Refinements of Complexity Results on Type Consistency for Object-Oriented Databases

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Received May 14, 1998; revised February 12, 2001

The method invocation mechanism is one of the essential features in object-oriented programming languages. This mechanism contributes to data encapsulation and code reuse, but there is a risk of a run-time type error. In the case of object-oriented databases (OODBs), a run-time error causes rollback. Therefore, it is desirable to ensure that a given OODB schema is consistent; i.e., no run-time type error occurs during the execution of queries under any database instance of the OODB schema. This paper discusses the computational complexity of the type-consistency problem. As a model of OODB schemas, we adopt update schemas introduced by R. Hull et al., which have all of the basic features of OODBs such as class hierarchy, inheritance, and complex objects. For several subclasses of update schemas, the complexity of the type-consistency problem is presented. Importantly, it turns out that nonflatness of the class hierarchy, recursion in the queries, and update operations in the queries each make the problem difficult.

1. INTRODUCTION

Among many features of object-oriented programming languages (OOPLs), the method invocation (or message passing) mechanism is an essential one. It is based on method name overloading and late binding by method inheritance along the class hierarchy. For a method name \( m \), different classes may have different
definitions (codes, implementations) of \( m \). When \( m \) is applied to an object \( o \), one of the definitions of \( m \) is selected depending on the class to which \( o \) belongs, and is bound to \( m \) at run-time (late binding or dynamic binding). This mechanism is important for data encapsulation and code reuse, but there is a risk of a run-time type error. For example, when a method \( m \) is invoked, the definition of \( m \) to be bound may not exist at run-time. Particularly with queries in object-oriented databases (OODBs), a run-time error causes rollback; i.e., all the modification up to the error must be cancelled.

In this paper, we discuss the computational complexity of the type-consistency problem for queries in OODBs. A database schema \( S \) is said to be consistent if no type error occurs during the execution of any method under any database instance, i.e.,

1. For every method invocation \( m \), the definition of \( m \) to be bound is uniquely determined through the class hierarchy with inheritance; and
2. No attribute-value update violates any type declaration given by \( S \).

In order to check type consistency, it is usually necessary to perform type inference, i.e., to examine whether for each class \( c \) and program construct \( x \) such as a variable in method implementation bodies, the value of \( x \) can be an object of class \( c \). It is quite advantageous for a given database schema to be consistent. First, since it is ensured at compile time that no type error occurs under any database instance, run-time type check can be omitted. Another advantage is an application to method-based authorization checking [5, 7, 16].

As a model of OODB schemas, we adopt update schemas introduced by [11]. Update schemas have all of the basic features of OODBs, such as class hierarchy, inheritance, and complex objects. Method implementations are based on a procedural OOPL model. Therefore, updating database instances is simply modeled as assignment of objects or basic values to attributes of objects. In [11], it is shown that the type-consistency problem for update schemas is undecidable.

The aim of this paper is to investigate the computational complexity of the type-consistency problem for several subclasses of OODB schemas. We focus on the following three factors and show that each of them creates difficulty in the problem (see also Fig. 1):

1. Nonflatness of the class hierarchy (Section 3.1). Define the height of the class hierarchy as the maximum length of a path in the hierarchy. If the height is zero, then all classes are completely separated and there is no superclass-subclass relation at all. For such a “flat” database schema, consistency is solvable in polynomial time. However, consistency for a nonflat schema is undecidable even if it is retrieval (i.e., no method definition in the schema contains any update operation) and the height of the class hierarchy is bounded by 1.

2. Recursion (Section 3.2). Consistency for a recursion-free schema is coNEXPTIME-complete, while consistency for a schema with recursion is undecidable even if it is terminating (i.e., the execution of every method terminates under every database instance) and the height of the class hierarchy is bounded by 1.
3. **Update operations** (Section 3.3). As stated above, consistency for a terminating (resp. recursion-free) schema with update operations is undecidable (resp. coNEXPTIME-complete), even if the height of the class hierarchy is bounded by 1. On the other hand, consistency for a terminating retrieval schema is solvable in polynomial time. Thus, update operations make the consistency problem difficult when the schema satisfies the termination property.

The following three restrictions are placed on the model of OODB schemas in this paper. First, schemas should be monadic (i.e., every method in a schema should be unary). Even if the arity is not bounded, consistency is expected to be still decidable for a flat schema, a recursion-free schema, and a terminating retrieval schema respectively. That is, in our conjecture, arity does not affect the decidability of consistency as long as we consider only the subclasses of schemas stated above. Secondly, there should be no program constructs such as a conditional branch and a while statement. However, using update operations, if-then statements can be simulated (see Example 3). Thirdly, the class hierarchy should be a forest (i.e., multiple inheritance is excluded). However, the results in this paper remain valid if an appropriate mechanism for multiple inheritance is incorporated into the model. That is, the third restriction is merely for simplicity.

There has been much research on the type-consistency problem for OOPLs. As a pioneer work, Abiteboul et al. [2, 3] introduced method schemas and studied the complexity of the type-consistency problem for a number of subclasses. In method

![Complexity of the type-consistency problem.](image)
schemas, each method is allowed to have more than one argument. However, method schemas cannot represent updates of a database instance since their method implementations are based on a functional OOPL model. The following is some of the main results and open problems of [2]:

1. Consistency is undecidable for a general method schema;
2. The complexity of consistency is open for a method schema with methods of arity at most two;
3. Consistency is coNP-complete for a recursion-free method schema provided that the arity of each method is bounded by a constant; and
4. Consistency is solvable in polynomial time for a monadic method schema.

Retrieve schemas of ours are a proper subclass of general method schemas and a proper superclass of monadic method schemas. Moreover, retrieve schemas are incomparable to method schemas with methods of arity at most two, and their intersection is not empty. In this paper, we provide a proof of undecidability for a retrieve schema that belongs to the intersection. That is, the open problem 2 above is shown to be undecidable. For more details, refer to the end of Section 3.1, where we briefly discuss the idea of how to translate a monadic retrieval schema into a method schema of arity at most two. In [17], an optimal incremental algorithm for the consistency checking of a recursion-free method schema is presented. In [1], the complexity of type consistency (and also the expressive power) for both update and method schemas is summarized.

As already stated, type inference is closely related to type consistency. In [14], a type inference algorithm for a procedural OOPL is proposed. The language presented in the article is polyadic and can express recursion and assignments to local variables. It also provides explicit new and if–then–else expressions. For each expression $e$ of a program, a type variable $\langle e \rangle$ that denotes the type of $e$ is introduced, and a sufficient condition for type consistency can be examined by computing the least solution of the equation that denote the relations among these type variables (also see [13, 15]). These articles provide type inference algorithms that check type safety for sufficiently practical languages, whereas this paper focuses on what properties of a language make exact type inference possible or impossible.

Our OOPL model is untyped in the sense that each variable has no type declaration. In contrast, type consistency for typed OOPLs has been discussed in several articles [4, 6, 8]. Since the language is typed in these articles, it can be assume that we know in advance the class to which the returned objects should belong for every method implementation body. Therefore, the consistency problem is simply to determine whether each method satisfies conditions such as covariance and contravariance. Therefore, for typed OOPLs, behavioral analysis of each method implementation body is unnecessary. These articles do not put an assumption to the ability of OOPLs for that reason. Type systems for OOPLs have also been extensively studied [9, 10]. For example, in [10], an elegant type system that relaxes contravariance restriction is proposed. The language presented in the article is polyadic and can express recursion and assignments to local variables, also providing explicit new expressions. In these approaches, an object is basically defined as
a record structure, each field of which represents an attribute (a state component) or a method of the object. The main focus is on providing a record structure with a static type system such that (1) the type system reflects inheritance and dynamic method binding, and (2) the type system is safe in the sense that the static type of an object \( o \) is always a supertype of the type of \( o \) assigned at run-time. These static type systems are defined provided that the signature of each method is statically given; that is, the class to which the returned objects should belong is known in advance. Hence, the computational complexity of type-consistency problem becomes trivial since the analysis of method bodies is unnecessary, as is the case in the consistency problem of typed OOPLs.

The remainder of the paper is organized as follows. In Section 2, we define database schemas and their instances, and show some examples. In Section 3, we show the computational complexity of the type-consistency problem for the subclasses of database schemas mentioned above. Lastly, in Section 4, we summarize the paper.

2. DATABASE SCHEMAS

2.1. Syntax

**Definition 1.** A database schema is a 6-tuple \( S = (C, \preceq, \text{Attr}, \text{Ad}, \text{Meth}, \text{Impl}) \) where:

1. \( C \) is a finite set of class names.
2. \( \preceq \) is a partial order on \( C \) representing a class hierarchy. If \( c' \preceq c \), then we say that \( c' \) is a subclass of \( c \) and \( c \) is a superclass of \( c' \). For simplicity, we assume that the class hierarchy is a forest on \( C \); that is, for all \( c_1, c_2, c \in C \), either \( c_1 \preceq c_2 \) or \( c_2 \preceq c_1 \) whenever \( c \preceq c_1 \) and \( c \preceq c_2 \).
3. \( \text{Attr} \) is a finite set of attribute names.
4. \( \text{Ad} : C \times \text{Attr} \rightarrow C \) is a partial function representing attribute declarations. By \( \text{Ad}(c, a) = c' \), we mean that the value of attribute \( a \) of an object of \( c \) must be an object of \( c' \) or its subclass.
5. \( \text{Meth} \) is a finite set of method names.
6. \( \text{Impl} : C \times \text{Meth} \rightarrow \text{WFP} \) is a partial function representing method implementations, where \( \text{WFP} \) is the set of well-formed programs defined below.

