triangleright p_2$ and $n_1 \triangleright n_2$. Let now $A, B$ be disjoint members of $\hat{R}_m$. For each $\delta = (p, n)$ in $\Delta$, there are $V_\delta$, $W_\delta$ in $\mathcal{R}$ such that

$$\hat{m}_p(A \Delta V_\delta) < 1/n, \quad \hat{m}_p(B \Delta W_\delta) < 1/n.$$

Now the nets $(V_\delta)$, $(W_\delta)$ converge to $A$, $B$, respectively, with respect to the uniformity $U^g_m$. If $Z_\delta = W_\delta \setminus V_\delta$, then $B \Delta Z_\delta \subset (A \Delta V_\delta) \cup (B \Delta W_\delta)$, which implies that $Z_\delta \rightarrow B$. If $F_\delta = V_\delta \cup Z_\delta$ and $D = A \cup B$, then

$$D \Delta F_\delta \subset (A \Delta V_\delta) \cup (B \Delta Z_\delta)$$

and hence $F_\delta \rightarrow D$. Thus

$$m'^\alpha(D) = \lim m(V_\delta \cup Z_\delta) = \lim[m(V_\delta) + m(Z_\delta)]$$

$$= \lim m(V_\delta) + \lim m(Z_\delta) = m'^\alpha(A) + m'^\alpha(B).$$

This completes the proof. \qed

Lemma 4.3.

(1) For $A, B \subset X$, we have

$$|\hat{m}_p(A) - \hat{m}_p(B)| \leq \hat{m}_p(A \Delta B).$$

(2) If $A, B \in \hat{R}_m$, then

$$|m'^\alpha_p(A) - m'^\alpha_p(B)| \leq m'^\alpha_p(A \Delta B).$$
Proof. (1) Suppose (say) that
\[ \hat{m}_p(A) - \hat{m}_p(B) > \hat{m}_p(A \triangle B). \]
Since \( A = (A \cap B) \cup (A \setminus B) \), we have
\[ \hat{m}_p(A) = \max\{\hat{m}_p(A \cap B), \hat{m}_p(A \setminus B)\} = \hat{m}_p(A \cap B) \leq \hat{m}_p(B), \]
a contradiction.

(2) The proof is analogous to that of (1).  \( \square \)

Lemma 4.4.

1. For \( G \in \mathcal{R} \), we have \( m_p^\alpha(G) = m_p(G) \).
2. If \( A \in \hat{\mathcal{R}}_m \), then \( m_p^\alpha(A) = \hat{m}_p(A) \).

Proof. 1. It is clear that \( m_p(G) \leq m_p^\alpha(G) \). On the other hand, let \( B \in \hat{\mathcal{R}}_m \) be contained in \( G \). There exists a net \((V_\delta)\) in \( \mathcal{R} \) converging to \( B \) in the uniformity \( \mathcal{U}_m^\alpha \). Then \( V_\delta \cap G \to B \cap G = B \). Thus
\[ p(m^\alpha(B)) = \lim p(m^\alpha(V_\delta \cap G)) = \lim p(V_\delta \cap G) \leq m_p(G), \]
which proves that \( m_p^\alpha(G) \leq m_p(G) \).

2. Let \( B \in \hat{\mathcal{R}}_m \), \( B \subset A \). There exists a net \((W_\delta)\) in \( \mathcal{R} \) converging to \( B \). But then \( W_\delta \cap A \to B \cap A = B \). Thus
\[ p(m^\alpha(B)) = \lim p(m^\alpha(W_\delta \cap A)). \]

Since
\[ p(m^\alpha(W_\delta \cap A)) \leq m_p^\alpha(W_\delta) = \hat{m}_p(W_\delta) \]
and
\[ \hat{m}_p(W_\delta) \to \hat{m}_p(B) \leq \hat{m}_p(A) \]
(by the preceding lemma), we get that \( m_p^\alpha(A) \leq \hat{m}_p(A) \). If \((V_\delta)\) is a net in \( \mathcal{R} \) which converges to \( A \) in the uniformity \( \mathcal{U}_m^\alpha \), then
\[ |\hat{m}_p(V_\delta) - m_p^\alpha(A)| = |m_p^\alpha(V_\delta) - m_p^\alpha(A)| \leq m_p^\alpha(A \triangle V_\delta) \leq \hat{m}_p(A \triangle V_\delta) \to 0 \]
and so \( \hat{m}_p(V_\delta) \to m_p^\alpha(A) \). Also \( \hat{m}_p(V_\delta) \to \hat{m}_p(A) \), by the preceding lemma, and hence \( m_p^\alpha(A) = \hat{m}_p(A) \).  \( \square \)

