The Structure of the Young Symmetrizers for Spin Representations of the Symmetric Group, I

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INTRODUCTION

In the first decade of this century, Alfred Young [29] introduced the concept of symmetrizers to describe the ordinary representations of the symmetric group; in the same period, Issai Schur [24] initiated the study into its projective representations. Recently, these classical theories have been brought together by Nazarov in a remarkable paper [18] developing the concept of a projective symmetrizer. The present series of papers is based on the author’s Ph.D. thesis [9] and examines a natural structure exhibited by these elements.

The first paper introduces the projective analogue of the classical Young symmetrizer; it is proved that this element has a multiplicative structure comparable with the $p_\lambda q_\lambda$-form observed by the ordinary symmetrizer. However, an intermediate factor $x_\lambda$ is required in our decomposition of the projective symmetrizer. The general results specify a “natural” expression for this additional element but, in practical circumstances, this object is too complicated to analyse further, thus requiring an alternative method-
A more efficient construction of the projective symmetrizer is available for certain classes of partitions. This alternative approach, discussed in the second paper, yields a “compact” expression for the intermediate factor. This compact element behaves in a more controlled manner than the natural expression and therefore supports further investigation.

The analysis of the projective symmetrizer continues in the third paper where we consider the problem of describing the simplest expression for the intermediate factor: a two-stage reduction procedure is presented to simplify any given expression for $x_\lambda$ to this “reduced” form. In practical instances, the evaluation of the reduced element is performed with the symbolic computation package Maple; our results are contained in this final paper and conclude the investigation into the symmetrizer.

The present paper begins with some preliminary material on the classical representation theory of the symmetric group: in Section 1, some aspects of the ordinary representation theory are discussed, including an alternative description of the Young symmetrizer given by Cherednik [1]; Section 2 establishes the projective representation theory and contains relevant information on the so-called “spin” representations. The combinatorial and algebraic ideas underlying the construction of our projective analogue of the Young symmetrizer appear in the third section; this enables us, in the following section, to concisely describe this construction. Section 5 examines projective analogues for the classical row symmetrizer $p_\lambda$ and column antisymmetrizer $q_\lambda$. The principal result (Theorem 6.5), proved in the final section, establishes a decomposition for the projective symmetrizer resembling the structure of the classical Young symmetrizer; this forms the foundation for further developments in the subsequent papers.

1. YOUNG SYMMETRIZERS FOR THE SYMMETRIC GROUP

In this opening section, the classical theory of Young symmetrizers is summarised. These objects were first described by Alfred Young [29] in 1901 and have a significant role in the representation theory of the symmetric group $S_n$ over the complex field $\mathbb{C}$. In particular, the Young symmetrizers are primitive idempotents in the group algebra $\mathbb{C}[S_n]$ labelled by partitions $\lambda \vdash n$; they are used to realize the irreducible representations of $S_n$ as minimal left ideals in $\mathbb{C}[S_n]$. The complete theory concerning Young symmetrizers appears in many standard texts, for example Weyl [28]. Additional references on the classical representation theory of the symmetric group include the texts [8, 20].
Let us recall that the symmetric group $S_n$ is the group of all permutations on symbols $1, 2, \ldots, n$ with the operation being the composition of mappings; here, we adopt the usual convention of multiplying permutations from right to left. It will be assumed that the reader is familiar with the combinatorial ideas such as partitions, Young diagrams and standard tableaux which are involved in the representation theory of the symmetric group. Many different approaches have been adopted to construct the irreducible representations of $S_n$ over the field $\mathbb{C}$: their unifying feature is the use of combinatorial techniques to associate an irreducible representation with each partition $\lambda \vdash n$. The ordinary representation theory was initially developed by Frobenius, but the greatest contribution to the early material came from an independent approach by Young based on the study of the group algebra $\mathbb{C}[S_n]$ and its idempotents. In the original work of Young, the symmetrizers were constructed as summations; more recently, Cherednik [1] has proposed a multiplicative description for these elements. The original papers by Young have been published in his collected works [30]; while Rutherford [21] has brought together this material in a modern account of Young's work.

Let us fix a partition $\lambda \vdash n$ and denote the number of standard $\lambda$-tableaux by $f_\lambda$. This integer specifies the degree of the irreducible representation associated with the partition $\lambda$. There are two well-known formulae for the number $f_\lambda$: the determinantal formula [8] was known to Frobenius and Young; the hook formula [3] was presented by Frame, Robinson, and Thrall in 1954. Note that there is a standard tableau with canonical form: the row tableau $\Lambda'$ is the $\lambda$-tableau obtained by inserting the symbols $1, 2, \ldots, n$ into the Young diagram by rows. The rows and columns in this row tableau describe a natural pair of Young subgroups in $S_n$; specifically, the row stabilizer $R_{\Lambda'}$ and the column stabilizer $C_{\Lambda'}$ are defined as the subgroups permuting the symbols within each row (respectively, column) of the tableau $\Lambda'$. The row stabilizer $R_{\Lambda'}$ is a Young subgroup of type $\lambda$, while the subgroup $C_{\Lambda'}$ is associated with the conjugate partition $\lambda'$. Illustrating this with the example $\lambda = (4, 2, 1)$, we have

\[
\Lambda' = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & & \\
7 & & & \\
\end{array}
\]

while the corresponding row and column stabilizers are

\[
R_{\Lambda'} = S_{\{1,2,3,4\}} \times S_{\{5,6\}} \times S_{\{7\}}, \quad C_{\Lambda'} = S_{\{1,5,7\}} \times S_{\{2,6\}} \times S_{\{3\}} \times S_{\{4\}}.
\]

Specify the row symmetrizer $p_\lambda$ and column antisymmetrizer $q_\lambda$ in the group algebra $\mathbb{C}[S_n]$ by

\[
p_\lambda = \sum_{r \in R_{\lambda'}} r, \quad q_\lambda = \sum_{c \in C_{\lambda'}} \text{sgn}(c) \cdot c.
\]
Then the Young symmetrizer $e_\lambda \in \mathbb{C}[S_n]$ is defined by the formula
\[
e_\lambda = \frac{f_\lambda}{n!} \cdot p_\lambda q_\lambda.
\] (1.1)

The present section describes only the basic facts in the theory of Young symmetrizers and follows the concise approach appearing in [4, Sect. 4.2]; the interested reader is referred to this text for further details, see also [28, Chap. IV]. The symmetrizer $e_\lambda$ is an idempotent in $\mathbb{C}[S_n]$ with right multiplication by $e_\lambda$ specifying a projection operator from the regular representation $\mathbb{C}[S_n]$ to an irreducible representation $V_\lambda$ of $S_n$. Furthermore, every irreducible representation of $S_n$ may be realized from some partition $\lambda \vdash n$ in this manner: this result will be stated formally in Theorem 1.3.

First we present some preliminary results. For any partition $\lambda \vdash n$, the symmetrizer $e_\lambda$ is a scalar multiple of a sum of certain permutations or their negatives: the sum taken over all permutations that can be expressed as a product $r \cdot c$. In particular, $e_\lambda \neq 0$ in the group algebra $\mathbb{C}[S_n]$.

**Lemma 1.1.** (a) $r \cdot p_\lambda = p_\lambda \cdot r = p_\lambda$ for all $r \in R_\lambda$;
(b) $\text{sgn}(c) c \cdot q_\lambda = q_\lambda \cdot \text{sgn}(c) c = q_\lambda$ for all $c \in C_\lambda$;
(c) the Young symmetrizer $e_\lambda$ satisfies the property
\[
r \cdot e_\lambda \cdot \text{sgn}(c) c = e_\lambda
\]
for all $r \in R_\lambda$, $c \in C_\lambda$; and, up to scalar multiplication, is the unique element in $\mathbb{C}[S_n]$ obeying this condition.

**Lemma 1.2.** (a) Suppose that $\lambda \succ \mu$ with respect to the lexicographic ordering $\succ$ on the partitions of $n$. Then $p_\lambda \cdot x \cdot q_\mu = 0$ for all $x \in \mathbb{C}[S_n]$; in particular, $e_\lambda \cdot e_\mu = 0$.
(b) The element $e_\lambda \cdot x \cdot e_\lambda$ is a scalar multiple of $e_\lambda$ for any $x \in \mathbb{C}[S_n]$; in particular, we have the equality $e_\lambda^2 = e_\lambda$.

That is, the symmetrizers $e_\lambda (\lambda \vdash n)$ are an orthogonal set of idempotents in $\mathbb{C}[S_n]$; the left ideals $V_\lambda = \mathbb{C}[S_n] \cdot e_\lambda$ generated by these idempotents are the subject of the next result.

**Theorem 1.3.** (a) $V_\lambda$ is an irreducible representation of $S_n$ for each partition $\lambda \vdash n$.
(b) Suppose that $\lambda \neq \mu$. Then the representations $V_\lambda$ and $V_\mu$ are non-isomorphic.

Since a distinct representation $V_\lambda$ is constructed for each partition $\lambda$ (conjugacy class in $S_n$) then they provide a complete set of irreducible representations for the symmetric group.

This opening section concludes with a brief discussion about an alternative description for the classical symmetrizer presented by Cherednik [1].
The original construction describes each Young symmetrizer $e_{\lambda}$ as a sum involving certain permutations or their negatives; more recently, a multiplicative description for $e_{\lambda}$ was realized via principal series representations of the degenerate affine Hecke algebra $H_{\mathcal{E}_{\theta}}$; see [1]. In particular, each symmetrizer can be constructed as the limiting value at a special point of a rational function in a real variable $v$ valued in $\mathbb{C}[S_n]$; see Proposition 1.4. This technique is called the fusion procedure.