A sentence is an expression that has one of the forms

1. \( y := y' \),
2. \( y := \text{self} \),
3. \( y := \text{self}.a \),
4. \( y := m(y') \),
5. \( \text{self}.a := y' \),
6. \( \text{return}(y') \),

where \( y, y' \) are variables, \( a \) is an attribute name, \( m \) is a method name, and \( \text{self} \) is a reserved word that denotes the object on which a method is invoked (or, to which a message is sent). A sentence of type 5 is called an update operation. The intuitive
meaning of each sentence seems obvious and the formal semantics will be presented in Section 2.2.

A program is a finite sequence of sentences. We say that a program \( s_1; s_2; \cdots; s_n \) is well formed when the following three conditions hold:

- No undefined variable is referred to. That is, for each \( s_i \) (\( 1 \leq i \leq n \)), if \( s_i \) is one of \( y := y' \), \( y := m(y') \), \( \text{self}.a := y' \), and \( \text{return}(y') \), then there exists a sentence \( s_j \) (\( j < i \)) that must be one of \( y' := y'' \), \( y' := \text{self} \), \( y' := \text{self}.a' \), and \( y' := m'(y'') \), where \( y' \) is a variable, \( a' \) is an attribute name, and \( m' \) is a method name.

- Only defined attributes are used. That is, for any sentence of type 3, \( a \) must be defined at that class or its superclasses. The formal definition of inheritance of attribute declarations will be described later.

- Only the last sentence \( s_n \) must have the form \( \text{return}(y') \) for some variable \( y' \). Thus the other sentences \( s_1, s_2, \ldots, s_{n-1} \) must be one of types 1 to 5.

Example 1. Consider the three programs in Fig. 2. Program (a) is well formed while (b) is not, since sentence \( s_{23} \) refers to variable \( y \) but no value is assigned to \( y \) in any of the preceding sentences \( s_{21} \) and \( s_{22} \). Neither is program (c) since the last sentence \( s_{34} \) is not in the form of \( \text{return}(y') \).

We often omit temporary variables for readability. For example, we write "\( y := m(\text{self}.a) \)" instead of "\( y' := \text{self}.a; y := m(y') \)," where \( y' \) is a temporary variable.

Definition 2. The description size of \( S \), denoted \( \|S\| \), is defined as

\[
\|S\| = |C| + |Attr| + |Meth| + (\text{the number of attribute declarations given by } Ad) + (\text{the total number of sentences given by } Impl),
\]

where \( |X| \) is the cardinality of a set \( X \).

2.2. Semantics

The inherited implementation of method \( m \) at class \( c \), denoted \( \text{Impl}^*(c, m) \), is defined as \( \text{Impl}(c', m) \) such that \( c' \) is the smallest superclass of \( c \) (with respect to the partial order \( \leq \)) at which an implementation of \( m \) exists; that is, if \( \text{Impl}(c'', m) \) is defined and \( c \leq c'' \), then it must hold that \( c' \leq c'' \). If such an implementation does not exist, then \( \text{Impl}^*(c, m) \) is undefined. Similarly, the inherited attribute declaration of attribute \( a \) at class \( c \), denoted \( \text{Ad}^*(c, a) \), is defined as \( \text{Ad}(c', a) \) such that \( c' \) is the

![FIG. 2. Example of programs.](image)
smallest superclass of \( c \) at which an attribute declaration of \( a \) exists. If such an attribute declaration does not exist, then \( \text{Ad}^*(c, a) \) is undefined.

A database instance of \( S \) is a pair \( I = (v, \mu) \), where:

1. To each class \( c \in C \), \( v \) assigns a disjoint, finite set \( v(c) \) of objects (or object identifiers). Each \( o \in v(c) \) is called an object of class \( c \). Let \( O_{K, I} = \bigcup_{c \in C} v(c) \). Let \( c(o) \) denote the class \( c \) such that \( o \in v(c) \).

2. To each object \( o \in v(c) \) and each attribute \( a \in \text{Attr} \) such that \( \text{Ad}^*(c, a) \) is defined, \( \mu \) assigns an object, denoted \( \mu(o, a) \), that is the value of attribute \( a \) (or simply \( a \)-value) of \( o \). If \( \text{Ad}^*(c, a) = c' \), then \( \mu(o, a) \) must belong to \( v(c') \) for some \( c' \) (\( c' \leq c' \)). Hereafter, \( \mu(o, a) \) is often denoted by \( o.a \).

The operational semantics of \( S \) is originally defined through a method execution tree [11]. In this paper, we present a more straightforward definition, in which the execution of a method is defined by rewriting rules on configurations of an interpreter for method implementations.

**Definition 3.** A configuration is one of the expressions

\[
\langle \mu, o \rangle, \quad \text{active}(\mu, o, m, i, \sigma), \quad \text{CF}-\text{await}(o, m, i, \sigma),
\]

where \( \mu \) is an assignment representing attribute values, \( o \) is an object, \( m \) is a method name, \( i \) is a positive integer, \( \sigma \) is an assignment of objects to the variables appearing in \( \text{Impl} \), and \( \text{CF} \) is a configuration. An initial configuration has the form \( \text{active}(\mu, o, m, 1, \sigma) \), where \( \text{Impl}^*(c(l(o)), m) \) is defined and \( \sigma \) is an assignment undefined everywhere.

Before presenting the formal semantics of configurations, we give an informal explanation here. \( \text{active}(\mu, o, m, i, \sigma) \) means that the interpreter is about to execute the \( i \)th sentence of \( \text{Impl}^*(c(l(o)), m) \), where \( \text{self} \) in \( \text{Impl}^*(c(l(o)), m) \) is interpreted as \( o \), the current variable assignment is given by \( \sigma \), and the current database instance is given by \( \mu \). \( \text{CF}-\text{await}(o, m, i, \sigma) \) represents that another method has been invoked at the \( i \)th sentence of \( \text{Impl}^*(c(l(o)), m) \). \( \langle \mu, o \rangle \) is the pair of the resulting database instance and the returned value after an execution of a method.

**Definition 4.** Let \( s(c, m, i) \) denote the \( i \)th sentence of \( \text{Impl}^*(c, m) \). Let \( f[(a_1, ... , a_n)/b] \) denote the function \( f' \) that is equal to \( f \) except that \( f'(a_1, ..., a_n) = b \). The one-step execution relation \( \rightarrow \) on configurations is defined by the rewriting rules shown in Fig. 3. Note that the execution is deterministic; that is, for every configuration \( \text{CF} \), there is at most one \( \text{CF}' \) such that \( \text{CF} \rightarrow \text{CF}' \).

**Definition 5.** Let \( o \in O_{K, I} \). A partial execution of method \( m \) for object \( o \) under instance \( I = (v, \mu) \) is a (possibly infinite but nonempty) sequence \( EX = \langle \text{CF}_0, \text{CF}_1, ... \rangle \) of configurations such that \( \text{CF}_0 \) is the initial configuration \( \text{active}(\mu, o, m, 1, \sigma) \) and \( \text{CF}_i \rightarrow \text{CF}_{i+1} \) for all \( i \).

A partial execution \( EX \) is said to be terminating if \( EX = \langle \text{CF}_0, ..., \text{CF}_n \rangle \) is a finite sequence and there is no \( \text{CF}_{n+1} \) such that \( \text{CF}_n \rightarrow \text{CF}_{n+1} \). If on the other hand \( EX \) is an infinite sequence, then \( EX \) is said to be nonterminating. Furthermore, \( EX \) is said to be complete if it is either terminating or nonterminating.


\begin{enumerate}
\item If \( s(cl(o), m, i) = "y := y'," \)
\[ \operatorname{active}(\mu, o, m, i, \sigma) \rightarrow \operatorname{active}(\mu, o, m, i + 1, \sigma[y/\sigma[y']]). \]
\item If \( s(cl(o), m, i) = "y := \text{self}," \)
\[ \operatorname{active}(\mu, o, m, i, \sigma) \rightarrow \operatorname{active}(\mu, o, m, i + 1, \sigma[y/o]). \]
\item If \( s(cl(o), m, i) = "y := \text{self}.a," \)
\[ \operatorname{active}(\mu, o, m, i, \sigma) \rightarrow \operatorname{active}(\mu, o, m, i + 1, \sigma[y/o(o,a)]). \]
\item If \( s(cl(o), m, i) = "y := m'(y')" \) and \( \text{Impl}'(cl(\sigma(y'))), m' \) is defined,
\[ \operatorname{active}(\mu, o, m, i, \sigma) \rightarrow \operatorname{active}(\mu, \sigma(y'), m', 1, \sigma_i) \circ \text{wait}(o, m, i, \sigma). \]
\item If \( s(cl(o), m, i) = "\text{self}.a := y'" \) and \( cl(\sigma(y')) \leq \text{Ad'}(cl(o), a) \),
\[ \operatorname{active}(\mu, o, m, i, \sigma) \rightarrow \operatorname{active}(\mu[\sigma(o,a)/\sigma(y')], o, m, i + 1, \sigma). \]
\item If \( s(cl(o), m, i) = "\text{return}(y')," \)
\[ \operatorname{active}(\mu, o, m, i, \sigma) \rightarrow (\mu, \sigma(y')). \]
\item If \( s(cl(o), m, i) = "y := m'(y')," \)
\[ (\mu, o') \circ \text{wait}(m, m, i, \sigma) \rightarrow \operatorname{active}(\mu, o, m, i + 1, \sigma[y/o']). \]
\end{enumerate}

\textbf{FIG. 3.} One-step execution relation.