Lemma 4.5. \( N_{m,p} = N_{m^\alpha,p} \).
Proof. Suppose that, for some \( x \in X \), we have \( N_{m,p}(x) > \alpha > N_{m,\sigma,p}(x) \). There exists \( A \in \mathcal{R}_m \), containing \( x \), such that \( m_p^{\sigma}(A) < \alpha \). Let \( V \in \mathcal{R} \) be such that \( m_p^{\sigma}(A \Delta V) = \hat{m}_p(A \Delta V) < \alpha \). There is a sequence \( (G_k) \) in \( \mathcal{R} \), with \( A \Delta V \subset \bigcup G_k \), such that \( m_p(G_k) < \alpha \) for all \( k \). As \( N_{m,p}(x) > \alpha \), we have that \( x \notin \bigcup G_k \) and so \( x \notin A \Delta V \), which implies that \( x \notin V \). Moreover, \( V \subset A \cup (V \setminus A) \) and thus

\[
N_{m,p}(x) = m_p(V) = m_p^{\sigma}(V) \leq \max\{ m_p^{\sigma}(A), m_p^{\sigma}(V \setminus A) \} < \alpha,
\]
a contradiction. \( \square \)

Lemma 4.6. \( \mathcal{R}_m \) is a \( \sigma \)-algebra and \( m^{\sigma} \) is \( \sigma \)-additive.

Proof. We prove first that \( m^{\sigma} \) is strongly additive. Indeed, let \( (A_n) \) be a sequence of pairwise disjoint members of \( \mathcal{R}_m \), \( p \in \text{cs}(E) \) and \( \epsilon > 0 \). For each \( n \), there exists \( V_n \in \mathcal{R} \) with \( m_p^{\sigma}(A_n \Delta V_n) < \epsilon \). Let \( W_1 = V_1, \; W_{n+1} = V_{n+1} \setminus \bigcup_{k=1}^{n} V_k \). Then

\[
A_{n+1} \Delta W_{n+1} \subset \bigcup_{k=1}^{n+1} A_k \Delta V_k
\]

and so \( m_p^{\sigma}(A_{n+1} \Delta W_{n+1}) < \epsilon \). The sets \( W_n, \; n = 1, 2, \ldots \), are pairwise disjoint. Since \( m \) is strongly additive, there exists \( n_0 \) such that \( m_p(W_n) < \epsilon \) for \( n \geq n_0 \). Now, for \( n \geq n_0 \), we have \( A_n = (A_n \cap W_n) \cup (A_n \setminus W_n) \) and hence

\[
m_p^{\sigma}(A_n) = \max\{ m_p^{\sigma}(A_n \cap W_n), m_p^{\sigma}(A_n \setminus W_n) \}
\]

\[
\leq \max\{ m_p^{\sigma}(W_n), \hat{m}_p(A_n \Delta W_n) \}
\]

\[
= \max\{ m_p(W_n), \hat{m}_p(A_n \Delta W_n) \} < \epsilon.
\]

This proves that \( m^{\sigma} \) is strongly additive. Next we show that \( \mathcal{R}_m \) is a \( \sigma \)-algebra. In fact, let \( (A_n) \) be a sequence in \( \mathcal{R}_m \) and \( A = \bigcup A_n \). We need to show that \( A \in \mathcal{R}_m \). We may assume that the sets \( A_n \) are pairwise disjoint. Let \( p \in \text{cs}(E) \) and \( \epsilon > 0 \). For each \( n \) there exists \( V_n \in \mathcal{R} \) such that \( \hat{m}_p(A_n \Delta V_n) < \epsilon \). Let \( W_1 = V_1, \; W_{n+1} = V_{n+1} \setminus \bigcup_{k=1}^{n} V_k \). Then \( \hat{m}_p(A_{n+1} \Delta W_{n+1}) < \epsilon \). Since the sets \( W_n \) are pairwise disjoint, there exists \( n \) such that \( m_p(W_n) < \epsilon \) for all \( k > n \). Now

\[
A \Delta \left( \bigcup_{k=1}^{n} W_k \right) \subset \left[ A \Delta \left( \bigcup_{k=1}^{\infty} W_k \right) \right] \cup \left[ \bigcup_{k>n} W_k \right].
\]

For each \( k \) there is a sequence \( (B_{ki})_{i=1}^{\infty} \) in \( \mathcal{R} \), such that \( A_k \Delta W_k \subset \bigcup_{i} B_{ki} \) and \( m_p(B_{ki}) < \epsilon \). If \( G = A \Delta (\bigcup_{k=1}^{\infty} W_k) \), then

\[
G \subset \bigcup_{k} A_k \Delta W_k \subset \bigcup_{k,i} B_{ki}
\]

and hence \( \hat{m}_p(G) \leq \sup_{k,i} m_p(B_{ki}) \leq \epsilon \). Also, for \( F = \bigcup_{k>n} W_k \), we have

\[
\hat{m}_p(F) \leq \sup_{k>n} m_p(W_k) \leq \epsilon.
\]
Thus

$$\hat{m}_p \left( A \Delta \left( \bigcup_{k=1}^{n} W_k \right) \right) \leq \max\{\hat{m}_p(F), \hat{m}_p(G)\} \leq \epsilon,$$

which proves that $A \in \mathcal{R}_m$.