We introduce some notation to describe the structure of the row tableau $\lambda'$. Let $(\lambda')^*$ denote the sequence of integers obtained by reading the symbols down the columns in $\lambda'$ from the left hand column to the right hand column; furthermore, for each $2 \leq j \leq n$, define $I_j$ as the subsequence in $(\lambda')^*$ consisting of all entries $i < j$ which occur before $j$ in this column sequence. Let us describe a collection of rational functions valued in the group algebra $\mathbb{C}[S_n]$. Suppose that $u$ and $u'$ are real variables. Then for any pair of integers $1 \leq i < j \leq n$, define the rational function $\epsilon_{i,j}$ by

$$\epsilon_{i,j}(u, u') = 1 + \frac{(i,j)}{u - u'},$$

where 1 is the identity in $\mathbb{C}[S_n]$ and $(i,j)$ denotes the transposition of the symbols $i$ and $j$.

Suppose that there are $l$ rows in the tableau $\lambda'$. Let $v$ be a real variable and assign a linear polynomial in $v$ to each entry in $\lambda'$ in the following manner: for each $1 \leq k \leq n$, put

$$u_k(v) = (q - p) + (l - p) v,$$

where $k = \lambda'(p, q)$. Here the integer $q - p$ is known as the content of the symbol $k$ in $\lambda'$. These integers describe the diagonals in $\lambda'$: the leading diagonal has symbols with zero content.

Now consider the rational function $e_{\lambda}$ in the variable $v$ defined by

$$e_{\lambda}(v) = \frac{f_{\lambda}}{n!} \cdot \prod_{j=2}^{n} \left( \prod_{i \in I_j} \epsilon_{i,j}(u_i, u_i) \right).$$

(1.2)

Note that the non-commuting factors $\epsilon_{i,j}$ appearing in (1.2) are arranged in the usual sense; that is, each product is written from left to right. The Young symmetrizer $e_{\lambda}$ is realized as the limiting value of the function (1.2) at $v = 0$; see [1, Theorem 1].

**Proposition 1.4.** The rational function $e_{\lambda}(v)$ has a limit at the point $v = 0$. This limiting value coincides with the Young symmetrizer $e_{\lambda}$ for each partition $\lambda \vdash n$.

In our example $\lambda = (4, 2, 1)$, the column sequence $(\lambda')^* = [1, 5, 7, 2, 6, 3, 4]$ determines subsequences $I_2 = [1], I_3 = [1, 2], I_4 = [1, 2, 3], I_5 = [1]$. 
Remark. (a) In the special cases \( \lambda = (n) \) and \( \lambda = (1^n) \), this approach gives the expressions

\[
e_{(n)} = \frac{1}{n!} \prod_{j=2}^{n} \left( \prod_{i=1}^{j-1} \left( 1 + \frac{(i \cdot j)}{j-i} \right) \right); \quad e_{(1^n)} = \frac{1}{n!} \prod_{j=2}^{n} \left( \prod_{i=1}^{j-1} \left( 1 - \frac{(i \cdot j)}{j-i} \right) \right).
\]

(b) The multiplicative description of the Young symmetrizer given in Proposition 1.4 had been suggested by Jucys [10]; see also [11] by the same author.

(c) The approach generalises to give a projective analogue of the symmetrizer; see Section 4.

2. SPIN REPRESENTATIONS OF THE SYMMETRIC GROUP

The projective representation theory of the symmetric group was developed by Issai Schur at around the same time as Young’s work on the ordinary representations. Schur had initially developed the general theory of projective representations for finite groups in [22, 23], before describing in his 1911 paper [24] all the basic details for the projective representations of the symmetric and alternating groups; these included a complete description of the irreducible projective characters complementing the corresponding results known for the ordinary characters. A class of symmetric functions introduced by Schur and subsequently called Schur \( Q \)-functions play a fundamental role in his approach. In the 1960s, a series of papers by Morris [13–15] presented an alternative approach using Clifford algebras and the so-called spin representations of orthogonal groups. More recently, methods developed by Nazarov [16] allow the explicit construction of every irreducible projective representation of \( S_n \).

In this second section, we summarise the classical results on the projective representations of the symmetric group \( S_n \). An excellent account of the projective representation theory is presented by Hoffman and Humphreys [5], while the combinatorics underlying this theory are examined in the exposition by Stembridge [26]. In the remarkable paper [24] introducing the

\[
I_6 = [1, 5, 2], \text{ and } I_7 = [1, 5]. \text{ Thus}
\]

\[
e_{(4,2,1)} = \lim_{v \to 0} \frac{35}{7!} \cdot e_{1,2}(2v + 1, 2v) e_{1,3}(2v + 2, 2v) e_{2,3}(2v + 2, 2v + 1) \\
\times e_{1,4}(2v + 3, 2v) e_{2,4}(2v + 3, 2v + 1) e_{3,4}(2v + 3, 2v + 2) \\
\times e_{1,5}(v - 1, 2v) e_{1,6}(v, 2v) e_{5,6}(v, v - 1) e_{2,6}(v, 2v + 1) \\
\times e_{1,7}(-2, 2v) e_{5,7}(-2, v - 1).
\]
projective representations of the symmetric and alternating groups, Schur computed the Schur multiplier $M(S_n)$ and gave presentations for the representation groups of $S_n$.

**Theorem 2.1 (Schur, 1911).** The Schur multiplier for the symmetric group $S_n$ is

$$M(S_n) = \begin{cases} 1 & \text{for } n \leq 3; \\ \mathbb{Z}_2 & \text{for } n \geq 4. \end{cases}$$

Let $\tilde{S}_n$ denote the group generated by the elements $t_1, t_2, \ldots, t_{n-1}$ together with a central involution $z$ subject to the relations

$$t_i^2 = z, \quad (t_i t_{i+1})^3 = z, \quad (t_i t_j)^2 = z \ (|i-j| \geq 2)$$

for all relevant $i, j$; and let $\hat{S}_n$ be the group generated by elements $t_1', t_2', \ldots, t_{n-1}'$ and a central involution $z'$ where

$$t_i'^2 = 1, \quad (t_i' t_{i+1}')^3 = 1, \quad (t_i' t_j')^2 = z' \ (|i-j| \geq 2).$$

**Theorem 2.2 (Schur, 1911).** For $n \geq 4$, the groups $\tilde{S}_n$ and $\hat{S}_n$ are representation groups for the symmetric group $S_n$ such that

(a) these groups are exactly the two non-isomorphic representation groups of $S_n$ for $n \neq 6$;

(b) when $n = 6$, $\tilde{S}_6 \cong \hat{S}_6$ is the unique (up to isomorphism) representation group of $S_6$.

Since the Schur multiplier $M(S_n)$ contains two cohomology classes for $n \geq 4$ then the ordinary representations $P$ of the representation group $\tilde{S}_n$ (or $\hat{S}_n$) split into two classes:

- $\{P \mid P(z) = I\}$, the ordinary (linear) representations of $S_n$;
- $\{P \mid P(z) = -I\}$, the so-called spin representations of $S_n$.

The spin representations are precisely those projective representations having factor sets which are non-cohomologous with the identity.

The irreducible linear representations of $S_n$ are parametrised by the partitions $\lambda \vdash n$. There is a similar result for the spin representations of $S_n$ involving the special class of strict partitions: a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is strict if $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$; that is, $\lambda$ has $l = l(\lambda)$ distinct parts. Let $SP(n)$ denote the set of all strict partitions of the integer $n$.

**Theorem 2.3 (Schur, 1911).** For each strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in SP(n)$, there exists an irreducible spin representation $P_\lambda$ of $S_n$ with degree given by

$$2^{(n-l)/2} \cdot \frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_l!} \cdot \prod_{1 \leq i < j \leq l} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$
where \([x]\) denotes the integer part of \(x\). If \(n - l\) is even, this representation \(P_\lambda\) is self-associate; whereas if \(n - l\) is odd then the associate representation \(P_\lambda^*\) provides a second irreducible spin representation of \(S_n\). Furthermore, ranging over all \(\lambda \in SP(n)\), the representations

\[
\{P_\lambda | n - l(\lambda) \text{ is even}\} \cup \{P_\lambda, P_\lambda^* | n - l(\lambda) \text{ is odd}\}
\]

are a complete set of inequivalent irreducible spin representations of the symmetric group \(S_n\).

In 1988, Nazarov [16] established a technique for explicitly constructing these irreducible spin representations by examining a representation group \(T_n\) which is isomorphic to the group \(\hat{S}_n\) introduced by Schur. The verification of this construction is given in a subsequent paper [17].

This section concludes with one further result concerning the spin representations of the symmetric group; this is a projective counterpart to the well-known result in the ordinary theory. The degree of the irreducible representation \(P_\lambda\) is given by the formula

\[
\dim(P_\lambda) = 2^{(|n - l(\lambda)|)/2} \cdot n_\lambda,
\]

where \(n_\lambda\) denotes the number of standard shifted tableaux of shape \(\lambda\). The expression for the integer \(n_\lambda\) appearing in Theorem 2.3 was known to Schur in 1911; but more recently, an analogue [15, Theorem 2.1] of the hook formula [3] gives an elegant calculation of this integer.

### 3. SOME PRELIMINARY IDEAS

This section presents the combinatorial and algebraic ideas involved in the construction of the projective analogue for the Young symmetrizer recently proposed by Nazarov in the remarkable paper [18]. The construction itself will appear in the following section. Additional references for the material in both these sections are contained in Nazarov’s paper.

The conventional approach towards the spin representations of the symmetric group using the “spin” groups introduced by Schur has been reviewed in the preceding section. However, Nazarov adopted an alternative approach to study these representations which exploits a relationship between the irreducible spin representations of \(S_n\) and a certain collection of irreducible projective representations for the hyperoctahedral group \(C_n\).