\textbf{Definition 6.} A terminating execution \( EX = \langle CF_0, \ldots, CF_n \rangle \) is \textit{successful} if \( CF_n = \langle \mu', o' \rangle \) for some \( y' \) and \( o' \), and \textit{aborted} otherwise.

Aborted executions are caused by two types of sentences "\( y := m'(y') \)" and "\( \text{self}.a := y' \)." By the rewriting rule (R4), an execution is aborted if method \( m' \) is undefined for the class of the object assigned to \( y' \). By (R5), an execution is aborted if the class of the object assigned to \( y' \) violates the attribute declaration given by \( Ad \). Both cases are viewed as \textit{type errors}. Now we are ready to define the notions of consistency and termination.

\textbf{Definition 7.} \( S \) is \textit{consistent} if every terminating execution is successful under every instance of \( S \), and \( S \) is \textit{terminating} if every complete execution is terminating under every instance of \( S \).

\textbf{Example 2.} Consider a database schema \( S_1 = (C_1, \leq_1, Attr_1, Ad_1, Meth_1, \text{Impl}_1) \), where

- \( C_1 = \{\text{director}, \text{manager}, \text{employee}\} \) and \( \text{director} \leq_1 \text{manager} \leq_1 \text{employee}; \)
- \( Attr_1 = \{\text{boss}, \text{supervisor}, \text{secretary}\} \) and \( Ad_1 \) is shown in Fig. 4; and
- \( Meth_1 = \{\text{get} \_ \text{secretary}, \text{query} 1\} \) and \( \text{Impl}_1 \) is shown in Fig. 5.

Intuitively, \( \text{boss} \) indicates the immediate boss of a person, and \( \text{supervisor} \) indicates the director of a person. The values of \( \text{supervisor} \) are supposed to be calculated by a method using the values of \( \text{boss} \). At first, the values of \( \text{supervisor} \) are instantiated with arbitrary values. Figure 6 illustrates a database instance.
FIG. 4. Definition of $Ad_1$.

\begin{verbatim}
(class employee)
   boss, supervisor : employee

(class manager)
   boss, supervisor : director

(class director)
   secretary : employee
\end{verbatim}

FIG. 5. Definition of $Impl_1$.

\begin{verbatim}
(employee, query1) :
   1 : y := get_secretary(self.boss);
   2 : return(y).

(director, get_secretary) :
   1 : return(self.secretary).
\end{verbatim}

FIG. 6. A database instance $I_1$.

\begin{verbatim}
(employee, calc_supervisor) :
   1 : y := calc_supervisor(self.boss);
   2 : return(y).

(director, calc_supervisor) :
   1 : return(self).

(employee, query2) :
   1 : self.supervisor := calc_supervisor(self);
   2 : y := get_secretary(self.supervisor);
   3 : return(y).
\end{verbatim}

FIG. 7. Definition of $Impl_1'$.
\( \mathbf{I}_1 = (v_1, \mu_1) \) of \( \mathbf{S}_1 \), where \( \text{Bob}, \text{Sara}, ... \) are objects and \( \text{Bob} \rightarrow \text{Sara} \) means \( \mu_1(\text{Bob, boss}) = \text{Sara} \). Consider the execution of \( \text{query}_1 \) for \( \text{Bob} \). Since \( \mu_1(\text{Bob, boss}) = \text{Sara} \in v_1(\text{manager}) \) and \( \text{Impl}^*_1(\text{manager, get_secretary}) \) is undefined, the execution is aborted. Also it is easily checked that \( \mathbf{S}_1 \) is terminating.

Let \( \mathbf{S}'_1 = (C_1, \preceq_1, \mathbf{Attr}_1, \mathbf{Ad}_1, \mathbf{Meth}_1, \text{Impl}^*_1) \), where \( \mathbf{Meth}_1 = \{ \text{calc_supervisor}, \text{get_secretary}, \text{query}_2 \} \) and \( \text{Impl}^*_1 \) is shown in Fig. 7. \( \mathbf{I}_1 \) is also an instance of \( \mathbf{S}'_1 \). The execution of \( \text{calc_supervisor} \) for \( \text{Bob} \) is successful and the last configuration is \( \langle \mu_1, \text{John} \rangle \); i.e., the returned value of the execution is \( \text{John} \). On the other hand, the execution of \( \text{calc_supervisor} \) for \( \text{Alice} \) is nonterminating. It can be shown that \( \text{calc_supervisor} \) returns an object of class \( \text{director} \) when it terminates. Next, consider the execution of \( \text{query}_2 \) for \( \text{Bob} \). When control reaches the second sentence of \( (\text{employee}, \text{query}_2) \) in Fig. 7, \( \text{Bob}.\text{supervisor} \) has been set to \( \text{John} \in v_1(\text{director}) \). Therefore the execution is successful. Consequently, it can be proved that \( \mathbf{S}'_1 \) is consistent.

In the next example, we show how to represent Boolean values in update schemas. In the example, a method that calculates \( \text{NOR} \) and a method that simulates \( \text{if-then} \) statements are presented. These methods imply the powerful expressiveness of update schemas.

**Example 3.** Consider a database schema \( \mathbf{S}_2 = (C_2, \preceq_2, \mathbf{Attr}_2, \mathbf{Ad}_2, \mathbf{Meth}_2, \text{Impl}_2) \), where

- \( C_2 = \{ c, c_t, c_f \} \) such that \( c_t \preceq_2 c \) and \( c_f \preceq_2 c \) (i.e., \( c \) is a superclass of both \( c_t \) and \( c_f \), see Fig. 8a); and

- \( \mathbf{Ad}_2 \) is shown in Fig. 8b.

We adopt the following Boolean-value representation: Let \( o \) be an object of class \( c_t \). Each attribute \( a \in \{ a_1, a_2, a', a'', a_f \} \) of \( o \) represents true if \( o.a = o \), and false otherwise. Note that \( o.a_f \) always represents false because of the declaration \( \text{Ad}_2(c_t, a_f) = c_f \).

Then we define two methods \( \text{nor}[a_1, a_2] \) and \( \text{if_then}[a_1, m] \) as shown in Fig. 8c. Method \( \text{nor}[a_1, a_2] \) calculates \( \text{NOR} \) of \( o.a_1 \) and \( o.a_2 \), and returns \( o \) if the result is...
true and $o.a_t$ otherwise. Here $a'$ is being used as a form of scratch paper. First, $o.a'$ is initialized with $o$, i.e., true. Then $o.a'$ is set $o.a_t$, i.e., false, if and only if either $o.a_1$ or $o.a_2$ represents true. Since every Boolean operator can be represented by NORs, we can construct a method that calculates a given Boolean formula using $\text{nor}[a_1, a_2]$. On the other hand, $if \ then \ [a_1, m]$ simulates $if \ then$ statements: $m$ is invoked on $o$ if and only if $o.a_1$. By the first two lines of $(c, if \ then \ [a_1, m])$, $o.a'$ is "normalized" so that $o.a'$ and hence $cl(o.a') \neq c_i$ whenever $o.a_i$ represents false.

2.3. Subclasses of the Database Schemas

In the last part of this section, we provide some notions for defining subclasses of the database schemas.

**Definition 8.** The height of $\leq$ in a schema is the maximum integer $n$ such that there exist distinct $c_0, c_1, \ldots, c_n \in C$ satisfying $c_0 \leq c_1 \leq \cdots \leq c_n$.

If the height of $\leq$ is zero, then the class hierarchy is flat. That is, all classes are completely separated and there is no superclass-subclass relation at all. We often say that $S$ is flat if $\leq$ is flat.

**Definition 9.** $Ad$ is covariant if $c_1 \leq c_2$ implies $Ad^*(c_1, a) \leq Ad^*(c_2, a)$ for all $c_1, c_2 \in C$ and $a \in \text{Attr}$ such that both $Ad^*(c_1, a)$ and $Ad^*(c_2, a)$ are defined.

Usually, covariance is defined as a property of method signatures. For example, in [2], a schema is said to be covariant if for each built-in method $m$ (assumed to be unary for simplicity) and for each pair $(m: c_1 \rightarrow c'_1), (m: c_2 \rightarrow c'_2)$ of signatures of $m$, we have that $c'_1 \leq c'_2$ whenever $c_1 \leq c_2$. In our model, an attribute $a$ can be regarded as a built-in method $m_a$ such that the signatures of $m_a$ are given by $Ad$ and the interpretation of $m_a$ is given by a database instance.

There are many situations in which it is natural to assume the covariance. For example, $\text{technical\_paper} \leq \text{literature}$ and $Ad^*(\text{technical\_paper}, \text{author}) \leq Ad^*(\text{literature}, \text{author})$, the latter of which means that the authors of technical papers are a subclass of those of general literatures.

**Definition 10.** $Impl$ is retrieval if it includes no update operation (i.e., sentence in the form of "$self.a := y$"). We often say that $S$ is retrieval if $Impl$ is retrieval.

**Definition 11.** The method dependency graph $G = (V, E)$ of $Impl$ is defined as [2]

- $V = \text{Meth}$; and
- An edge from $m$ to $m'$ is in $E$ if and only if there is a class $c$ such that $m$ appears in $Impl(c, m')$.