Finally, $m^\sigma$ is $\sigma$-additive. In fact, let $(A_n) \subset \mathcal{R}_m$, $A_n \downarrow \emptyset$. Let $B_n = A_n \setminus A_{n+1}$. The sets $B_n$ are pairwise disjoint. Since $m^\sigma$ is strongly additive, there exists $N$ such that $m^\sigma_p(B_n) = \hat{m}_p(B_n) < \epsilon$ for all $n \geq N$. Let $n \geq N$. Then $A_n = \bigcup_{k \geq n} B_k$. For each $k \geq n$, there is a sequence $(G_{ki})_{i=1}^{\infty}$ in $\mathcal{R}$ such that $B_k \subset \bigcup_i B_{ki}$ and $m_p(B_{ki}) < \epsilon$. Now $A_n \subset \bigcup_{k \geq n} \bigcup_i B_{ki}$ and so

$$m^\sigma_p(A_n) \leq \sup_{k \geq n} \sup_i m_p(B_{ki}) \leq \epsilon.$$ 

This proves that $m_p^\sigma(A_n) \to 0$ and so $m^\sigma$ is $\sigma$-additive. This completes the proof. □

Combining the preceding lemmas, we get the following extension.

**Theorem 4.7.** Let $m \in M_0(\mathcal{R}, E)$ be strongly additive. If $\mathcal{R}^\sigma$ is the $\sigma$-algebra generated by $\mathcal{R}$, then there exists a unique extension $m^\sigma \in M_0(\mathcal{R}^\sigma, E)$ of $m$. Moreover, $N_m.p = N_{m^\sigma}.p$.

**Proof.** Since $\mathcal{R}_m$ is a $\sigma$-algebra, it follows that $\mathcal{R}^\sigma$ is contained in $\mathcal{R}_m$. Thus the restriction $m^\sigma$ of $m^\sigma$ to $\mathcal{R}^\sigma$ is a $\sigma$-additive extension of $m$. To prove the uniqueness, let $\mu \in M_0(\mathcal{R}^\sigma, E)$ be an extension of $m$ and let

$$\mathcal{F} = \{A \in \mathcal{R}^\sigma: \mu(A) = m^\sigma(A)\}.$$ 

It is easy to see that, if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$. The family $\mathcal{F}$ is a monotone class. Indeed, let $(A_n)$ be a sequence in $\mathcal{F}$ with $A_n \downarrow A$. Then

$$\mu(A) = \lim \mu(A_n) = \lim m^\sigma(A_n) = m^\sigma(A).$$

Similarly, if $(B_n) \subset \mathcal{F}$ and $B_n \uparrow B$, then $m^\sigma(B) = \mu(B)$. Hence $\mathcal{F}$ is a monotone class. Since $\mathcal{R}^\sigma$ is the monotone class generated by $\mathcal{R}$ (by [6], Theorem B on p. 27), it follows that $\mathcal{F} = \mathcal{R}^\sigma$ and so $\mu = m^\sigma$. The equality $N_{m,p} = N_{m^\sigma,p}$ is a consequence of Lemma 4.5. □

**Theorem 4.8.** For $A \in \mathcal{R}^\sigma$ and $p \in cs(E)$, we have $m^\sigma_p(A) = m_p^\sigma(A)$.

**Proof.** It is clear that $m^\sigma_p(A) \leq m^\sigma_p(A)$. On the other hand, let $B \in \mathcal{R}_m$, $B \subset A$. There exists a net $(V_\delta)$ in $\mathcal{R}$ converging to $B$ with respect to the uniformity $\mathcal{U}_m$. But then $V_\delta \cap A \to B \cap A = B$. Hence

$$p(m^\sigma(B)) = \lim_\delta p(m^\sigma(A \cap V_\delta)) = \lim_\delta p(m^\sigma(A \cap V_\delta)).$$

592
Since \( p(m^\sigma (A \cap V_k)) \leq m^\sigma_p (A) \), it follows that \( p(m^\sigma (B)) \leq m^\sigma_p (A) \), which proves that \( m^\sigma_p (A) \geq m^\sigma_p (A) \). This completes the proof. \( \square \)

**Theorem 4.9.** Let \( m \in M_\sigma (R, E) \) and \( \mu \in M_\sigma (R) \) both be strongly additive and suppose that \( m \ll \mu \). Then \( m^\sigma \ll \mu^\sigma \).