In particular, the description of a covering group \(D_n\) for \(C_n\) naturally describes a twisted group algebra \(M_n\), while a central involution \(\zeta\) in the group \(D_n\) plays a parallel role to the involution \(z\) in Schur’s classical theory. The method adopted by Cherednik [1] to study the ordinary representations of the symmetric group then generalises by introducing an analogue of the de-
generate affine Hecke algebra corresponding to $D_n$; this object is called the degenerate affine Sergeev algebra and denoted by $Se_n$.

In the opening part of this section, we present the combinatorial ideas required in the construction of the projective Young symmetrizer; these concepts apply to strict partitions and are analogous to the classical objects introduced by Young to develop the ordinary representation theory; see [8]. The standard reference for the following material is [12]. Let us fix a strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in SP(n)$.

**Definition 3.1.** The **shifted Young diagram** of shape $\lambda$ is the array of $n$ boxes into $l$ rows with $\lambda_i$ boxes in the $i$th row ($1 \leq i \leq l$) such that each row is shifted by one position to the right relative to the preceding row.

**Definition 3.2.** A **shifted tableau** of shape $\lambda$ is an array obtained by inserting the symbols $1, 2, \ldots, n$ bijectively into the $n$ boxes of the shifted Young diagram for $\lambda$. Furthermore, a shifted tableau is said to be **standard** if the symbols increase along each row (from left to right) and down each column.

Let $\mathcal{S}_\lambda$ denote the set of all standard shifted tableaux with shape $\lambda$; in particular, there are two distinguished standard tableaux in $\mathcal{S}_\lambda$ having special significance, namely

— the **row tableau** $\Lambda^r$ created by inserting the symbols $1, 2, \ldots, n$ consecutively by rows into the shifted diagram, and

— the **column tableau** $\Lambda^c$ where the symbols $1, 2, \ldots, n$ occur consecutively by columns.

Consider the example $\lambda = (4, 3, 1)$ where the row and column tableaux are

$$\Lambda^r = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 \\
8 &
\end{array}, \quad \Lambda^c = \begin{array}{ccc}
1 & 2 & 4 & 7 \\
3 & 5 & 8 \\
6 &
\end{array}.$$

The description for the projective symmetrizer uses sequences obtained from these special tableaux; however, we will present the following notation for an arbitrary standard tableau. Let $\Lambda \in \mathcal{S}_\lambda$ be fixed and denote by $\Lambda$ the sequence of integers obtained from this standard tableau by reading the symbols along the rows from left to right ordered from the top row to the bottom row; similarly, let $(\Lambda)^*$ denote the sequence determined from $\Lambda$ by reading the symbols down the columns taken from the left column to the right column. For each integer $2 \leq k \leq n$, let $\mathcal{A}_k$ and $\mathcal{A}'_k$ be the sub-sequences of $\Lambda$ consisting of all entries $j < k$ which occur respectively before and after $k$ in this sequence. Similarly, let $\mathcal{B}_k$ and $\mathcal{B}'_k$ denote the
subsequences in \((\Lambda)^*\) consisting of the entries \(j < k\) which occur respectively before and after \(k\) in the column sequence. Furthermore, let \(a_k, a'_k\) and \(b_k, b'_k\) be the lengths of the sequences \(\mathcal{A}_k, \mathcal{A}'_k\) and \(\mathcal{B}_k, \mathcal{B}'_k\), respectively.

There is a bijection between the set of all shifted \(\lambda\)-tableaux and the symmetric group \(S_n\) described in the following manner: given any shifted tableau \(\Lambda\), we define \(w_\Lambda \in S_n\) by

\[
w_\Lambda = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_n & p_{n-1} & \cdots & p_1 \end{pmatrix},
\]

where \(p_1, \ldots, p_n\) denote the ordered entries in the column sequence \((\Lambda)^*\); that is, the element \(w_\Lambda\) is constructed in permutation list notation by reversing the column sequence \((\Lambda)^*\). The remainder of this section now introduces the algebraic objects employed in the realization of the projective symmetrizer; specifically, we will describe a central \(\mathbb{Z}_2\)-extension \(D_n\) for the hyperoctahedral group \(C_n\).

**Definition 3.3.** The hyperoctahedral group \(C_n\) is the group of signed permutations on the symbols \(\pm 1, \pm 2, \ldots, \pm n\); that is, permutations \(\sigma\) with the constraint \(\sigma(\pm i) = -\sigma(\pm i)\) for all \(i\).

The hyperoctahedral group can be viewed as the semi-direct product \(S_n \rtimes \mathbb{Z}_2^n\), or equivalently as the wreath product \(\mathbb{Z}_2 \wr S_n\); the group \(C_n\) also has an interpretation as the Weyl group for the root system of type \(B_n\). A detailed account describing the projective representations of this group has been provided by Stembridge [27]. The Schur multiplier \(M(C_n)\) for the hyperoctahedral group was determined by Ihara and Yokonuma [6] in 1965.

**Proposition 3.4.**

\[
M(C_n) = \begin{cases} 
\mathbb{Z}_2 & \text{for } n = 2; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 & \text{for } n = 3; \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{for } n \geq 4.
\end{cases}
\]

Restricting consideration to the case \(n \geq 4\), the Schur multiplier is identified with the group of triples \((x, y, z)\) where \(x, y, z \in \{\pm 1\}\); that is, the eight cohomology classes for the projective representations of \(C_n\) \((n \geq 4)\) are labelled by \((\pm 1, \pm 1, \pm 1)\). In [18], Nazarov employed one particular class of representations, labelled by \((+1, +1, -1)\) in the conventional theory [27], by introducing a covering group \(D_n\) for the hyperoctahedral group; this is constructed as a semi-direct product of the symmetric group and a Clifford group.

We introduce the Clifford group \(Cl_n\) generated by the elements \(c_1, \ldots, c_n\) and the central involution \(\zeta\) subject to the relations

\[
c_i^2 = \zeta; \quad c_i c_j = \zeta c_j c_i \quad (j \neq i)
\]
for all possible $i$ and $j$. The symmetric group $S_n$ acts on these Clifford generators $c_1, \ldots, c_n$ via the permutation of their indices: $\sigma \cdot c_i = c_{\sigma(i)} \cdot \sigma$ for all $\sigma \in S_n$. Let $D_n$ denote the semi-direct product of the Clifford group $C_n$ by $S_n$ under this action; then $D_n$ is a covering group for a certain class of projective representations of $C_n$. We will consider those representations of $D_n$ where the central involution $\zeta \mapsto -I$ is represented faithfully. Let us introduce the appropriate factor algebra $M_n = C[D_n]/\langle \zeta = -1 \rangle$; this is generated as a $\mathbb{C}$-algebra by elements $c_1, \ldots, c_n$ subject to the above relations with $\zeta = -1$ and elements $s_1, \ldots, s_{n-1}$ obeying the standard relations for the symmetric group $S_n$ together with the relations imposed by the semi-direct product structure; see Lemma 4.1. Similarly, we construct an algebra from the Clifford group: define the Clifford algebra $Z_n = C[C_n]/\langle \zeta = -1 \rangle$ as the $\mathbb{C}$-algebra generated by the elements $c_1, \ldots, c_n$ where $\zeta = -1$. Note that the algebra $Z_n$ has dimension $2^n$ with a natural $\mathbb{C}$-basis $\mathcal{C}$ given by $\{c_{i_1}c_{i_2}\cdots c_{i_p} | 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}$.

The irreducible representations $\rho$ of the group $D_n$ such that $\rho(\zeta) = -I$ were parametrised in [27] by the pairs $(\lambda, \delta)$ where $\lambda \in SP(n)$ and $\delta = (\pm 1)^{(h)}$. A proof of construction of these representations is given in [18, Sect. 1] utilising the irreducible spin representations of the symmetric group; in particular, the irreducible representations $\rho_{\lambda, \delta}$ are determined from the spin representations by specializing a general theorem given in [2] for the irreducible representations of the semi-direct product of finite groups. This relationship between these two families of representations provides the basis for employing this approach to investigate the projective representations of the symmetric group.

This preliminary section is now completed by introducing the fundamental object underlying Nazarov's construction; namely, the degenerate affine Sergeev algebra $S_{n, n}$. This is an analogue for the group $D_n$ of the degenerate affine Hecke algebra $H_{n, n}$; see [1].

**Definition 3.5.** The degenerate affine Sergeev algebra $S_{n, n}$ is the associative algebra with identity generated over $M_n$ by pairwise-commuting elements $x_1, \ldots, x_n$ subject to the relations

\[
x_i s_i - s_i x_{i+1} = -1 - c_i c_{i+1}; \quad x_i c_i = -c_i x_{i};
\]
\[
x_{i+1} s_i - s_{i} x_{i+1} = 1 - c_i c_{i+1}; \quad x_j c_i = c_i x_j \quad (j \neq i);
\]
\[
x_j s_i = s_i x_j \quad (j \neq i, i + 1).
\]

The technique used by Cherednik to obtain the multiplicative description for the classical symmetrizer has been generalised in [18]: a projective analogue for the Young symmetrizer is determined by investigating the principal series representations of the Sergeev algebra $S_{n, n}$ relative to its maximal commutative subalgebra $C[x_1, \ldots, x_n]$. 
4. THE PROJECTIVE ANALOGUE FOR THE YOUNG SYMMETRIZER

In the opening section, we described a multiplicative presentation for the Young symmetrizer realized by Cherednik in [1] via the principal series representations for the degenerate affine Hecke algebra $\mathcal{H}_n$. This approach to the symmetrizer has been generalised in the remarkable paper “Young’s symmetrizers for projective representations of the symmetric group” [18]. The construction of this projective analogue for the classical symmetrizer is concisely described in the present section; the reader is referred to Nazarov’s paper [18] for further details, including proofs for the stated results; see also [19].