If the method dependency graph of $Impl$ is acyclic, then $Impl$ is recursion-free. We often say that $S$ is recursion-free if $Impl$ is recursion-free. Note that $S$ is terminating whenever it is recursion-free.
3. COMPLEXITY OF THE TYPE-CONSISTENCY PROBLEM

3.1. Nonflatness of the Class Hierarchy

In this section, we show how nonflatness of the class hierarchy affects the complexity of the type-consistency problem. First, the following theorem claims that consistency for a flat schema is solvable in polynomial time.

THEOREM 1. Let $S = (C, \leq, Attr, Ad, Meth, Impl)$ be a database schema. If $S$ is flat, then consistency for $S$ is solvable in polynomial time.

Proof. Define an instance $\bar{I} = (\bar{v}, \bar{\mu})$ of $S$ as:

- $\bar{v}(c) = \{o_c\}$ for each $c \in C$; and
- $\bar{\mu}(o_c, a) = o_{c'}$ if $Ad^*(c, a) = c'$.

As we shall see, $I$ works as a representative of all the instances of $S$. Note that $\bar{\mu}$ is never altered during any execution even if $S$ is not retrieval, since $\leq$ is flat and each class has exactly one object.

First, we show that there is an aborted execution under $\bar{I}$ if and only if $S$ is inconsistent. The "only if" part is obvious. Conversely, let $I = (\bar{v}, \bar{\mu})$ be an arbitrary instance of $S$ and $h: O_{S1} \rightarrow O_{S1}$ be a homomorphism such that $h(o) = o_c$ for each $o \in \bar{v}(c)$. It can be shown that for every (partial) execution $EX$ under $I$, $h(EX)$ is a (partial) execution under $\bar{I}$ by induction on the length of $EX$. Then it can be easily proved that $h(EX)$ is aborted whenever $EX$ is aborted. Thus, in order to decide the consistency of $S$, we have only to check whether there is an aborted execution under $I$.

To check whether there is an aborted execution under $I$, we consider every execution of each $m$ for each $o \in O_{S1}$. We compute only the last configuration of the execution of each $m$ for each $o$, not the entire execution, since computing the entire executions takes exponential time in general. In order to remember the intermediate results of our computation, we use a table $T$, where $T(o_c, m, i)$ represents the last configuration of the partial execution from the first sentence up to the $i$th sentence in $Imp^*(c, m)$. In the following, we show how to compute each entry of $T$. Define $T(o_c, m, 0)$ as $active(\bar{\mu}, o_c, m, 1, \sigma_1)$. If $s(c, m, i + 1)$ is not $y := m'(y')$, compute $T(o_c, m, i + 1)$ from $T(o_c, m, i)$ through the corresponding rewriting rule in Fig. 3. Suppose that $s(c, m, i + 1) = "y := m'(y')"$. Also suppose that $T(o_c, m, i) = active(\bar{\mu}, o_c, m, i, \sigma)$ for some $\sigma$. If $Imp^*(ch(\sigma(y')))$, $m'$ is undefined, then the execution of $m$ for $o_c$ is aborted. That is, $S$ turns out to be inconsistent. Otherwise, there are the following three cases. Let $n$ be the number of sentences in $Imp^*(ch(\sigma(y'))), m')$.

1. If we have already obtained $T(\sigma(y'), m', n)$, then compute $T(o_c, m, i + 1)$ through (R7).

2. Suppose that $T(\sigma(y'), m', n)$ has not been obtained yet.
   (a) If we have already tried to compute $T(\sigma(y'), m', 1)$, then give up computing $T(\sigma(y'), m', n)$ (and thus $T(o_c, m, i + 1)$), since the execution is nonterminating. It is ensured that a type error never occurs during that method execution.
   (b) Otherwise, try to compute $T(\sigma(y'), m', 1), ..., T(\sigma(y'), m', n)$.
In summary, for all $c$ and $m$ such that $\text{Impl}^*(c, m)$ is defined, compute $T(o_c, m, i)$ in a depth-first manner. If an aborted execution is detected during the computation, then $S$ is inconsistent. Otherwise, if the computation is completed for each $m$ and $o_c$, then $S$ is consistent. Since each $T(o_c, m, i)$ is computed at most once, this algorithm terminates in a linear time of the size of $T$, and $T$ has a linear size of the total number of sentences given by $\text{Impl}$ since flatness implies $\text{Impl} = \text{Impl}^*$.

On the other hand, the following theorem says that consistency for a nonflat schema is undecidable even if it is retrieval and the height of the class hierarchy is bounded by one.

**Theorem 2.** Let $S = (C, \leq, \text{Attr}, \text{Ad}, \text{Meth}, \text{Impl})$ be a nonflat database schema. Consistency for $S$ is undecidable, even if $S$ is retrieval, the height of $\leq$ is one, and $\text{Ad}$ is covariant.

This theorem is proved by showing a reduction from the Post’s Correspondence Problem (PCP) to the consistency problem for a database schema with the conditions in the theorem. Let $\langle w, u \rangle$ ($w = \langle w_1, ..., w_n \rangle$, $u = \langle u_1, ..., u_n \rangle$) be an instance of the PCP over alphabet $\Sigma = \{0, 1\}$. We construct a database schema $S_{w,u}$ such that

- $S_{w,u}$ is retrieval;
- the height of $\leq$ of $S_{w,u}$ is one;
- $\text{Ad}$ of $S_{w,u}$ is covariant; and
- $S_{w,u}$ is inconsistent if and only if $\langle w, u \rangle$ has a solution.

The idea for $S_{w,u}$ to satisfy the last condition is as follows. Let $\text{post}$ be a method in $S_{w,u}$, which plays the principal role in the reduction. Each pair of a database instance $I$ and an object $o_1 \in O_{\text{Inst}}$ is regarded as a candidate for a solution of $\langle w, u \rangle$. If $(I, o_1)$ is actually a solution of $\langle w, u \rangle$, then the execution of $\text{post}$ for $o_1$ under $I$ is aborted. Otherwise, the execution of $\text{post}$ for $o_1$ under $I$ is nonterminating (therefore no type error occurs during the execution). By ensuring that no type error occurs during the execution of any method except $\text{post}$, we can conclude that $S_{w,u}$ satisfies the last condition.

Now we show the construction of $S_{w,u}$. Suppose that

$$w_1 = w_{1,1} w_{1,2} \cdots w_{1,d_1}, \quad ..., \quad w_m = w_{m,1} w_{m,2} \cdots w_{m,d_m},$$

$$u_1 = u_{1,1} u_{1,2} \cdots u_{1,e_1}, \quad ..., \quad u_n = u_{n,1} u_{n,2} \cdots u_{n,e_n},$$

where all of the $w_{i,j}$’s and $u_{i,j}$’s are in $\Sigma$. Figures 9 and 10 show the definition of $\leq$ and $\text{Ad}$ of $S_{w,u}$, respectively. Class $c_i$ ($1 \leq i \leq n$) represents the $i$th pair $\langle w_i, u_i \rangle$, and class $c'_i$ (resp. $c''_i$) represents symbol 0 (resp. 1). Note that the height of $\leq$ is one and $\text{Ad}$ is covariant. Next, define methods $\text{post}$, $m_w$, $\text{is0}$, $\text{is1}$, and $\text{ise}$ as Figs. 11–14 (also define method $m_y$ similarly to $m_w$). The underlined part (e.g., the second line of $(c_i, m_y)$) is a macro notation, and all of them can be expanded when $\langle w, u \rangle$ is reduced to $S_{w,u}$. Note that $S_{w,u}$ is retrieval (i.e., there is no sentence in the form of $\text{self}.a := y$). Moreover,
FIG. 9. \( \leq \) of \( S_{\circ, w} \).

<table>
<thead>
<tr>
<th>Class ( c_i ) (1 ( \leq i \leq n ))</th>
<th>Class ( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{\circ w} : c )</td>
<td>( a_{\circ w} : c' )</td>
</tr>
</tbody>
</table>

FIG. 10. \( Ad \) of \( S_{\circ, w} \).

\[
(c_i, \text{post}) \ (1 \leq i \leq n) : \\
1: \ y := m_w(\text{self}); \\
2: \ y := \text{isc}'(y); \\
3: \ y := m_w(\text{self}); \\
4: \ y := \text{isc}'(y); \\
5: \ y := \text{test}(\text{self}); \\
6: \ \text{return}(\text{self});
\]

FIG. 11. Definition of method post.

\[
(c_i, m_w) \ (1 \leq i \leq n) : \\
1: \ y := m_w(\text{self}, a_{\circ w}); \\
2: \ \text{if} \ w_{i, w} = 0 \\
\quad \text{then} \ y := \text{is0}(y); \\
\quad \text{else} \ y := \text{is1}(y); \\
\quad \vdots \\
\end{align*}
\[
\begin{align*}
\text{d}_i + 1: & \ \text{if} \ w_{i, w} = 0 \\
\text{then} \ y := \text{is0}(y); \\
\text{else} \ y := \text{is1}(y); \\
\text{d}_i + 2: & \ \text{return}(y).
\end{align*}
\]

FIG. 12. Definition of method \( m_w \).

\[
(c_i', \text{is0}) : \\
1: \ \text{return}(\text{self}, a_{\circ w}).
\]

\[
(c_i', \text{is0}) : \\
1: \ \text{loop forever}.
\]

\[
(c_i, \text{is0}) : \\
1: \ \text{return}(\text{self}).
\]

\[
(c_i', \text{is1}) : \\
1: \ \text{return}(\text{self}, a_{\circ w}).
\]

\[
(c_i', \text{is1}) : \\
1: \ \text{loop forever}.
\]

\[
(c_i, \text{is1}) : \\
1: \ \text{return}(\text{self}).
\]

FIG. 13. Definition of methods \text{is0} and \text{is1}.
each method except post and test has its definition at every class;

- method post is not invoked by another method; and

- method test, which appears at the fifth line of \((c_i, \text{post})\), has no definition at any class, and can be invoked only by post.