**Proof.** Let \( p \in cs (E) \) and \( \epsilon > 0 \). There exists a \( \delta > 0 \) such that, for \( A \in R \), if \( |\mu|(A) < \delta \), then \( m_p (A) < \epsilon \). Assume now \( A \in R^\sigma \) and \( |\mu^\sigma|(A) < \delta \). There is a sequence \( (V_n) \) in \( R \) such that \( A \subset \bigcup V_n \) and \( |\mu|(V_n) < \delta \) for every \( n \). But then \( m^\sigma_p (V_n) = m_p (V_n) < \epsilon \), for every \( n \), and so

\[
m^\sigma_p (A) \leq \sup_n m^\sigma_p (V_n) \leq \epsilon.
\]

This clearly completes the proof. \( \square \)

**Theorem 4.10.** \( \hat{R}_m = R^\sigma_m \) and \( m^\sigma = \overline{m}^\sigma \).

**Proof.** Let \( A \in \hat{R}_m \). Given \( p \in cs (E) \) and \( \epsilon > 0 \), there exists \( V \in R \) such that \( m_p (A \Delta V) < \epsilon \). Next, there is a sequence \( (G_n) \) in \( R \) such that \( A \Delta V \subset G = \bigcup G_n \) and \( m_p (G_n) < \epsilon \) for all \( n \). Then \( G \in R^\sigma \). If \( B = V \cap G^c \) and \( F = V \cup G \), then \( B \subset A \subset F \) and \( F \setminus B = G \). Moreover,

\[
m^\sigma_p (G) = \sup_n m^\sigma_p (G_n) = \sup_n m_p (G_n) \leq \epsilon.
\]

This proves that \( A \in R^\sigma_m \). Moreover, if \( A_1 = B \), \( A_2 = G \) and \( A_3 = F^c \), then \( \{A_1, A_2, A_3\} \) is an \( R^\sigma \)-partition of \( X \) and, for \( f = \chi_A \), we have that

\[
|f(x) - f(y)| \cdot m^\sigma_p (A_k) \leq \epsilon,
\]

if \( x, y \in A_k \). If \( x_k \in A_k \), then

\[
\epsilon \geq p \left( \int f \, dm^\sigma - \sum_{k=1}^{3} f(x_k) m^\sigma(A_k) \right) = p \left( \int f \, dm^\sigma - m^\sigma (B) - f(x_2) m^\sigma (G) \right).
\]

But

\[
p(m^\sigma (A) - m^\sigma (B)) = p(m^\sigma (A \setminus B)) \leq m^\sigma_p (A \setminus B) \leq m^\sigma_p (F \setminus B) = m^\sigma_p (G) \leq \epsilon
\]

and \( p(m^\sigma (G)) \leq m^\sigma_p (G) \leq \epsilon \). Thus

\[
p \left( \int f \, dm^\sigma - m^\sigma (A) \right) \leq \epsilon.
\]
It follows that
\[ m^\alpha(A) = \int f \, dm^\alpha = \overline{m}^\sigma(A). \]

Conversely, let \( A \in \mathcal{R}_m^\sigma \). There are \( B, F \in \mathcal{R}^\sigma \), with \( B \subset A \subset F \) and \( m^\sigma_p(F \setminus B) < \epsilon \).

Now \( A \Delta B \subset F \setminus B \) and
\[ \hat{m}_p(A \Delta B) \leq \hat{m}_p(F \setminus B) = m^\sigma_p(F \setminus B) < \epsilon, \]
which proves that \( A \) is in the closure of \( \mathcal{R}^\sigma \) with respect to the uniformity \( \mathcal{U}_m^\sigma \) and hence \( A \in \hat{\mathcal{R}}_m \). This completes the proof. \( \square \)

5. EXTENSIONS OF \( \tau \)-ADDITIVE MEASURES

Note. Throughout this section, unless it is stated explicitly otherwise, \( m \) will be an element of \( M_\tau(\mathcal{R}, E) \) which is strongly additive.

For \( p \in cs(E) \), we define
\[ \check{m}_p : P(X) \to \mathbb{R}, \quad \check{m}_p(A) = \inf \sup m_p(G_i), \]
where the infimum is taken over the collection of all families \( (G_i)_{i \in I} \) of members of \( \mathcal{R} \) which cover \( A \). It is easy to show that
\[ \check{m}_p(A \cup B) = \max\{\check{m}_p(A), \check{m}_p(B)\}. \]

Let
\[ \check{d}_p : P(X) \times P(X) \to \mathbb{R}, \quad \check{d}_p(A, B) = \check{m}_p(A \Delta B). \]

Then \( \check{d}_p \) is an ultrapseudometric on \( P(X) \) and \( \check{d}_p \leq \hat{d}_p \). Let \( \mathcal{U}_m^\tau \) be the uniformity induced by the pseudometrics \( \check{d}_p \), \( p \in cs(E) \). Then \( \mathcal{U}_m^\tau \) is coarser than \( \mathcal{U}_m^\sigma \). If \( \hat{\mathcal{R}}_m \) is the closure of \( \mathcal{R} \) in \( P(X) \) with respect to the topology induced by \( \mathcal{R}_m^\sigma \), then \( \hat{\mathcal{R}}_m^\sigma \subset \hat{\mathcal{R}}_m \).