An element $\psi_\lambda \in M_n$ will be constructed for each standard shifted tableau $\lambda \in P_n$ as the limiting value of a certain function; note that the limiting procedure is technical and determines a complicated expression restricting further analysis. We give special attention to the element $\psi_{\lambda'}$ associated with the row tableau $\lambda'$ since it has a crucial role in the description of the projective symmetrizer $v_\lambda \in M_n$. It is conjectured that this symmetrizer is an idempotent; this result has been verified for the special class of one-part partitions $\lambda = (n)$. The first result gives a presentation for the twisted group algebra $M_n$ defined in Section 3.

**Lemma 4.1.** The algebra $M_n$ is an associative algebra with identity generated over $\mathbb{C}$ by the elements $s_1, \ldots, s_{n-1}$ and $c_1, \ldots, c_n$ subject to the following relations. First, the symmetric group relations

\[ s_i^2 = 1, \quad (s_is_{i+1})^3 = 1, \quad (s_is_j)^2 = 1 \quad (|i-j| \geq 2) \]

on the generators $s_1, \ldots, s_{n-1}$. Second, the elements $c_1, \ldots, c_n$ satisfy the Clifford relations

\[ c_i^2 = -1, \quad c_i c_j = -c_j c_i \quad (j \neq i). \]

Finally, there is the semi-direct action

\[ s_i c_i = c_{i+1} s_i, \quad s_i c_{i+1} = c_i s_i, \quad s_i c_j = c_j s_i \quad (j \neq i, i+1). \]

**Proof.** This presentation follows immediately from the definition. 

Suppose that $u$ and $u'$ are real numbers with $u \pm u' \neq 0$. For any integer $1 \leq k \leq n - 1$, define the element $\psi_k(u, u') \in M_n$ by

\[ \psi_k(u, u') = s_k + \frac{1}{u - u'} - \frac{c_k c_{k+1}}{u + u'}. \]

Each element will be actually considered as a rational function of the real variables $u, u'$ valued in the algebra $M_n$. This collection of elements satisfy some special relations.
Lemma 4.2. The rational functions $\psi_k(u, u')$ obey the relations

$$
\psi_k(u, u') \psi_l(v, v') = \psi_l(v, v') \psi_k(u, u'), \quad (|k - l| \geq 2);
$$

$$
\psi_k(u, u') \psi_{k+1}(u'', u') \psi_k(u''', u) = \psi_{k+1}(u'', u) \psi_k(u''', u') \psi_{k+1}(u, u')
$$

for all relevant $k, l$. Furthermore, for each $1 \leq k \leq n - 1$, we have the equality

$$
\psi_k(u, u')^2 = \frac{2}{u - u'} \psi_k(u, u') + \left(1 - \frac{1}{(u - u')^2} - \frac{1}{(u + u')^2}\right).
$$

Proof. These relations are established in [18] using Lemmas 4.4 and 4.5 in that paper.

Consider the following condition on the pair $(u, u')$:

$$
\frac{1}{(u + u')^2} + \frac{1}{(u - u')^2} = 1. \quad (4.1)
$$

This constraint (equivalent to the equality (4.11) in [18]) will be referred to as the idempotent condition on $(u, u')$ in view of the following consequence of Lemma 4.2.

Lemma 4.3. (a) Suppose that $(u, u')$ satisfies the condition (4.1) with $u \pm u' \neq 0$. Then the element $\psi_k(u, u')$ is an idempotent in the algebra $M_n$ up to some scalar factor, namely

$$
\psi_k(u, u')^2 = \frac{2}{u - u'} \psi_k(u, u').
$$

(b) Suppose that $(u, u')$ does not satisfy (4.1) and $u \pm u' \neq 0$ then $\psi_k(u, u')$ is invertible; in particular

$$
\psi_k(u, u')^{-1} = \left(1 - \frac{1}{(u - u')^2} - \frac{1}{(u + u')^2}\right)^{-1} \cdot \psi_k(u', u).
$$

Next, let us examine the product $\psi_k(u, u') \psi_{k+1}(u'', u') \psi_k(u''', u)$ on the left hand side of the third relation in Lemma 4.2. This element can be regarded as a function of the variables $u, u'$ and $u''$; it is well-defined only if $(u \pm u')(u \pm u')(u' \pm u') \neq 0$. Furthermore, on restriction to $(u, u', u'')$ such that the pair $(u, u')$ satisfies the idempotent condition (4.1) then the limit exists at $u'' = u'$; before stating this result formally, we present some additional notation. Suppose that $u \pm u' \neq 0$. Then for any integer $1 \leq k \leq n - 2$, define $\theta_k(u, u') \in M_n$ as

$$
\psi_k(u, u') \psi_{k+1}(u', u) + \psi_k(u, u') \left(-\frac{1}{(u - u')^2} + \frac{c_k c_{k+1}}{(u + u')^2} + \frac{c_{k+1} c_{k+2}}{u^2 - u'^2} + \frac{c_{k+2} c_k}{u^2 - u'^2}\right).
$$
As with \( \psi_k(u, u') \), the above expression is considered as a function on \( u, u' \) valued in \( M_n \).

**Lemma 4.4.** Suppose that \( (u, u') \) satisfies the equality (4.1) where \( u \pm u' \neq 0 \). Then the product \( \psi_k(u, u') \psi_{k+1}(u', u') \psi_k(u', u) \) has a limit at \( u' = u' \) equal to the value \( \theta_k(u, u') \).

**Proof.** See Lemma 5.1 in [18].

Given an \( l(\lambda) \)-tuple \( r = (r_1, \ldots, r_l) \) with each \( r_i \in [0, 1) \), define \( v(i, j) \) to be \( j - i + r_i \) and let

\[
u(i, j) = \sqrt{v(i, j)(v(i, j) + 1)}.
\]

Now create a shifted array of shape \( \lambda \) with real entries by placing the expression \( u(i, j) \) in the \( (i, j) \)-position of the array. Consider two adjacent entries \( u = u(i, j) \) and \( u' = u(i, j + 1) \) in some row of this array. Then \( v(i, j + 1) - v(i, j) = 1 \) and thus the pair \( (u, u') \) satisfies (4.1); furthermore, we have \( u \pm u' \neq 0 \). Now consider arbitrary entries \( u(i, j) \) and \( u(i', j') \): since the contents \( j - i \) and \( j' - i' \) are non-negative integers with \( 0 \leq r_i, r_i' < 1 \), it follows that

\[
\langle (i, j)^2 - u(i', j')^2 = 0 \iff j - i = j' - i' \text{ and } r_i = r_i'.
\]

We assign a function \( \psi_\lambda(r) \) with each standard shifted tableau \( \lambda \) of shape \( \lambda \): let \( \lambda \in \mathcal{S}_\lambda \) be fixed and define

\[
\psi_\lambda(r) = \prod_{k=2}^n \prod_{p=1}^{b_k} \psi_{k-p}(u_k, u_{\beta_k(p)}),
\]

where \( u_k = u(i, j) \) for \( k = \lambda(i, j) \). This element is regarded as a function of the \( l \)-tuple \( r \); it is well-defined provided that the components \( r_i \in r \) are all distinct, since condition (4.2) then shows that \( u_k \pm u_k' \neq 0 \) for all \( k \neq k' \).

It is natural to introduce the set of degenerate vectors

\[
\Delta = \{(r_1, \ldots, r_l) \mid r_i = r_j \text{ for some } i \neq j; \ 0 \leq r_i < 1\}.
\]

Employing this notation, the element \( \psi_\lambda(r) \) is well-defined for all \( r \not\in \Delta \); meanwhile, Nazarov has demonstrated that this function does not have any singularities within the set \( \Delta \). This continuation of the function \( \psi_\lambda(r) \) onto the set \( \Delta \) is called the **fusion procedure**.

**Theorem 4.5** [18, Theorem 5.6]. The function \( \psi_\lambda(r) \) has a non-zero limit at each \( r \in \Delta \).

The limiting procedure presented in Nazarov’s constructive proof realizes a complicated expression for each limit; a complete description of this procedure appears in [18].
Theorem 4.5 is only required at the point $r = 0$; in particular, define the element $\psi_3 \in M_n$ as the limiting value of $\psi_3(r)$ at this special point. The collection of elements $\psi_3 (\Lambda \in \mathcal{P}_n^\prime)$ play a principal role in Nazarov’s paper. We will give special attention to the element $\psi_3 \in M_n$ associated with the row tableau $\Lambda'$ since this expression is fundamental to the construction of our projective symmetrizer. The limiting values of the scalars $u(i, j)$ are denoted by $u_0(i, j) = \sqrt{j - i) (j - i + 1)}$; note that each value only depends on the content $j - i$ of the position $(i, j)$ in the array.