Thus, a type error occurs if and only if the control reaches the fifth line of \((c_i, \text{post})\) during the execution of post. Therefore, in order to prove the correctness of the reduction, it suffices to show that \((w, u)\) has a solution if and only if there is an instance \(I\) such that the control reaches the fifth line of \((c_i, \text{post})\) during the execution of post for some \(o_i \in O_{k+1}\) under \(I\).

Let \(I = (v, \mu)\) and \(o_1 \in v(c_1) \cup \cdots \cup v(c_n)\). In what follows, we explain the behavior of the execution of post for \(o_1\) under \(I\). First, assume that \(I\) is in the following form (F1) (see also Fig. 15):

\[
\begin{align*}
\text{(F1)} & \quad \bullet a_{i} \cdot a_x = a_{i+1} \in v(c_i) \cup \cdots \cup v(c_n) \quad (1 \leq i \leq k - 1), \\
& \quad \bullet a_{k+1} \cdot a_x = a' \in v(c_i), \\
& \quad \bullet a_{j} \cdot a_x = a'_{j+1} \in v(c_j) \cup v(c'_j) \quad (1 \leq j \leq l - 1), \\
& \quad \bullet a_{l} \cdot a_x = a'_{l+1} \in v(c_e).
\end{align*}
\]

In \(I\), sequence \(o_1 \cdot \cdots \cdot o_k\) represents a candidate for a solution of \((w, u)\), and sequence \(o'_1 \cdot \cdots \cdot o'_l\) represents a word over \(\Sigma\). Let \(w_n\) and \(u_n\) denote the words represented by \(o_i\) (i.e., \(w_n = w_{o_i}\) and \(u_n = u_{o_i}\) if \(o_i \in v(c_i)\)), and let \(x_j\) denote the symbol represented by \(o'_j\) (i.e., \(x_j = 0\) if \(o'_j \in v(c'_j)\), and \(x_j = 1\) if \(o'_j \in v(c_j)\)). The following two lemmas claim that the execution of the first two lines of \((c_i, \text{post})\) terminates if and only if \(w_n \cdot w_{o_1} = x_1 \cdot \cdots \cdot x_l\).

**Lemma 1.** Suppose that \(I\) is in the form of (F1). If there is \(I'\) (\(l' \leq l\)) such that \(w_n \cdot w_{o_1} = x_{l'} \cdot \cdots \cdot x_{l'}\), then the execution of \(m_n\) for \(o_1\) terminates and returns \(o'_{l' + 1}\). Otherwise, the execution of \(m_n\) for \(o_1\) does not terminate.

**Proof.** The lemma is proved by induction on \(k\). Without loss of generality, \(o_i\) is assumed to be an object of class \(c_1\).

[ Basis ] Suppose that \(k = 1\). By the first line of \((c_1, m_n)\), method \(m_n\) is recursively invoked on \(a_{i} \cdot a_x\), which is an object of class \(c\) since \(k = 1\). By \((c, m_n)\), this invocation results in \(o'_{i}\), and it is assigned to \(y\) at the first line of \((c_1, m_n)\). Suppose that \(w_{1, o_1} = 0\). By the second line of \((c_1, m_n)\), method is0 is invoked on \(o'_{i}\). From the
Fig. 15. A database instance of $S_w,u$. definition of $is\emptyset$, the execution of $is\emptyset$ for $o_1' \rightarrow o_\omega$ if $o_1' \in v(c')$, and does not terminate if $o_1' \not\in v(c') \cup v(c_1')$. Since a similar property holds when $w_1 \cdot a_1 = 1$, we can conclude that the execution of the second line of $(c_1,m_n)$ terminates and $o_1'$ is assigned to $y$ if and only if $w_0 \cdot a_0 \cdot a_\omega = o_\omega$. Also by induction on $d_1$, we obtain that the execution of $m_n$ for $o_1$ terminates and returns $o_1' \cdot d_{n+1}$ if $w_0 \cdot a_0 \cdot a_\omega \cdot x_1 \cdot d_1 \leq l$, and does not terminate otherwise.

[Inductive Step] Suppose that $k > 1$. By the first line of $(c_1,m_n)$, method $m_n$ is recursively invoked on $o_1' \rightarrow o_\omega$ ( = $o_2'$). From the inductive hypothesis, the execution of $m_n$ for $o_1$ terminates and returns $o_1' \cdot d_{n+1}$ if $w_0 \cdot a_0 \cdot a_\omega \cdot x_1 \cdot d_1 \leq l$, and does not terminate otherwise. In and after the second line of $(c_1,m_n)$, it is checked that $w_0 \cdot a_0 \cdot a_\omega \cdot x_1 \cdot l' \leq l$. Thus, the lemma holds when $k > 1$.

Lemma 2. Suppose that $I$ is in the form of (F1). The execution of $isc'$ for $o_1' \cdot a_\omega$ terminates if and only if $o_1' = o_1' \cdot a_\omega$ (i.e., $l' = l$).

Proof. Obvious from the definition of $isc'$.

Thus, the third line of $(cl(o_1), post)$ is executed if and only if $w_0 \cdot w_\omega = x_1 \cdots x_1$. Similar properties hold for the third and fourth lines of $(cl(o_1), post)$. Therefore, the control reaches the fifth line of $(cl(o_1), post)$ if and only if $w_0 \cdot w_\omega = u_0 \cdots u_\omega = x_1 \cdots x_1$.

Next, suppose that $I$ is not in the form of (F1). Then $I$ must be in one of the following forms:

(F2) The "$a_\omega$-chain" forms a cycle. That is, there is $o_1 \in v(c_1) \cup \cdots \cup v(c_n)$ such that $o_1' \cdot a_\omega \cdots a_\omega = o$ and $o \cdot a_\omega \cdots a_\omega = o$.

(F3) The "$a_\omega$-chain" does not form a cycle but the "$a_\omega$-chain" forms a cycle. That is, there are $o_1 \in v(c)$ and $o_1' \in v(c_1') \cup v(c_1')$ such that $o_1' \cdot a_\omega \cdots a_\omega = o$, $o \cdot a_\omega \cdots a_\omega = o'$, and $o' \cdot a_\omega \cdots a_\omega = o'$. 
In the case of (F2), the recursive call of \(m_w\) at the first line of \((c_i, m_w)\) does not terminate. In the case of (F3), the execution of is0 or is1 in \((c_i, m_w)\) or isc\(^c\) in \((cl(o_1), post)\) does not terminate. Therefore, if \(I\) is not in the form of (F1), then the control does not reach the fifth line of \((cl(o_1), post)\).

Suppose that \(\langle w, u \rangle\) has a solution. Then there is an instance \(I\) in the form of (F1) such that \(w_{o_1} = u_{o_1} = x_{i_1} \cdots x_1\). During the execution of post for \(o_1\) under \(I\), the control reaches the fifth line of \((cl(o_1), post)\). Conversely, suppose that there is an instance \(I\) such that the control reaches the fifth line of \((cl(o_1), post)\) during the execution of post for \(o_1\) under \(I\). Then \(I\) must be in the form of (F1) and satisfy that \(w_{o_1} = u_{o_1} = x_{i_1} \cdots x_1\). Obviously, \(o_1, \ldots, o_k\) represent the solution of \(\langle w, u \rangle\). This concludes the proof of Theorem 2.

As stated in Section 1, method schemas [2, 3] are based on a functional OOPL model. Since \(S_{w,u}\) is retrieval, it can be translated into a method schema. For example,

\[
\text{Impl}(c_i, post) = \text{test}(\text{isc}'(m_w(self))), \text{isc}'(m_u(self))),
\]

\[
\text{Impl}(c_i, m_w) = \text{isc}_i(\cdots \text{isc}_i(d(m_w(m_a(x_{i_1})))) \cdots),
\]

where isc\(_{i,j}\) is either is0 or is1 according to \(w_{i,j}\), and \(m_a\) is a method that returns the \(a\)th-value of the argument object. It is easily verified that \(S_{w,u}\) can be translated into a method schema with methods of arity two. Thus, we have the following result, which was open in [2]:

**Corollary 1.** Consistency for a method schema with methods of arity two is undecidable.

### 3.2. Recursion

Intuitively, recursion makes the length of the execution unbounded. In this section, we show that the complexity of the type-consistency problem is affected by this unboundedness.

**Theorem 3.** Let \(S = (C, \leq, Attr, Ad, Meth, Impl)\) be a database schema with recursion. Consistency for \(S\) is undecidable, even if \(S\) is terminating, the height of \(\leq\) is one, and Ad is covariant.

To prove Theorem 3, for a given input string \(x\) of a fixed deterministic Turing machine \(M\), we construct a schema \(S_{M,x}\) satisfying the following conditions:

- \(S_{M,x}\) is terminating;
- the height of \(\leq\) of \(S_{M,x}\) is one;
- Ad of \(S_{M,x}\) is covariant; and
- \(S_{M,x}\) is inconsistent if and only if \(M\) accepts \(x\).