Lemma 5.1. For \( A \in \mathcal{R} \), we have that \( m_p(A) = \check{m}_p(A) \).

Proof. Clearly \( m_p(A) \geq \check{m}_p(A) \). On the other hand, if \( (G_i) \) is a family of members of \( \mathcal{R} \) covering \( A \), then \( m_p(A) \leq \sup_i m_p(G_i) \), since \( m \) is \( \tau \)-additive, which implies that \( m_p(A) \leq \check{m}_p(A) \). \( \square \)

Now for the map \( m : \mathcal{R} \to E \), we have
\[ p(m(A) - m(B)) \leq m_p(A \Delta B) = \check{m}_p(A \Delta B). \]

Thus \( m \) is uniformly continuous for the uniformity induced on \( \mathcal{R} \) by \( \mathcal{U}_m^\tau \). Hence, there exists a unique uniformly continuous extension \( m^\tau : \hat{\mathcal{R}}_m \to E \).

594
The proofs of the following two lemmas are analogous to the ones of Lemmas 4.2 and 4.3, respectively.

**Lemma 5.2.** \( \tilde{\mathcal{R}}_m \) is an algebra of subsets of \( X \) and \( m' \in M(\mathcal{R}_m, E) \).

**Lemma 5.3.**

1. For \( A, B \) subsets of \( X \), we have
   \[
   |\tilde{m}_p(A) - \tilde{m}_p(B)| \leq m_p(A \Delta B).
   \]

2. If \( A, B \in \mathcal{R}_m \), then
   \[
   |m'_p(A) - m'_p(B)| \leq m'_p(A \Delta B).
   \]

**Lemma 5.4.**

1. For \( A \in \mathcal{R} \), we have \( m_p(A) = \tilde{m}_p(A) \).
2. If \( A \in \mathcal{R}_m \), then \( \tilde{m}_p(A) = m'_p(A) \).

**Proof.**

1. Let \( A \in \mathcal{R} \). By Lemma 5.1 we have \( m_p(A) = \tilde{m}_p(A) \). Clearly \( m'_p(A) \geq m_p(A) \). On the other hand, if \( B \in \tilde{\mathcal{R}}_m \) is contained in \( A \), then there exists a net \( (V_B) \) in \( \mathcal{R} \) which converges to \( B \) for the uniformity \( U_m \). But then \( V_B \cap A \rightarrow B \cap A = B \) and so
   \[
   p(m'(B)) = \lim p(V_B \cap A) \leq m_p(A).
   \]
   This proves that \( m'_p(A) \leq m_p(A) \).

2. Let \( A \in \tilde{\mathcal{R}}_m \). First we show that \( m'_p(A) \leq \tilde{m}_p(A) \). Indeed, let \( B \in \tilde{\mathcal{R}}_m \) be contained in \( A \). There exists a net \( (W_B) \) in \( \mathcal{R} \) converging to \( B \) for the uniformity \( U_m \). Then \( W_B \cap A \rightarrow B \cap A = B \) and so
   \[
   p(m'(B)) = \lim p(m'(W_B \cap A)).
   \]
   But
   \[
   p(m'(W_B \cap A)) \leq m'_p(W_B) = \tilde{m}_p(W_B) \rightarrow \tilde{m}_p(B) \leq \tilde{m}_p(A)
   \]
   and hence \( p(m'(B)) \leq \tilde{m}_p(A) \), which proves that \( \tilde{m}_p(A) \geq m'_p(A) \). Since \( A \in \tilde{\mathcal{R}}_m \), there exists a net \( (V_B) \) in \( \mathcal{R} \) converging to \( A \). Then
   \[
   |\tilde{m}_p(V_B) - m'_p(A)| = |m'_p(V_B) - m'_p(A)|
   \leq m'_p(V_B \Delta A) \leq \tilde{m}_p(V_B \Delta A) \rightarrow 0,
   \]
   which implies that \( m'_p(A) = \lim \tilde{m}_p(V_B) \). Also, by the preceding Lemma, \( \tilde{m}_p(V_B) \rightarrow \tilde{m}_p(A) \). Hence \( \tilde{m}_p(A) = m'_p(A) \). This completes the proof. \( \Box \)
The proof of the next lemma is analogous to the one of Lemma 4.5.