In the special case $\lambda = (n)$, there exists only one standard tableau, namely the row tableau and the appropriate limit can be evaluated immediately giving

$$\psi_3 = \prod_{k=2}^{n} \prod_{j=1}^{k-1} \psi_{k-j} \left( \frac{\sqrt{(k-1)k}}{\sqrt{(j-1)j}} \right). \quad (4.4)$$

Unfortunately, the limiting procedure for general partitions is substantially more complicated than the simple technique used here; although an explicit expression for any $\psi_3 (\Lambda \in \mathcal{P}_n^\prime)$ can always be obtained via the construction given in [18]. For example, the elements $\psi_3$ and $\psi_3$ associated with the row and column tableaux, respectively, for $\lambda = (4, 3, 1)$ are given by

$$\psi_3 = \frac{\sqrt{2}}{2} \cdot \theta_1(\sqrt{2}, 0) \psi_3(\sqrt{6}, 0) \psi_4(\sqrt{2}, 0) \theta_2(\sqrt{6}, \sqrt{2}) \theta_4(\sqrt{2}, 0)$$
$$\times \psi_3(0, \sqrt{6}) \theta_3(\sqrt{2}, 0) \psi_6(\sqrt{12}, 0) \psi_5(\sqrt{12}, \sqrt{2}) \psi_7(\sqrt{6}, 0)$$
$$\times \psi_6(\sqrt{6}, \sqrt{2}) \theta_4(\sqrt{12}, \sqrt{6}) \psi_3(\sqrt{6}, 0) \psi_2(\sqrt{6}, \sqrt{2}) \psi_7(\sqrt{6}, 0)$$
$$\times \psi_4(\sqrt{12}, 0) \psi_3(\sqrt{12}, \sqrt{2}) \psi_2(\sqrt{12}, 0) \psi_6(\sqrt{6}, 0);$$

$$\psi_3 = \frac{9}{2} \cdot \psi_4(0, \sqrt{12}) \psi_3(0, \sqrt{6}) \psi_5(\sqrt{2}, \sqrt{12}) \psi_7(0, \sqrt{6}) \psi_6(0, \sqrt{12}) \cdot \psi_3.$$  

This example clearly illustrates that elaborate patterns can appear in the expressions for these elements; this complexity restricts further analysis even for some elementary partitions. In the second paper of this series, we will present an alternative procedure for dealing with the singular factors in the function $\psi_3(r)$ when $\lambda$ has a special form; the expression for $\psi_3$, given by this new approach is both easier to compute and it illustrates a natural generalisation of the $p_2 \cdot q_3$-structure for the classical Young symmetrizer. The following results appear in [18] as Proposition 5.7 (the special case $\Lambda = \Lambda'$ in the limit $r \to 0$) and Theorem 6.3, respectively.

**Proposition 4.6.** Suppose that $k = \Lambda(i, j)$ and $k + 1 = \Lambda(i, j + 1)$ occupy the same row in the tableau $\Lambda'$. Then $\psi_3(u_k, u_{k+1}) \cdot \psi_3 = 0$.

**Proposition 4.7.** Suppose that $k = \Lambda(i, j)$ and $k + 1 = \Lambda(i + 1, j)$ occupy the same column in the tableau $\Lambda'$. Then $\psi_3(u_k, u_{k+1}) \cdot \psi_3 = 0$ and $\psi_3 \cdot \psi_3(u_k, u_{k+1}) = 0$.  

This is an appropriate point to outline the significance of the row element \( \psi_{\lambda'} \) in the representation theory of the degenerate affine Sergeev algebra \( \mathcal{S}_n \); see [18, Sect. 7]. Given any \( \lambda \in SP(n) \), define \( V_\lambda = M_n \cdot \psi_{\lambda'} \) as the left ideal in \( M_n \) generated by the element \( \psi_{\lambda'} \). This ideal forms an \( \mathcal{S}_n \)-submodule in a certain principal series representation of the algebra \( \mathcal{S}_n \). The space \( V_\lambda \) is reducible as a module for the subalgebra \( M_n \subset \mathcal{S}_n \); its decomposition into irreducible components of type \( \rho_{\lambda, \delta} \) is described by the next result.

**Theorem 4.8 [18, Theorem 8.3].** The representation of the group \( D_n \) in \( V_\lambda \) decomposes as

\[
V_\lambda \cong \begin{cases} 
\oplus 2^{l(\lambda) - 1} \text{ copies of } \rho_{\lambda, 1} & \text{ if } l(\lambda) \text{ is even;} \\
\oplus 2^{l(\lambda) - 1} \text{ copies of } \rho_{\lambda, 1} \oplus \rho_{\lambda, -1} & \text{ if } l(\lambda) \text{ is odd.}
\end{cases}
\]

The section now concludes with a description for a projective analogue \( v_\lambda \) of the classical symmetrizer. This analogue is known to be idempotent in the special case \( \lambda = (n) \); this result is conjectured for any strict partition \( \lambda \). The next definition first appeared in [18].

**Definition 4.9.** For any \( \lambda \in SP(n) \), define the projective symmetrizer \( v_\lambda \in M_n \) by

\[
v_\lambda = \frac{n_\lambda}{n!} \cdot \psi_{\lambda'} w_{\lambda'}^{-1},
\]

where \( n_\lambda \) denotes the number of standard shifted tableaux with shape \( \lambda \).

Since the element \( \psi_{\lambda'} \) can be expanded as a sum with leading term \( w_{\lambda'} \), then

\[
v_\lambda = \frac{n_\lambda}{n!} + \sum_{\sigma \in S_n \setminus \{1\}} c_\sigma \cdot \sigma \quad (4.5)
\]

for some \( c_\sigma \in \mathbb{Z}_n \). In the example \( \lambda = (n) \), the symmetrizer \( v_{(n)} \) can be described explicitly as

\[
v_{(n)} = \frac{1}{n!} \cdot \prod_{k=2}^{n} \left( \prod_{j=1}^{k-1} \left( 1 + \frac{(j \ k)}{u_k - u_j} - \frac{c_j c_k (j \ k)}{u_k + u_j} \right) \right),
\]

where, in this special instance, the scalars are given by \( u_l = \sqrt{l(l-1)} \) for each \( 1 \leq l \leq n \). Returning to the general case, let \( Z_\lambda \) denote the subalgebra in \( \mathbb{Z}_n \) generated by the Clifford elements \( c_{\lambda(1,1)}, \ldots, c_{\lambda((\lambda), (\lambda))} \) corresponding to the leading diagonal in the row tableau. The next theorem has been established in [18].

**Theorem 4.10 [18, Theorem 9.2].** Given any \( \lambda \in SP(n) \), the symmetrizer \( v_\lambda \) satisfies the equality \( v_\lambda^2 = z_\lambda v_\lambda \) for some non-zero element \( z_\lambda \in Z_\lambda \).
In the special case $\lambda = (n)$, this result can be strengthened:

**Corollary 4.11 [18, Corollary 9.3]** The symmetrizer $v_{(n)}$ is an idempotent in $M_n$.

Furthermore, it is conjectured that the equality $z_\lambda = 1$ is satisfied for any strict partition $\lambda$. The final result on the projective symmetrizer $v_\lambda$ in this section should be compared with Proposition 4.6 and Proposition 4.7 for the elements $\psi_\lambda$ and $\psi_{\lambda'}$, respectively.

**Theorem 4.12.** (a) Suppose that $k = \lambda'(i, j)$ and $k + 1 = \lambda'(i, j + 1)$ are adjacent entries in the same row of the tableau $\lambda'$. Then

$$
(k \ k + 1) - \frac{1}{u_{k+1} - u_k} - \frac{c_k c_{k+1}}{u_{k+1} + u_k} \cdot v_\lambda = 0.
$$

(b) Suppose that $k = \lambda'(i, j)$ and $l = \lambda'(i + 1, j)$ are adjacent entries in the same column of the tableau $\lambda'$. Then

$$
v_\lambda \cdot (k \ l) + \frac{1}{u_k - u_l} + \frac{c_k c_l}{u_k + u_l} = 0.
$$

**Proof.** See Theorem 9.5 in [18].

This result implies that the symmetrizer $v_\lambda$ is divisible by certain elements in $M_n$; an idea developed formally in Section 6. It will provide a decomposition for $v_\lambda$ analogous to the $p_\lambda q_\lambda$-structure of the Young symmetrizer. This description gives an alternative construction with the advantage that the element should be easier to determine as a combinatorial object.

5. THE ELEMENTS $P_\lambda$ AND $Q_\lambda$

The construction for the projective symmetrizer $v_\lambda$ presented in the preceding section involves some technical concepts; the element $\psi_{\lambda'}$ is realized by an elaborate limiting procedure and the resulting complicated expression for $v_\lambda$ generally prohibits further analysis of the problem. Therefore, the natural idea is to inquire whether the $p_\lambda q_\lambda$-structure exhibited by the Young symmetrizer has any generalisation to the projective case; this question will be answered in the remaining sections by a decomposition of the element $v_\lambda$ into a standard form resembling the classical structure. In this section, we introduce elements $P_\lambda$ and $Q_\lambda$ in the algebra $M_n$ regarded as projective analogues of the row symmetrizer $p_\lambda$ and column antisymmetrizer $q_\lambda$, respectively. It is proved that these elements are idempotents up to certain integer scalars; this fundamental result demonstrates an obvious analogy with the classical situation.

We begin this section by describing a family of elements in the algebra $M_n$ behaving in a similar manner to the elements $\psi_k(u, u')$ considered in
Section 4. These new elements are labelled by pairs of collinear entries or cocolumnar entries in the row tableau \(\Lambda^r\) and have a fundamental role in our subsequent investigation into the structure of the symmetrizer \(v_\lambda\). In particular, let us suppose that the symbols \(j\) and \(k\) appear in the same row or the same column in \(\Lambda^r\). Then \(u_k \pm u_j \neq 0\) and we may introduce the element \(\phi(k, j) \in M_n\) by

\[
\phi(k, j) = 1 + \frac{(j \; k)}{u_k - u_j} - \frac{c_j c_k (j \; k)}{u_k + u_j}.
\]

In the remaining analysis, we take the limiting values for the scalars \(u_1, \ldots, u_n\) described in Section 4 since the elements \(\phi(k, j)\) are employed subsequent to the realization of the limit \(r \to 0\); this has particular significance for the \(\phi\)-terms associated with cocolumnar entries.