First of all, we define a Turing machine and an instantaneous description.
Definition 12. A deterministic Turing machine $M$ is a triple $(Q, \Sigma, \delta)$, where

- $Q$ is a finite set of states. $Q$ contains two special states: the initial state $q_0$ and the accepting state $q_{yes}$.
- $\Sigma$ is a finite set of symbols. $\Sigma$ contains two special symbols: the blank symbol $B$ and the first symbol $f$. The first symbol is always placed at the leftmost cell of the tape.
- $\delta$ is a function that maps $(Q - \{q_{yes}\}) \times \Sigma \times \{\leftarrow, \rightarrow, \cdot\}$ to $Q \times \{\leftarrow, \rightarrow\}$. We assume that if $\delta(q, \cdot) = (q', \gamma, d)$, then $\gamma = \triangleright$ and $d = \rightarrow$. Therefore, the tape head never falls off the left end of the tape.

An instantaneous description (ID) $I$ of $M$ is a finite sequence $(q_1, \gamma_1), \ldots, (q_k, \gamma_k)$, where $q_i \in Q$ and $\gamma_i \in \Sigma$. It is required that $\gamma_1 = \triangleright$, and exactly one $q_i$ is in $Q$ ($i$ denotes the head position). The $i$th pair $(q_i, \gamma_i)$ of an ID $I$ is denoted by $I[i]$. The transition relation $\triangleright M$ over the set of IDs is defined as usual.

We only describe the outline of the reduction (see the Appendix for a complete proof). First, in order to ensure that the execution of each recursively defined method $m$ is terminating, we use an attribute, say $a_{ws}$, which "marks" an object. Suppose that an object $o$ is visited by a recursive invocation of $m$. If $o.a_{ws}$ represents true (see Example 3), then $m$ sets $o.a_{ws}$ false and continues the execution. Otherwise, $m$ returns from the invocation. Consequently, $o.a_{ws}$ represents true only if $o$ has not been visited. Since the set $O_{S_{M,x}}$ of objects is finite, it can be shown that $S_{M,x}$ is terminating. Moreover, by setting $o.a_{ws}$ true when $m$ returns, other recursively defined methods can reuse $a_{ws}$. See Lemma 3 in the Appendix for a formal description of this technique.

Let $TM$ be a method in $S_{M,x}$, which plays the principal role in the reduction. $TM$ simulates $M$ on $x$ as follows. Each database instance $I$ of $S_{M,x}$ is considered as a working space to compute the IDs of $M$ on $x$. $TM$ simulates $M$ on $x$ exactly $r$ steps, where $r > 0$ is a constant dependent on $I$. If the ID after $r$-step transitions contains the accepting state $q_{yes}$, then $TM$ causes a type error. Otherwise, the execution of $TM$ is successful. By ensuring that no type error occurs during the execution of any method except $TM$, the following property holds: If $M$ accepts $x$, then there is an instance $I$ such that both the number of steps $r$ and the size of the working space determined by $I$ are large enough to find an aborted execution of $TM$ under $I$ (i.e., $S_{M,x}$ is inconsistent). Otherwise, there is no aborted execution of $TM$ under any instance (i.e., $S_{M,x}$ is consistent).

Define $\equiv$ and $Ad$ of $S_{M,x}$ as shown in Figs. 16 and 17, respectively. In Fig. 17, $\vec{a}$ denotes a tuple $(a_1, \ldots, a_K)$ of attributes, where $K = \lceil \log((|Q| + 1) \cdot |\Sigma|) \rceil$ (i.e., the number of bits to represent an element of an ID). $Ad(c_i, \vec{a}) = c$ means that $Ad(c_i, a_i) = c$ for each $i$ ($1 \leq i \leq K$). An element of an ID is stored in $\vec{a}$ as the binary
encoded form stated in Example 3. Attributes $\bar{a}'$ and $\bar{a}''$ are used for storing intermediate results during the computation of an ID. Attribute $a_{\text{cont}}$ is used for determining $r$, i.e., the number of steps to be simulated. Attributes $a_{\text{pos}}$ and $a'_{\text{pos}}$ are used for checking whether $M$ is in the accepting state after the simulation. Note that the height of $\leq$ is one and $Ad$ is covariant. Next, define method $TM$ as shown in Fig. 18. All the methods except $\text{test}$ is defined at every class. Method $\text{test}$ is defined only at class $c_f$. Since we can define all the methods so that no update operation causes a type error (see the method definitions presented in the Appendix), a type error occurs if and only if the control reaches the fifth line of $(c_t, TM)$ and $\text{test}$ is about to be invoked on an object of class $c_t$, $c_i$, or $c'_i$. 

In what follows, we explain the behavior of $TM$. Let $I = (\nu, \mu)$ be a database instance of $S_{M,x}$ and $o_1 \in \nu(c_i)$. Suppose that $TM$ is invoked on $o_1$. Then $\text{get ws}$ is executed for $o_1$ by the first line of $(c_t, TM)$. This obtains objects $o_2, ..., o_{k+1}$ satisfying $o_i.a = o_{i+1} \ (1 \leq i \leq k)$ by following attribute $a_{\nu}$ of each $o_i$, where $k$ is a constant dependent on $I$ and satisfies $k \geq 1$. The objects $o_2, ..., o_{k+1}$ will be used as a working space to simulate $M$. Since attribute $a_{\nu}$ is defined only at class $c_i$, the class of $o_2, ..., o_k$ must be $c_i$. By a technical reason, we want $o_{k+1}$ to be an object of class $c'_i$. To achieve this, if the $a_{\nu}$-chain from $o_1 (1)$ ends up with an object of class $c_i$ or $c$, or (2) forms a cycle, then $\text{get ws}$ changes the value of $o_{k+1}$ to an object of class $c'_i$ (see Fig. 19). Lemma 5 in the Appendix provides a formal description of the behavior of $\text{get ws}$.

Let $I_0$ be the initial ID of $M$ on $x$, and $n$ be the length of $I_0$. By executing $\text{init ws}$ for $o_1$ at the second line of $(c_i, TM)$, each $I_0[i] \ (1 \leq i \leq k)$ is stored in $o_i.\bar{a}$, where $o_i.\bar{a}$ denotes the tuple $(o_i.a_1, ..., o_i.a_k)$. Therefore, if $k < n$, then elements $I_0[k+1], ..., I_0[n]$ are abandoned. Conversely, if $n < k$, then $\langle \perp, B \rangle$ is stored in $o_{n+1}.\bar{a}, ..., o_k.\bar{a}$. (Actually, this is done by $\text{get ws}$; see the definitions of $\text{get ws}$ and

```
(c_i, TM) :
1 : y := get ws(self);
2 : y := init ws(self);
3 : y := step(self);
4 : y := accept(self);
5 : y := test(y);
6 : return(self);
```

FIG. 18. Definition of method $TM$. 
FIG. 19. A database instance after invoking method get_ws on $o_1$.

FIG. 20. Working space to simulate $M$. 
Method \textit{step} simulates $r$-step transitions of $M$. Let $I$ denote the $j$th ID of $M$ on $x$ (counting from 0). Suppose that the first $k-j$ elements of $I_i$ are stored in $o_{j+1}, a, \ldots, a, d$. More precisely, $I_i[j] (1 \leq i \leq k-j)$ is stored in $a_{j+i}, d$. Note that the initial ID $I_0$ satisfies this condition. Consider a database instance shown in Fig. 20a. Let us compute the next ID $I_{j+1}$. Note that $I_{j+1}[i]$ can be computed from $I_{j}[i-1], I_{j}[i], \text{ and } I_{j}[i+1]$. Therefore, if these three adjacent elements are stored in one object, we can compute $I_{j+1}[i]$ using \texttt{nor}[$*, *$] stated in Example 3. To do this, for every object $o$ in the $a_n$-chain, we copy the element of the ID stored in $o$ to $o.a_n$ and $o.a_n.a$ as shown in Fig. 20b. (It seems impossible to copy the data in $o.a_n$ to $o$, although we do not know its formal proof.) Method \texttt{copy}[a_1, a_2] defined in Fig. 21 copies the Boolean value represented by $o.a_1$ to $o.a_n.a_2$ when it is invoked on $o$. Thus we can obtain the next ID, and the place where the ID is stored is "shifted to right" (see Fig. 20c, where $\delta(q, 1) = (q', 0, \rightarrow)$).

Next we explain attribute $a_{\text{cont}}$. This attribute indicates whether the simulation should be continued. Let $o$ be the object in which the first element of the current ID is stored. If $o.a_{\text{cont}}$ represents true, then the simulation of $M$ is continued. Otherwise, the simulation stops. For example, in the case of Fig. 20c, the simulation stops after two steps (Fig. 20d). See Lemmas 7 and 8 in the Appendix for a formal description of the behavior of \textit{step}.

Method \textit{accept} checks whether $q_{\text{yes}}$ is in the last ID by using \texttt{nor}[$*, *$] and \texttt{copy}[$*, *$]. It returns $a_{k+1} \in v(c_i)$ if $q_{\text{yes}}$ is in the last ID, and $a_{k+1} \in v(c_i)$ otherwise. See Lemma 9 in the Appendix for a formal description of the behavior of \text{accept}.

Method \textit{test} is invoked on the returned value of \textit{accept}. Since \textit{test} is defined only at class $c_i$, this invocation causes a type error if and only if $q_{\text{yes}}$ is in the last ID.