**Lemma 5.5.** $N_{m,p} = N_{m'^*,p}$. 

**Lemma 5.6.** $\tilde{\mathcal{R}}_m$ is a $\sigma$-algebra which contains the $\sigma$-algebra $\mathcal{R}^{bo}$ of all $\tau_\mathcal{R}$-Borel sets. Moreover, $m'^*$ is $\tau$-additive.

**Proof.**

**Claim I.** For each family $(A_i)_{i \in I}$ of subsets of $X$ and $A = \bigcup A_i$, we have $\tilde{m}_p(A) = \sup_i \tilde{m}_p(A_i) = d$.

In fact, let $\alpha > d$. For each $i$, there exists a family $F_i$ of members of $\mathcal{R}$ such that $A_i \subset \bigcup F_i$ and $m_p(B) < \alpha$ for every $B \in F_i$. If $F = \bigcup_i F_i$, then $A \subset \bigcup F$ and $m_p(B) < \alpha$, for each $B \in F$, which implies that $\tilde{m}_p(A) \leq \alpha$. It follows that $\tilde{m}_p(A) \leq d \leq \tilde{m}_p(A)$.

**Claim II.** $m'^*$ is strongly additive.

Indeed, let $(A_n)$ be a sequence of pairwise disjoint members of $\tilde{\mathcal{R}}_m$ and let $p \in cs(E)$ and $\epsilon > 0$. For each $n$, there exists $V_n \in \mathcal{R}$ such that $\tilde{m}_p(A_n \Delta V_n) < \epsilon$. Let $W_1 = V_1$ and $W_{n+1} = V_{n+1} \setminus \bigcup_{k=1}^n V_k$. Then $\tilde{m}_p(A_{n+1} \Delta W_{n+1}) < \epsilon$. The sets $W_n$ are pairwise disjoint. Since $m$ is strongly additive, there exists $n_o$ such that $m_p(W_n) < \epsilon$ for all $n \geq n_o$. Now, for $n \geq n_o$, we have $A_n = (A_n \cap W_n) \cup (A_n \setminus W_n)$ and hence

$$m'^*(A_n) \leq \max\{m'^*(A_n \cap W_n), m'^*(A_n \setminus W_n)\}$$

$$\leq \max\{m'_p(W_n), m'_p(A_n \Delta W_n)\} < \epsilon,$$

and the claim follows.

**Claim III.** $\tilde{\mathcal{R}}_m$ is a $\sigma$-algebra.

In fact, let $(A_n)$ be a sequence in $\tilde{\mathcal{R}}_m$ and $A = \bigcup A_n$. We may assume that the sets $A_n$ are pairwise disjoint. Given $p \in cs(E)$ and $\epsilon > 0$, there is a sequence $(W_n)$ of pairwise disjoint members of $\mathcal{R}$ such that $\tilde{m}_p(A_n \Delta W_n) < \epsilon$ for every $n$. As the sets $W_n$ are pairwise disjoint, there exists $n$ such that $m_p(W_k) < \epsilon$ for all $k > n$. Now

$$A \Delta \left( \bigcup_{k=1}^n W_k \right) \subset A \Delta \left( \bigcup_{k=1}^\infty W_k \right) \cup \left[ \bigcup_{k>n} W_k \right],$$

$$\subset \left[ \bigcup_{k=1}^\infty A_k \Delta W_k \right] \cup \left[ \bigcup_{k>n} W_k \right].$$

In view of Claim I, we get that $\tilde{m}_p(A \Delta (\bigcup_{k=1}^n W_k)) \leq \epsilon$. This proves that $A \in \tilde{\mathcal{R}}_m$. 596
Claim IV. Every $\tau_\mathcal{R}$-closed set is in $\mathcal{R}_m$ and so $\mathcal{R}^{bo}$ is contained in $\mathcal{R}_m$.

Indeed, if $A$ is $\tau_\mathcal{R}$-closed, then the exists a decreasing net $(V_\delta)$ in $\mathcal{R}$ with $A = \bigcap V_\delta$. Since $m$ is strongly additive, there exists $\delta_0$ such that $m_p(V_{\delta_0} \setminus V_\delta) \leq \epsilon$ for all $\delta \geq \delta_0$. Since $A^c = \bigcup_{\delta \geq \delta_0} V_\delta^c$, we get that

\[
A^c \cap V_{\delta_0} = \bigcup_{\delta \geq \delta_0} V_{\delta_0} \setminus V_\delta \quad \text{and so} \quad m_p(A^c \cap V_{\delta_0}) \leq \epsilon.
\]

But $A^c \cap V_{\delta_0} = A \triangle V_{\delta_0}$. This proves that $A \in \mathcal{R}_m$ and the claim follows.

Claim V. $m^{\tau'}$ is $\tau$-additive.