The \(\phi\)-terms have a close relationship with the \(\psi\)-terms examined in the preceding section; indeed, the element \(\phi(k, j)\) on any adjacent pair of symbols in \(\Lambda^r\) can be expressed in the established notation using the following equalities.

**Lemma 5.1.** (a) Suppose that \(k\) and \(k + 1\) are adjacent entries in the same row of \(\Lambda^r\). Then

\[
\phi(k + 1, k) = \psi_k(u_{k+1}, u_k) \cdot s_k; \quad \phi(k, k + 1) = s_k \cdot \psi_k(u_k, u_{k+1}).
\]

(b) Suppose that \(j < k\) are adjacent entries in the same column of \(\Lambda^r\). Then

\[
\phi(k, j) = w_{\Lambda^r}(s_p \cdot \psi_p(u_k, u_j)) w_{\Lambda^r}^{-1}; \quad \phi(j, k) = w_{\Lambda^r}(\psi_p(u_j, u_k) \cdot s_p) w_{\Lambda^r}^{-1},
\]

where the integer \(p\) is given by \(w_{\Lambda^r}(p) = k\).

**Proof.** The equalities in (a) follow directly from the definitions; the results in (b) are less obvious and use a property of the permutation \(w_{\Lambda^r}\). Since the symbols \(j\) and \(k\) are adjacent in the column sequence \((\Lambda^r)^*\) then \(k = w_{\Lambda^r}(p)\) and \(j = w_{\Lambda^r}(p + 1)\) for some positional integer \(p\); the stated correspondences follow.

The correspondence described in Lemma 5.1 is reflected in the next result; cf. Lemma 4.2.

**Lemma 5.2.** The elements \(\phi(k, j)\) obey the following relations in the algebra \(M_n\),

\[
\phi(k, j) \phi(j, k) = 1 - \frac{1}{(u_k - u_j)^2} - \frac{1}{(u_k + u_j)^2};
\]

\[
\phi(k, j) \phi(k', j') = \phi(k', j') \phi(k, j), \quad \text{if } j, k, j', k' \text{ are distinct};
\]

\[
\phi(i, j) \phi(k, j) \phi(k, i) = \phi(k, i) \phi(k, j) \phi(i, j), \quad \text{if } i, j, k \text{ are distinct}
\]
for all relevant values of the arguments. Furthermore, there is the equality
\[ \phi(k, j)^2 = 2 \phi(k, j) - \left( 1 - \frac{1}{(u_k - u_j)^2} - \frac{1}{(u_k + u_j)^2} \right). \]

**Proof.** These relations are obtained by a straightforward calculation; see [9, Lemma 4.1.3].

It is known that the element \( \psi_k(u, u') \) is either invertible or idempotent (up to a scalar multiple) depending on the real-valued arguments \( u \) and \( u' \); see Lemma 4.3. A parallel result for the elements \( \phi(k, j) \) is stated as a consequence of Lemma 5.2.

**Lemma 5.3.** (a) Suppose that the entries \( j \) and \( k \) occupy adjacent positions in some row or column of \( \Lambda' \). Then \( \phi(k, j) \) is idempotent up to a scalar: \( \phi(k, j)^2 = 2 \phi(k, j) \).

(b) Suppose that the entries \( j \) and \( k \) are not adjacent within some row or column of \( \Lambda' \). Then \( \phi(k, j) \) is invertible in the algebra \( M_n \); its inverse is given by
\[ \phi(k, j)^{-1} = \left( 1 - \frac{1}{(u_k - u_j)^2} - \frac{1}{(u_k + u_j)^2} \right)^{-1} \cdot \phi(j, k). \]

**Proof.** (a) Since the symbols \( j \) and \( k \) are adjacent within some row or column of \( \Lambda' \) then the pair of scalars \( (u_j, u_k) \) satisfies the idempotent condition (4.1) and the stated result follows from the last equality in Lemma 5.2. (b) Here the pair \( (u_j, u_k) \) does not satisfy the idempotent condition and the first relation in Lemma 5.2 gives the required result.

Note that Lemma 5.3(a) follows since the scalars \( u_1, \ldots, u_n \) are considered in the limit \( r \to 0 \). Indeed any pair \( (u_j, u_k) \) corresponding to adjacent column entries only satisfies the equality (4.1) on these limiting values, although (4.1) is obeyed by adjacent row symbols for any non-limiting scalars as well.

Next we will introduce two elements in the algebra \( M_n \) appearing in our decomposition of the symmetrizer \( v_{\lambda} \). These are defined for any strict partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in SP(n) \).

**Definition 5.4.** (a) Define the row element \( P_{\lambda'} \in M_n \) as the product
\[ P_i = \prod_{j=i+1}^{\lambda_i+i-1} \left( \prod_{j'=i}^{j-1} \phi(\Lambda'(i, j), \Lambda'(i, j')) \right) \]
for each row index \( 1 \leq i \leq l \).
(b) Define the column element $Q_{3r} \in M_n$ as the product $Q_1 Q_2 \cdots Q_{l'}$ where $l' = \lambda_1$ and
\[ Q_j = \prod_{i=2}^{l'} \left( \prod_{i=1}^{j-1} \phi(\lambda'(i, j), \lambda'(i', j)) \right) \]
for each column index $1 \leq j \leq l'$; here the integer $\lambda'_j$ is the height of column $j$ in $\lambda'$.

**Remark.** Since the $\phi$-terms are generally non-commutative then the ordering within these products is significant: unless stated otherwise, our notation will assume the usual convention for ordering factors; that is, each product is written from left to right. There are instances when a reversed ordering is indicated by a reversal in the product indices.

This definition is illustrated with an elementary example: the row tableau for the partition $\lambda = (4, 3, 1)$ has been constructed in Section 3 and gives elements
\[ P_{\lambda'} = \phi(2, 1) \phi(3, 1) \phi(3, 2) \phi(4, 1) \phi(4, 2) \phi(4, 3) \]
\[ \times \phi(6, 5) \phi(7, 5) \phi(7, 6); \]
\[ Q_{\lambda'} = \phi(5, 2) \phi(6, 3) \phi(8, 3) \phi(8, 6) \phi(7, 4). \]

The subsequent results will demonstrate that the elements $P_{\lambda'}$ and $Q_{\lambda'}$ in the algebra $M_n$ may be considered as projective analogues for the classical elements $p_{\lambda'}$ and $q_{\lambda'}$, respectively in the group algebra $\mathbb{C}[S_n]$. We proceed by using the relations in Lemma 5.2 to obtain alternative descriptions for the projective elements; meanwhile, some quite elementary results established below have crucial significance in several aspects of the later analysis.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in SP(n)$ be fixed. Note that the components $P_1, \ldots, P_l$ in the row element $P_{\lambda'}$ are pairwise commutative, since any two products $P_i$ and $P_j$ ($i \neq j$) are determined by different rows in $\lambda'$ with their factors evaluated on disjoint sets of symbols. The column components $Q_1, \ldots, Q_{l'}$ in $Q_{\lambda'}$ are pairwise commutative by the same reasoning. Some alternative presentations for the elements $P_{\lambda'}$ and $Q_{\lambda'}$ are given by taking different orderings within these components $P_j$ and $Q_j$. Here it will be convenient to adopt some generalised notation to describe the rows and columns in the tableau $\lambda'$. Let $r_i$ ($1 \leq i \leq l$) and $c_j$ ($1 \leq j \leq l'$) respectively denote row $i$ and column $j$ in $\lambda'$; the $m$ entries in any given row or column are denoted by $a_1, a_2, \ldots, a_m$ in accordance with the graphical presentation
\[
\begin{array}{c}
1 \quad 2 \quad \cdots \quad m \\
r_i & = & \begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
a_m
\end{array}
\end{array}
\quad \text{or} \quad
\begin{array}{c}
1 \quad 2 \quad \cdots \quad m \\
c_j & = & \begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
a_m
\end{array}
\end{array}
\]
Consider a fixed row $r_i$ in $\Lambda'$ with the component $P_i$ in the row element $P_{\Lambda'}$ given by

$$P_i = \prod_{k=2}^{m} \left( \prod_{k' = 1}^{k-1} \phi(a_k, a_{k'}) \right).$$  \hspace{1cm} (5.1)

The next result proposes equivalent expressions for this component.

**Lemma 5.5.** The component $P_i$ associated with row $r_i$ in $\Lambda'$ has the alternative descriptions:

$$P_i = \prod_{k=1}^{m-1} \left( \prod_{k' = 1}^{m} \phi(a_k, a_{k'}) \right); \hspace{1cm} (5.2)$$

$$P_i = \prod_{k=m}^{2} \left( \prod_{k' = k-1}^{1} \phi(a_k, a_{k'}) \right). \hspace{1cm} (5.3)$$

**Proof.** These equalities are verified by induction on the number $m$ of entries in the row $r_i$. Clearly the result holds when $m = 2$ since the expressions in (5.1), (5.2), and (5.3) determine $P_i$ equivalently as the single factor $\phi(a_2, a_1)$. Let us assume the alternative descriptions on any row with length $m - 1$; we will obtain the formulae (5.2) and (5.3) for the row $r_i$ separately.