Suppose that $M$ accepts $x$. Then $M$ halts after finite steps. Therefore, there is a database instance $I$ such that both $k$ and $r$ are large enough to cause a type error under $I$. Conversely, suppose that $M$ does not accept $x$. Since $q_{\text{yes}}$ never appears in the $a_n$-chain, invocation of \textit{test} causes no type error. Thus, Theorem 3 has been proved.

In contrast to the above result, consistency for a recursion-free schema with update operations is coNEXPTIME-complete.

\begin{figure}[h]
\centering
\begin{minipage}{0.45\textwidth}
\begin{verbatim}
\{c, set.f[a2] : 1 : self.a2 := self.a;
  2 : return(self).
\}
\{c, set.t[a2] : 1 : self.a2 := self.a;
  2 : return(self).
\}
\end{verbatim}
\end{minipage} \hspace{0.5cm}
\begin{minipage}{0.45\textwidth}
\begin{verbatim}
\{c, copy[a1, a2] : 1 : y := set.f[a2]([self.a_n]);
  2 : y := if.then[a1, set.t[a2]([self]);
  3 : return(self).
\}
\{c, copy[a1, a2] : 1 : y := set.f[a2]([self.a_n]);
  2 : return(self).
\}
\end{verbatim}
\end{minipage}
\caption{Definition of method \texttt{copy}[a_1, a_2].}
\end{figure}
Theorem 4. Let \( S = (C, \leq, \text{Attr}, \text{Ad}, \text{Meth}, \text{Impl}) \) be a recursion-free schema with update operations. Then consistency for \( S \) is in coNEXPTIME.

Proof. Since \( S \) is recursion-free, the length of any execution under any instance of \( S \) is bounded by \( N^{|\text{Meth}|} \), where \( N \) is the maximum number of sentences of a method in \( \text{Impl} \). Therefore, to find inconsistency for \( S \), nondeterministically guess an instance of size at most \( N^{|\text{Meth}|} \leq |S| \leq 2^{|S| \log |S|} \), which causes a type error. That is, consistency for \( S \) is in coNEXPTIME.

Theorem 5. Let \( S = (C, \leq, \text{Attr}, \text{Ad}, \text{Meth}, \text{Impl}) \) be a recursion-free schema with update operations. Consistency for \( S \) is coNEXPTIME-hard, even if the height of \( \leq \) is one and \( \text{Ad} \) is covariant.

Sketch of Proof. Let \( M \) be a fixed \( 2^{p(n)} \)-time bounded nondeterministic Turing machine for a polynomial \( p \), and let \( x \) be an input string for \( M \) with length \( n \). We construct, in polynomial time \( p(n) \), a recursion-free schema that is inconsistent if and only if \( M \) accepts \( x \).

The idea of simulating \( M \) on \( x \) is similar to Theorem 3. However, two problems still remain. First, we must simulate a nondeterministic transition of \( M \). To do this, we introduce new attributes for each object in the \( a_{\infty} \)-chain. The \( j \)-th nondeterministic choice \( \text{ch}_j \) is represented by the new attributes of object \( o \) in which the first element of the \((j-1)\)th ID \( I_{j-1} \) is stored. Then we can compute \( I_J[i] \) from \( I_{j-1}[i-1], I_{j-1}[i], I_{j-1}[i+1], \) and \( \text{ch}_j \).

The other problem is how to simulate \( 2^{p(n)} \) steps of \( M \) with a recursion-free schema containing at most \( p(n) \) methods. To solve this problem, we use methods \( \text{step}_i \) \((0 \leq i \leq p(n)) \) defined as

\[
\begin{align*}
(c_i, \text{step}_i) & (1 \leq i \leq p(n)) : \\
& y := \text{step}_{i-1}(\text{self}); \\
& y := \text{step}_{i-1}(y); \\
& \text{return}(y).
\end{align*}
\]

It is easily verified that if \( \text{step}_{p(n)} \) is invoked on an object \( o \) in the \( a_{\infty} \)-chain, then \( \text{step}_0 \) is sequentially invoked on the first \( 2^{p(n)} \) objects in the \( a_{\infty} \)-chain from \( o \). A method that simulates one-step transition is defined in the same way since it must access \( 2^{p(n)} \) objects in the working space. Thus, \( 2^{p(n)} \) steps of \( M \) are simulated by executing \( \text{step}_{p(n)} \). The other recursively defined methods (such as \( \text{get.ws} \) and \( \text{accept} \)) in the proof of Theorem 3 are also implemented in the same manner.

3.3. Update Operations

The following theorem can be obtained from Theorem 2 of [16]:

Theorem 6. Let \( S = (C, \leq, \text{Attr}, \text{Ad}, \text{Meth}, \text{Impl}) \) be a schema that is terminating. If \( S \) is retrieval, then consistency for \( S \) is solvable in polynomial time.
By Theorems 3 and 6, we can conclude that update operations make the type-consistency problem difficult if a given schema is terminating.

4. CONCLUSIONS

We have discussed the complexity of the type-consistency problem for some subclasses of OODB schemas. Moreover, by comparing the results, we have shown how the complexity is affected by nonflatness of the class hierarchy, recursion, and update operations.

When we classify OODB schemas in view of nonflatness, recursion, and update operations, the type-consistency problem is undecidable or intractable for most of practical OODB schemas. Therefore, as future works, it is desirable to find another subclass of OODB schemas that is practical and for which consistency is tractable. For example, consistency is expected to be decidable for acyclic database schemas [12], which are considered as an object-oriented extension of nested relational database schemas. It is also important to develop an incremental algorithm for type-consistency checking.

APPENDIX: COMPLETE PROOF OF THEOREM 3

Let \( M \) be a Turing machine and \( x = x_1 \ldots x_n \) an input string for \( M \). We abbreviate \( \text{self}.a := \text{self} \) and \( \text{self}.a := \text{self}.a \) to \( \text{self}.a := \text{true} \) and \( \text{self}.a := \text{false} \), respectively. Methods \( \text{test} \), \( \text{get.ws} \), \( \text{init.ws} \), \( \text{step} \), \( \text{accept} \) are defined as shown in Figs. 22–26, respectively.

First, we show that \( S_{M,x} \) is terminating.

**Lemma 3.** Let \( I = (v, \mu) \) be an arbitrary database instance of \( S_{M,x} \), and \( o_1 \) be an arbitrary object in \( O_{S_{M,x}} \). The execution of \( \text{get.ws} \) for \( o_1 \) is terminating under \( I \).

**Proof.** If \( o_1 \in v(c_i) \cup v(c_j) \cup v(c) \), then the execution is terminating since \( (c, \text{get.ws}) \) is executed for \( o_1 \). Thus in the following we consider the remaining case, i.e., \( o_1 \in v(c_i) \). First of all, by the first line of \( (c_i, \text{get.ws}) \), \( o_1.a_{ws} \) is set to true. Then \( \text{get.ws}' \) is invoked on \( o_1 \). By the second line of \( (c_i, \text{get.ws}') \), \( \text{get.ws}'' \) is invoked on \( o_1 \) since \( o_1.a_{ws} \) is true. By \( (c_i, \text{get.ws}'') \), \( \text{get.ws}' \) sets \( o_1.a_{ws} \) false and recursively invokes \( \text{get.ws}' \) on \( o_1.a_{ws} \).

Consider the case that \( \text{get.ws}' \) is recursively invoked on an object \( o \). There are three cases to be considered:

1. If \( o \in v(c_i) \cup v(c_j) \cup v(c) \), then the recursive invocation of \( \text{get.ws}'' \) terminates since \( (c, \text{get.ws}'') \) is executed for \( o \).

   \[
   (c_i, \text{test}) : \\
   1 : \text{return}(\text{self}).
   \]

   **FIG. 22.** Definition of method \( \text{test} \).
(2) If $o \not\in v(c_i)$ and $o.a_{ws}$ is false, then no more recursive invocation occurs from the definition of $(c_i, \text{get ws}')$.

(3) If $o \in v(c_i)$ and $o.a_{ws}$ is true, then get ws" is invoked on $o$ by the second line of $(c_i, \text{get ws}')$. Method get ws" sets $o.a_{ws}$ false and recursively invokes get ws on $o.a_{ws}$. Thus, every time get ws' is recursively invoked, the number of objects $o$ such that $o.a_{ws}$ is true decreases. Since $O_{S,M,x}$ is finite, one of the conditions (1) and (2) above holds eventually.

Therefore, the execution of get ws on $o_1$ is terminating. ■

Similarly, it can be proved that the execution of every recursively defined method (such as step, delta, accept, etc.) in $S_{M,x}$ is terminating. Thus we have the following lemma:

**Lemma 4.** $S_{M,x}$ is terminating.

---

(FIG. 23. Definition of method get ws.)

(FIG. 24. Definition of method initws.)
In what follows, we show that TM simulates M on x correctly. Hereafter, we mean \( o.a = o \) by \( o.a = \text{true} \).