In fact, let $(A_\delta)$ be a net in $\mathcal{R}_m$ with $A_\delta \downarrow \emptyset$. Since $m^{\tau'}$ is strongly additive, given $p \in \mathcal{C}s(E)$ and $\epsilon > 0$, there exists $\delta_0$ such that $m^{\tau'}_p(A_{\delta_0} \setminus A_\delta) < \epsilon$ for all $\delta \geq \delta_0$. As $A_{\delta_0} = \bigcup_{\delta \geq \delta_0} A_{\delta_0} \setminus A_{\delta}$, we get that

\[
m^{\tau'}_p(A_{\delta_0}) = m_p(A_{\delta_0}) = \sup_{\delta \geq \delta_0} m_p(A_{\delta_0} \setminus A_\delta) = \sup_{\delta \geq \delta_0} m^{\tau'}_p(A_{\delta_0} \setminus A_\delta) \leq \epsilon.
\]

Hence $\lim m^{\tau'}_p(A_\delta) = 0$ and so $m^{\tau'}$ is $\tau$-additive. ☐

Theorem 5.7. If $m^{\tau}$ is the restriction of $m^{\tau'}$ to $\mathcal{R}^{bo}$, then $m^{\tau} \in M_\tau(\mathcal{R}^{bo}, E)$ is the unique $\tau$-additive extension of $m$ to $\mathcal{R}^{bo}$. Moreover, $m^{\tau}|_{\mathcal{R}^{bo}} = m^{\sigma}$.

Proof. Assume that $\mu \in M_\tau(\mathcal{R}^{bo}, E)$ is an extension of $m$. We first show that $\mu(A) = m^{\tau}(A)$ for each $\tau_\mathcal{R}$-closed set $A$. Indeed, there exists a decreasing net $(V_\delta)$ in $\mathcal{R}$ with $A = \bigcap V_\delta$. Let $B_\delta = A^c \cap V_\delta$. Then $B_\delta \downarrow \emptyset$ and so $m^{\tau}(B_\delta) \rightarrow 0$ and $\mu(B_\delta) \rightarrow 0$. Since $V_\delta = A \cup B_\delta$, we have that

\[
\mu(A) - m^{\tau}(A) = m^{\tau}(B_\delta) - \mu(B_\delta) \rightarrow 0,
\]

and hence $\mu(A) = m^{\tau}(A)$. Also $\mu(B) = m^{\tau}(B)$ for each $\tau_\mathcal{R}$-open set $B$ since $\mu(X) = m^{\tau}(X)$. For $A$ a $\tau_\mathcal{R}$-open set and $B$ a $\tau_\mathcal{R}$-closed set, we have that

\[
\mu(A \cap B) = \mu(A) - \mu(A \cap B^c) = m^{\tau}(A) - m^{\tau}(A \cap B^c) = m^{\tau}(A \cap B).
\]

It is easy to show that the family $\mathcal{F}$ of all finite unions of sets of the form $A \cap B$, where $A$ is $\tau_\mathcal{R}$-open and $B$ $\tau_\mathcal{R}$-closed, is an algebra. Moreover, every member of $\mathcal{F}$ is a finite union of pairwise disjoint members of $\mathcal{F}$. Thus $\mu(G) = m^{\tau'}(G)$ for every member $G$ of $\mathcal{F}$. It is clear that $\mathcal{R}^{bo}$ coincides with the $\sigma$-algebra generated by $\mathcal{F}$. As $\mathcal{F}$ is an algebra, $\mathcal{R}^{bo}$ coincides with the monotone class generated by $\mathcal{F}$ (by Halmos [6], Theorem B on p. 27). The class $\mathcal{F}_1$, of all members $A$ of $\mathcal{R}^{bo}$ for which $\mu(A) = m^{\tau}(A)$, is monotone. It follows that $\mu = m^{\tau}$ on $\mathcal{R}^{bo}$. Finally, if $m_1 = m^{\tau}|_{\mathcal{R}^{bo}}$, then $m_1$ is a $\sigma$-additive extension of $m$ and thus $m_1 = m^{\sigma}$ by the uniqueness part of Theorem 4.7. This completes the proof. ☐

The proof of the following theorem is analogous to the one of Theorem 4.8.
Theorem 5.8. For $A \in \mathcal{R}^{bo}$ and $p \in cs(E)$, we have $m^r_p(A) = m^\ast_p(A)$.

Theorem 5.9. For $A \in \mathcal{R}^\sigma$ and $p \in cs(E)$, we have $m^\sigma_p(A) = m^r_p(A)$.

Proof. There exists a net $(V_\delta)$ in $\mathcal{R}$ which converges to $A$ with respect to the uniformity $\mathcal{U}_m^\sigma$. Since $\mathcal{U}_m^\sigma$ is coarser than $\mathcal{U}_m^r$, $(V_\delta)$ converges to $A$ with respect to $\mathcal{U}_m^r$. Now

$$m^\sigma_p(V_\delta) \to m^\sigma_p(A) \quad \text{and} \quad m^r_p(V_\delta) \to m^r_p(A).$$

Since $m^\sigma_p(V_\delta) = m^r_p(V_\delta) = m_p(V_\delta)$, the theorem follows. □

Theorem 5.10. $\mathcal{R}_m \supseteq \mathcal{R}^{bo}$ and $m^\tau = \overline{m}^r$.