(a) Deleting the entry $a_m$ in row $r_i$ creates a row with length $m - 1$; the inductive hypothesis then enables the component $P_i$ to be expressed as

$$m-2 \prod_{k' = 1}^{m-1} \left( \prod_{k' = 1}^{m} \phi(a_k, a_{k'}) \right) \cdot \prod_{k' = 1}^{m-1} \phi(a_m, a_{k'}).$$

The factor $\phi(a_m, a_1)$ commutes with each term $\phi(a_k, a_{k'})$ where $2 \leq k' < k \leq m - 1$ by the second relation in Lemma 5.2 giving

$$\prod_{k=2}^{m} \phi(a_k, a_1) \cdot \prod_{k'=2}^{m-2} \left( \prod_{k' = k+1}^{m-1} \phi(a_k, a_{k'}) \right) \cdot \prod_{k'=2}^{m-1} \phi(a_m, a_{k'}).$$

Repeating this procedure on each factor in the final product yields the expression (5.2).

(b) In the second instance, the induction hypothesis gives the element $P_i$ as

$$\prod_{k=m-1}^{2} \left( \prod_{k'=k+1}^{1} \phi(a_k, a_{k'}) \right) \cdot \prod_{k'=1}^{m-1} \phi(a_m, a_{k'}).$$

Each factor in the double product on the left can be transferred across the right hand product by the second and third relations in Lemma 5.2. For instance, translating the term $\phi(a_2, a_1)$ interchanges the factors...
\( \phi(a_m, a_1) \) and \( \phi(a_m, a_2) \) in the latter product. This procedure can be formally described by placing the second arguments in the factors of the right hand product into a sequence; initially the product is represented by \([a_1, a_2, \ldots, a_{m-1}]\). On transferring the \( \phi(a_2, a_1) \) term through the product, this sequence is modified to \([a_2, a_1, a_3, \ldots, a_{m-1}]\). The next step moving \( \phi(a_3, a_1) \) interchanges the entries \( a_1 \) and \( a_3 \). This procedure continues for each factor in the double product: it is observed that the sequence is ultimately reversed to \([a_{m-1}, \ldots, a_1]\); thus the ordering of factors is reversed giving the expression (5.3).

The preceding result can be reformulated for any column component \( Q_j \) since its factors behave in precisely the same manner as those in the component \( P_i \). Specifically, let us now fix a column \( c_j \) in the tableau \( \Lambda^r \) describing its \( m \) entries using the standardised notation; then the component \( Q_j \) in the column element \( Q_{3r} \) is given by

\[
Q_j = \prod_{k=2}^{m} \left( \prod_{k'=1}^{k-1} \phi(a_k, a_{k'}) \right).
\]

This description is identical to (5.1) for the row component \( P_i \) although obviously these expressions are defined on different sets of entries \( \{a_1, \ldots, a_m\} \). Thus Lemma 5.5 has an immediate counterpart giving some alternative presentations for the column element \( Q_{X'} \).

**Lemma 5.6.** The component \( Q_j \) associated with column \( c_j \) in \( \Lambda^r \) has alternative descriptions

\[
Q_j = \prod_{k'=1}^{m-1} \left( \prod_{k=k'+1}^{m} \phi(a_k, a_{k'}) \right); \quad Q_j = \prod_{k=m}^{2} \left( \prod_{k'=k-1}^{1} \phi(a_k, a_{k'}) \right).
\]

The next result has similarities with Theorem 4.12 for the projective symmetrizer \( v_\lambda \).

**Proposition 5.7.** (a) Suppose that \( k = \Lambda^r(i, j) \) and \( k + 1 = \Lambda^r(i, j + 1) \) are adjacent entries within some row in \( \Lambda^r \). Then

\[
(i) \left( (k + 1) - \frac{1}{u_{k+1} - u_k} \right) \cdot P_{\Lambda^r} = 0;
\]

\[
(ii) \ P_{\Lambda^r} \cdot \left( (k + 1) - \frac{1}{u_{k+1} + u_k} \right) = 0.
\]

(b) Suppose that \( k = \Lambda^r(i, j) \) and \( l = \Lambda^r(i + 1, j) \) are adjacent entries within some column in \( \Lambda^r \). Then

\[
(i) \left( (l) + \frac{1}{u_k - u_l} \right) \cdot Q_{\Lambda^r} = 0;
\]

\[
(ii) \ Q_{\Lambda^r} \cdot \left( (l) + \frac{1}{u_k + u_l} \right) = 0.
\]
Proof. (a) Using the relations in Lemma 5.2, the element $P_{\lambda'}$ can take either of the forms

$$P_{\lambda'} = \phi(k + 1, k) \cdot P', \quad P_{\lambda'} = P'' \cdot \phi(k + 1, k)$$

(5.5)

for some elements $P', P'' \in M_n$. Since $k$ and $k + 1$ are adjacent entries in $\Lambda'$ then $(u_{k+1}, u_k)$ satisfies the condition (4.1); therefore $\phi(k + 1, k) \cdot \phi(k, k + 1) = \phi(k, k + 1) \cdot \phi(k + 1, k) = 0$. From the descriptions in (5.5), it then follows that

$$\phi(k + 1, k) \cdot P_{\lambda'} = 0, \quad P_{\lambda'} \cdot \phi(k + 1, k) = 0.$$

The stated equalities are given via an appropriate multiplication by the transposition $(k k + 1)$.

(b) The parallel results for the column element $Q_{\lambda'}$ are verified in exactly the same manner, since the product $Q_{\lambda'}$ can be written in either of the forms

$$Q_{\lambda'} = \phi(l, k) \cdot Q', \quad Q_{\lambda'} = Q'' \cdot \phi(l, k)$$

(5.6)

for certain elements $Q', Q'' \in M_n$ where the scalars $(u_l, u_k)$ satisfy condition (4.1).

Proposition 5.7 in this form should be compared with Theorem 4.12 for the projective symmetrizer $v_{\lambda'}$; this similarity will be exploited in Section 6 to prove that the symmetrizer is divisible by the elements $P_{\lambda'}$ and $Q_{\lambda'}$. Before we proceed any further in the present section, it will be useful to give an alternative interpretation of the results in Proposition 5.7. An equivalent statement of the result appears below; here we observe comparisons with Lemma 1.1 in the classical case. This justifies our claim that the elements $P_{\lambda'}$ and $Q_{\lambda'}$ are projective analogues of the row symmetrizer $p_{\lambda'}$ and column antisymmetrizer $q_{\lambda'}$ in the algebra $\mathbb{C}[S_n]$.

Corollary 5.8. (a) Suppose that $k$ and $k + 1$ are adjacent entries within a row in $\Lambda'$. Then

(i) $(k k + 1) \cdot P_{\lambda'} = \left( \frac{1}{u_{k+1} - u_k} + \frac{c_k c_{k+1}}{u_{k+1} + u_k} \right) \cdot P_{\lambda'}$;

(ii) $P_{\lambda'} \cdot (k k + 1) = P_{\lambda'} \cdot \left( \frac{1}{u_{k+1} - u_k} - \frac{c_k c_{k+1}}{u_{k+1} + u_k} \right)$.

(b) Suppose that $k < l$ are adjacent entries in the same column in $\Lambda'$. Then

(i) $(k l) \cdot Q_{\lambda'} = \left( -\frac{1}{u_k - u_l} + \frac{c_k c_l}{u_k + u_l} \right) \cdot Q_{\lambda'}$;

(ii) $Q_{\lambda'} \cdot (k l) = Q_{\lambda'} \cdot \left( -\frac{1}{u_k - u_l} - \frac{c_k c_l}{u_k + u_l} \right)$. 
The elementary nature of Corollary 5.8 conceals a powerful result: it is fundamental in both the theoretical developments contained in the present paper and in the more practical aspects of the investigation discussed in the final paper of this series.

The remainder of this section establishes that the projective elements $P_{3r}$ and $Q_{3r}$ are idempotents in $M_n$ up to certain integer scalars. This result will be stated as Theorem 5.12; it utilises a collection of equalities generated by two preliminary lemmata. We describe the rows and columns in the row tableau $\Lambda'$ using the standardised notation presented earlier; in particular, the symbols $a_1, \ldots, a_m$ denote the entries in a given row or column of $\Lambda'$.

The following preliminary result will be required in the subsequent proofs.

**Lemma 5.9.** Let $1 \leq r, s \leq m$ be integers with $s \geq r + 2$. For each $1 \leq k < s - 1$, we have

$$\left( \frac{1}{u_{a_k} - u_{u_{a_k}} + u_{u_{a_k}} + u_{a_k}} \right) \left( \frac{1}{u_{a_k} - u_{a_k} + u_{a_k} + u_{a_k}} \right) \left( \frac{1}{u_{a_k} - u_{a_k} + u_{a_k} + u_{a_k}} \right) \left( \frac{1}{u_{a_k} - u_{a_k} + u_{a_k} + u_{a_k}} \right) = \left( \frac{1}{u_{a_k} - u_{a_k} + u_{a_k} + u_{a_k}} \right) \left( \frac{1}{u_{a_k} - u_{a_k} + u_{a_k} + u_{a_k}} \right).$$

**Proof.** Each equality is verified by expanding both sides and collecting the terms. 

The next result describes a collection of elements in the algebra $M_n$ preserving $P_{3r}$ under left multiplication; here the symbols $a_1, \ldots, a_m$ denote the entries in a fixed row in $\Lambda'$.

**Lemma 5.10.** The row element $P_{3r}$ is invariant under left multiplication by the element

$$\prod_{k=r+1}^{s-1} \left\{ \frac{1}{u_{a_k} - u_{a_k} + u_{a_k} + u_{a_k}} \right\} (a_{r_k} a_{k})$$

$$\times \left( \frac{1}{u_{a_k} - u_{a_k} + u_{a_k} + u_{a_k}} \right) (a_{r_k} a_{k})$$

for any choice of integers $1 \leq r < s \leq m$.