**Lemma 5.** Let \( I = (v, \mu) \) be an arbitrary database instance of \( S_{M, x} \), and \( o_1 \in v(c_i) \) be an arbitrary object. After the execution of \( \text{get.ws} \) for \( o_1 \) under \( I \), there exists a positive integer \( k \) that satisfies the following condition (C1):

\[
\begin{align*}
\text{(C1-1)} & \quad o_1 \in v(c_i), \quad o_i.a_{m} = a_{i+1} \in v(c_i) \quad (1 \leq i \leq k-1), \quad \text{and} \quad o_k.a_{m} = a_{k+1} \in v(c_i); \\
\text{(C1-2)} & \quad o_i.a_{m} = a_{i+1} = \text{true} \quad (1 \leq i \leq k); \\
\text{(C1-3)} & \quad o_i, \overline{a} \quad (1 \leq i \leq k) \quad \text{represents} \quad \langle \bot, B \rangle.
\end{align*}
\]

**Proof.** Suppose that \( \text{get.ws} \) is invoked \( k \) times by the second line of \( (c_i, \text{get.ws}) \) during the complete execution of \( \text{get.ws} \) for \( o_1 \). In what follows, we show that \( k \) satisfies condition (C1).
First, we prove that \( k \geq 1 \). By the second line of \((c_1, \text{get}_\text{ws})\), \( \text{get}_\text{ws}' \) is invoked on \( o_1 \). Since \( o_1.a_{ws} \) is true by the first line of \((c_1, \text{get}_\text{ws})\), \( \text{get}_\text{ws}'' \) is invoked on \( o_1 \) by the second line of \((c_1, \text{get}_\text{ws}'')\). Then, by the second line of \((c_1, \text{get}_\text{ws}'')\), \( \text{get}_\text{ws}'' \) is invoked on \( o_1.a_{ws} = a_2 \). Thus \( k \geq 1 \).

Next, we prove \((C1-1)\). Consider the \( i \)th invocation \((1 \leq i < k)\) of \( \text{get}_\text{ws}' \) from the second line of \((c_1, \text{get}_\text{ws})\). Let \( o_{i+1} \) be the self object of the invocation. Note that \( o_{i+1}.a_{ws} = o_{i+1} \in v(c_i) \) and \( o_{i+1}.a_{ws} \) is true since \( i < k \) (see the condition \((3)\) in the proof of Lemma 3). By the first and third lines of \((c_i, \text{get}_\text{ws}')\), the returned value of this invocation is \( o_{i+1} \). Therefore, by the second and third lines of \((c_i, \text{get}_\text{ws}'')\), it holds that \( o_i.a_{ws} = o_{i+1} \in v(c_i) \). Next, consider the \( k \)th invocation of \( \text{get}_\text{ws}' \), and let \( o \) be the self object of the invocation. In this case, one of the conditions \((1)\) and \((2)\) in the proof of Lemma 3 holds. If \((1)\) holds, then \( o.a'_i \) is returned as the returned value of this invocation since \((c, \text{get}_\text{ws}')\) is executed for \( o \) (see also Fig. 19(1)). If \((2)\) holds, then \( o.a'_i \) is returned by the first and third lines of \((c_i, \text{get}_\text{ws}')\) (see also Fig. 19(2)). Thus, in either case, \( o.a'_i \in v(c_i) \) is returned and assigned to \( o_k.a_{ws} \) by the third line of \((c_i, \text{get}_\text{ws}'')\). By letting \( o_{k+1} = o.a'_i \), condition \((C1-1)\) is satisfied.

Conditions \((C1-2)\) and \((C1-3)\) hold by the fourth, fifth, and sixth lines of \((c_i, \text{get}_\text{ws}'')\).

The following lemma holds evidently from the definition of method \text{initws} (see Fig. 24).

**Lemma 6.** Suppose that \( I = (v, \mu) \) satisfies condition \((C1)\) for some \( k \) \((k \geq 1)\). Then, after the execution of \text{initws} for \( o_1 \) under \( I \), the following condition \((C2)\) holds:

\[
\begin{align*}
(C2-1) & \quad \text{The same as } (C1-1); \\
(C2-2) & \quad \text{The same as } (C1-2); \\
(C2-3) & \quad \text{For each } i (1 \leq i \leq k), o_i.a \text{ represents the } i \text{th element } I_0[i] \text{ of the initial ID of } M \text{ on } x.
\end{align*}
\]

The following lemma, which states the behavior of method \text{delta} (see Fig. 25), is also easily obtained from the explanation in Section 3.2. Intuitively, it states that \text{delta} computes a one-step transition of \( M \) correctly.

**Lemma 7.** Suppose that \( I = (v, \mu) \) satisfies the following condition \((C3)\) for some \( k \) \((k \geq 1)\):

\[
\begin{align*}
(C3-1) & \quad \text{The same as } (C2-1); \\
(C3-2) & \quad o_i.a'_{ws} \text{ is true } (1 \leq i \leq k); \\
(C3-3) & \quad \text{There exists } j (0 \leq j \leq k-1) \text{ such that for each } i (1 \leq i \leq k-j), o_{j+i}.a \text{ represents } I_j[i].
\end{align*}
\]

Then, after the execution of \text{delta} for \( o_{k+1} \) under \( I \), the following condition \((C3')\) holds:

\[
\begin{align*}
(C3'-1) & \quad \text{The same as } (C3-1); \\
(C3'-2) & \quad \text{The same as } (C3-2); \\
(C3'-3) & \quad \text{For each } i (1 \leq i \leq k-j), o_{(j+1)+i}.a \text{ represents } I_{j+i}[i].
\end{align*}
\]
Lemma 8. Suppose that \( I = (v, \mu) \) satisfies condition (C2) for some \( k \geq 1 \). Then, after the execution of \( \text{step} \) for \( o_1 \) under \( I \), the following condition (C4) holds:

(C4-1) The same as (C2-1);
(C4-2) The same as (C2-2);
(C4-3) Let \( r \) be the largest index such that for each \( l (1 \leq l \leq r) \), \( o_l.a_{\text{cont}} \) is true, i.e., \( r = \max\{0 \} \cup \{ j \mid \exists i, a_{\text{cont}} = a_0 \} \). Then, for each \( i (1 \leq i \leq k - r) \), \( o_{r+i}.\delta \)

represents \( I_l[i] \).

Proof. From the definition of methods \( \text{step} \) and \( \delta \), the value of \( o_r.a_{\text{ws}} \) (1 \leq i \leq k) is never altered. Thus, (C4-1) holds by the assumption (C2-1).

Next we show that (C4-3) is satisfied. By (C2-2), \( o_r.a_{\text{ws}} \) is true for each \( i (1 \leq i \leq k) \). Therefore, by the definitions of (c1, \text{step}), (c1, \text{step'}), and (c1, \text{step''}), it is easily verified that \( \delta \) is sequentially invoked on \( o_1, ..., o_r \) during the execution of \( \text{step} \) for \( o_1 \). Moreover, we claim that:

- (C2) implies (C3) since (C2-3) is obtained by letting \( j = 0 \) in (C3-3); and
- (C3') implies (C3) since (C3-3) is obtained by replacing \( j + 1 \) in (C3'-3) by \( j \).

Since \( \text{step} \) can alter \( o_r.a \) and \( o_r.a_{\text{ws}} \) only by invoking \( \delta \). Lemma 7 can be applied \( r \) times. Consequently, after the execution of \( \text{step} \) for \( o_1 \) under \( I \), \( o_r.a_i \) represents \( I_l[i] \) for each \( i (1 \leq i \leq k - r) \). That is, (C4-3) holds.

Lastly, (C4-2) is satisfied because of (C3'-2) and the fourth line of (c1, \text{step'}).

Lemma 9. Suppose that \( I = (v, \mu) \) satisfies condition (C4) for some \( k \geq 1 \). Then the returned value of the execution of \( \text{accept} \) for \( o_1 \) under \( I \) is \( o_{k+1} \) if there is some object \( o_i (1 \leq i \leq k) \) such that \( o_i.a \) contains the accepting state \( q_{\text{yes}} \), and \( o_{k+1}.a \) otherwise.

Proof. By the first line of (c1, \text{accept}), \( o_1.a_{\text{yes}} \) is set to false (i.e., \( o_1.a_t \)). Then \( \text{accept} \) is invoked on \( o_1 \). Since \( o_1.a_{\text{ws}} \) is true by (C4-2), \( \text{accept} \) is invoked on \( o_1 \). Inductively, consider the execution of \( \text{accept} \) for \( o_j (1 \leq j \leq k) \). By the second line of (c1, \text{accept}), \( o_j.a_{\text{yes}} \) is set to true (i.e., \( o_j \)) if \( o_j.a \) contains \( q_{\text{yes}} \), and false (i.e., \( o_j.a_t \)) otherwise. By the third and fourth lines, \( o_{j+1}.a_{\text{yes}} \) is set to \( o_{j}.a_{\text{yes}} \lor o_{j}.a_{\text{yes}} \). Therefore, by the inductive hypothesis, \( o_{j+1}.a_{\text{yes}} \) is set to true (i.e., \( o_{j+1} \)) if there is some object \( o_j (1 \leq j \leq f) \) such that \( o_j.a \) contains \( q_{\text{yes}} \), and \( o_{j+1}.a_{\text{yes}} \) is set to false (i.e., \( o_{j+1}.a_t \)) otherwise.

Lastly, since \( o_{k+1} \in q_{c_1} \) by condition (C4-1), \( (c_1', \text{accept'}) \) is executed for \( o_{k+1} \). Therefore, the returned value of the execution of \( \text{accept} \) for \( o_k \) is \( o_{k+1}.a_{\text{yes}} \). Thus, the lemma holds.

By Lemmas 5–9 and the explanation in Section 3.2, the following lemma holds.

Lemma 10. \( S_{M, x} \) is inconsistent if and only if \( M \) accepts \( x \).

Theorem 3 is obtained by Lemmas 4 and 10.
REFERENCES