Proof. Let $A \in \mathcal{R}_m$. Given $p \in cs(E)$ and $\epsilon > 0$, there exists $V \in \mathcal{R}$ with $\tilde{m}_p(A \Delta V) < \epsilon$. Let $(G_i)$ be a family in $\mathcal{R}$ such that $A \Delta V \subseteq G = \bigcup G_i$ and $m_p(G_i) < \epsilon$ for every $i$. If $B = V \cap G^c$ and $F = V \cup G$, then $B \subseteq A \subseteq F$. Moreover, $F \setminus B = G$ and

$$m^r_p(G) = \sup_i m^r_p(G_i) = \sup_i m_p(G_i) \leq \epsilon.$$ 

This proves that $A \in \mathcal{R}^{bo}_m$. Moreover, if $B_1 = B, B_2 = G$ and $B_3 = F^c$, then $\{B_1, B_2, B_3\}$ is a $\mathcal{R}^{bo}$-partition of $X$ and, for $f = \chi_A$, we have

$$|f(x) - f(y)| \cdot m^r_p(B_k) \leq \epsilon$$

if $x, y \in B_k$. If $x_k \in B_k$, then

$$\epsilon \geq p \left( \int f \, dm^r - \sum_{k=1}^3 f(x_k)m^r(B_k) \right)$$

$$= p(\overline{m}^r(A) - m^r(B) - f(x_2)m^r(G)).$$

But

$$p(m^\tau(A) - m^r(B)) = p(m^\tau(A \setminus B)) \leq \tilde{m}_p(G) = m^r_p(G) \leq \epsilon$$

and $p(m^r(G)) \leq m^r_p(G) \leq \epsilon$. Thus

$$p(\overline{m}^r(A) - m^\tau(A)) \leq \epsilon.$$ 

This proves that $\overline{m}^r(A) = m^\tau(A)$.

Conversely, let $A \in \mathcal{R}^{bo}_m$. Then there are $B, F \in \mathcal{R}^{bo}$ such that $B \subseteq A \subseteq F$ and $m^r_p(F \setminus B) < \epsilon$. Now $A \Delta B \subseteq F \setminus B$ and

$$\tilde{m}_p(A \Delta B) \leq \tilde{m}_p(F \setminus B) = m^r_p(F \setminus B) < \epsilon,$$

598
which proves that $A$ is in the closure of $\mathcal{R}^{bo}$ in $P(X)$ with respect to the uniformity $U^*_m$. Hence $A \in \mathcal{R}_m$. This completes the proof. \(\Box\)

The proof of the following theorem is analogous to the one of Theorem 4.9.

**Theorem 5.11.** Let $m \in M_+(\mathcal{R}, E)$ and $\mu \in M_+(\mathcal{R})$ be both strongly additive. If $m \ll \mu$, then $m^r \ll \mu^r$.

**Theorem 5.12.** Let $m \in M(\mathcal{R}, E)$ be strongly additive and let $f \in \mathcal{K}^X$ be $m$-integrable. Then:

1. If $m$ is $\sigma$-additive, then $f$ is $m^\sigma$-integrable and $\int f \, dm = \int f \, dm^\sigma$.
2. If $m$ is $\tau$-additive, then $f$ is $m^\tau$-integrable and $\int f \, dm = \int f \, dm^\tau$.

**Proof.** (1) Assume that $m$ is $\sigma$-additive and let $p \in cs(E)$ and $\epsilon > 0$. Since $f$ is $m$-integrable, there exists an $\mathcal{R}$-partition $\{A_1, A_2, \ldots, A_n\}$ of $X$ such that

$$|f(x) - f(y)| \cdot m_p(A_k) < \epsilon$$

if $x, y \in A_k$. Since $m^\sigma_p(A_k) = m_p(A_k)$, it follows that $f$ is $m^\sigma$-integrable.

Moreover, if $x_k \in A_k$, then

$$p\left(\int f \, dm - \sum_{k=1}^{n} f(x_k)m(A_k)\right) \leq \epsilon$$

and

$$p\left(\int f \, dm^\sigma - \sum_{k=1}^{n} f(x_k)m^\sigma(A_k)\right) \leq \epsilon,$$

which implies that $p(\int f \, dm - \int f \, dm^\sigma) \leq \epsilon$. It follows that $\int f \, dm = \int f \, dm^\sigma$ since $E$ is Hausdorff.

(2) The proof is analogous to that of (1). \(\Box\)

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