**Proof.** We expand the product appearing in (5.7) by distributing this product over the sums in its $s - r - 1$ factors via expansions of the type $(1 - m_\alpha)(1 - m_\beta) = 1 - m_\alpha - m_\beta + m_\alpha m_\beta$ for $m_\alpha, m_\beta \in M_n$. Thus the element (5.7) can be expressed as a summation with $2^{s-r-1}$ terms labelled by the subsets of $\{a_{r+1}, \ldots, a_{s-1}\}$. Let us describe the summand labelled...
by the subset \( \{k_1, k_2, \ldots, k_p\} \) for some \( 0 \leq p \leq s - r - 1 \). Put \( k_0 = a_r \) and \( k_{r+1} = a_s \) with
\[
k_0 < a_{r+1} \leq k_1 < k_2 < \cdots < k_p \leq a_{s-1} < k_{p+1}.
\]

Then the corresponding summand in the expansion of (5.7) is
\[
(-1)^{p+1} \prod_{i=1}^{p+1} \left( \frac{1}{u_{k_0} - u_{k_i}} + \frac{c_{k_0} c_{k_i}}{u_{k_0} + u_{k_i}} \right) (k_0, k_i)
\]
\[
= (-1)^{p+1} \prod_{i=1}^{p+1} \left( \frac{1}{u_{k_0} - u_{k_i}} + \frac{c_{k_i-1} c_{k_i}}{u_{k_0} + u_{k_i}} \right) \cdot (k_0, k_1)(k_0, k_2) \cdots (k_0, k_{p+1}).
\]

Note that the product of the transpositions can be expressed as the cycle \((k_0, k_{r+1}, k_p, \cdots, k_1)\). Meanwhile, for each index \( 1 \leq i \leq p \), we write \( k_i = a_j \) for some \( r + 1 \leq j_i \leq s - 1 \) and denote the entry \( a_{j_i+1} \) by \( k_i^+ \). Then the permutation \((k_0, k_{p+1}, k_p, \cdots, k_1)\) can be decomposed into the adjacent transpositions
\[
(a_{s-1} a_j)(a_{s-2} a_{s-1}) \cdots (a_r a_{r+1}) \\
\cdots (a_{r+1} a_{r+2}) \cdots (k_1 k_1^+) \cdots (k_p k_p^+) \cdots (a_{s-1} a_s),
\]
where \((k_i k_i^+)\) indicates the omission of each transposition \((k_i k_i^+)\) from the product.

Since each transposition in this decomposition permutes adjacent entries within our row in \( \Lambda' \) then Corollary 5.8 provides the equality \((k_0, k_{p+1}, k_p, \cdots, k_1) \cdot P_{\Lambda'} = \tilde{\omega} \cdot P_{\Lambda'} \) where
\[
\tilde{\omega} = \prod_{\substack{j = r + 1 \ldots s - 1 \\{a_j\} \neq k_1, \ldots, k_p}} \left( \frac{1}{u_{a_{j+1}} - u_{a_j}} + \frac{c_{a_{j+1}} c_{a_j}}{u_{a_{j+1}} + u_{a_j}} \right) \cdot \prod_{j=r}^{s-1} \left( \frac{1}{u_{a_{j+1}} - u_{a_j}} + \frac{c_{a_j} c_{a_{j+1}}}{u_{a_{j+1}} + u_{a_j}} \right);
\]
here \(\alpha(a_j)\) denotes the number of entries in \( \{k_1, \ldots, k_p\} \) which precede the symbol \( a_j \) in the row of \( \Lambda' \). There are \( s - r - p - 1 \) factors in the first product while the second component contains \( s - r \) factors; this gives a total of \( 2(s - r) - (p + 1) \) factors in the expression \( \tilde{\omega} \). Let us extract a \((-1)\) term from each of these factors noting the identity \( 2(s - r) - (p + 1) \equiv p + 1 \) \( \pmod{2} \).

Given any \( \{k_1, k_2, \ldots, k_p\} \subseteq \{a_{r+1}, \ldots, a_{s-1}\} \) with \( a_{r+1} \leq k_1 < k_2 < \cdots < k_p \leq a_{s-1} \) then define \( \{l_1, l_2, \ldots, l_q\} \) as the complement \( \{a_{r+1}, \ldots, a_{s-1}\} \setminus \{k_1, k_2, \ldots, k_p\} \) where
\[
a_{r+1} \leq l_1 < l_2 < \cdots < l_q \leq a_{s-1}.
\]
Note that \( p + q = s - r - 1 \). Using this notation, the expression \( \bar{\omega} \) can be rewritten as
\[
\bar{\omega} = (-1)^{p+1} \prod_{i=q}^{1} \left( \frac{1}{u_i - u_i^*} - \frac{c_{k_{al_1}} c_{l_1}}{u_i + u_i^*} \right) \prod_{j=r}^{s-1} \left( \frac{1}{u_j - u_{j+1}} - \frac{c_{u_j} c_{u_j}}{u_j + u_{j+1}} \right).
\]
Thus left multiplication of \( P_{\lambda'} \) by the summand in (5.7) labelled by the subset \( \{ k_1, \ldots, k_p \} \) is described by
\[
\prod_{i=1}^{p+1} \left( \frac{1}{u_k - u_k} + \frac{c_{k_{i-1}} c_{k_i}}{u_k + u_k} \right) \prod_{i=q}^{1} \left( \frac{1}{u_l - u_l^*} - \frac{c_{k_{al_1}} c_{l_1}}{u_l + u_l^*} \right) \cdot \eta \cdot P_{\lambda'},
\]
where
\[
\eta = \prod_{j=r}^{s-1} \left( \frac{1}{u_j - u_{j+1}} - \frac{c_{u_j} c_{u_j}}{u_j + u_{j+1}} \right).
\]

The action of (5.7) on the element \( P_{\lambda'} \) is therefore obtained by summing the expressions (5.8) over all subsets \( \{ k_1, \ldots, k_p \} \subseteq \{ a_{r+1}, \ldots, a_{s-1} \} \). These subsets partition into two classes according to the criterion: (a) \( k_p = a_{s-1} \) or (b) \( k_p < a_{s-1} \). Furthermore, the subsets in (a) and (b) are in one-to-one correspondence: if \( \{ k_1, \ldots, k_p \} \) is in class (b) then \( \{ k_1, \ldots, k_p, a_{s-1} \} \) belongs to (a); while the complements in \( \{ a_{r+1}, \ldots, a_{s-1} \} \) are given by \( \{ l_1, \ldots, l_q, a_{s-1} \} \) and \( \{ l_1, \ldots, l_q \} \), respectively. In this manner, the summation over subsets of \( \{ a_{r+1}, \ldots, a_{s-1} \} \) can be reduced to a summation over all subsets \( \{ k_1, \ldots, k_p \} \subseteq \{ a_{r+1}, \ldots, a_{s-2} \} \); specifically, let us construct the summation of the expressions (5.8) over the restricted subsets with the symbol \( a_{s-1} \) first appended to the \( k \)-set and then appended to the \( l \)-set.

Each combined summand can be written explicitly as
\[
\prod_{i=1}^{p} \left( \frac{1}{u_k - u_k} + \frac{c_{k_{i-1}} c_{k_i}}{u_k + u_k} \right) \cdot \xi \cdot \prod_{i=q}^{1} \left( \frac{1}{u_l - u_l^*} - \frac{c_{k_{al_1}} c_{l_1}}{u_l + u_l^*} \right) \cdot \eta \cdot P_{\lambda'},
\]
where \( \xi \) denotes the expression
\[
\left( \frac{1}{u_k - u_{a_{s-1}}} + \frac{c_{k} c_{a_{s-1}}}{u_k + u_{a_{s-1}}} \right) \left( \frac{1}{u_k - u_{a_s}} + \frac{c_{a_{s-1}} c_{a_s}}{u_k + u_{a_s}} \right)
+ \left( \frac{1}{u_k - u_{a_s}} + \frac{c_{k} c_{a_s}}{u_k + u_{a_s}} \right) \left( \frac{1}{u_{a_{s-1}} - u_{a_s}} + \frac{c_{k} c_{a_{s-1}}}{u_{a_{s-1}} + u_{a_s}} \right).
\]
Putting \( a_k = k_p \) in the identity in Lemma 5.9 gives
\[
\xi = \left( \frac{1}{u_{a_{s-1}} - u_{a_s}} + \frac{c_{a_{s-1}} c_{a_s}}{u_{a_{s-1}} + u_{a_s}} \right) \left( \frac{1}{u_k - u_{a_{s-1}}} + \frac{c_{k} c_{a_{s-1}}}{u_k + u_{a_{s-1}}} \right);
\]
here the first term commutes with the preceding factors in the above summand, while the remaining terms in this summand coincide with the expression (5.8) on the restricted subset \( \{k_1, \ldots, k_p\} \subseteq \{r+1, \ldots, s-2\} \). Thus the summation is restricted to the smaller indexing set.

This procedure can be performed recursively for \( k_p < a_{s-2}, k_p < a_{s-3}, \) etc.; in this manner, the action of the element (5.7) on \( P_{\lambda'} \) simplifies to the expression

\[
\prod_{j=r+1}^{s-1} \left( \frac{1}{u_{a_j} - u_{a_{j+1}}} + \frac{c_{a_j} c_{a_{j+1}}}{u_{a_j} + u_{a_{j+1}}} \right)
\times \prod_{j=r}^{s-1} \left( \frac{1}{u_{a_j} - u_{a_{j+1}}} - \frac{c_{a_j} c_{a_{j+1}}}{u_{a_j} + u_{a_{j+1}}} \right) \cdot P_{\lambda'}. \tag{5.9}
\]

Meanwhile, for each \( r \leq j \leq s - 1 \), the pair \( (a_j, u_{a_{j+1}}) \) satisfies the idempotent condition (4.1); as a consequence, it can be shown that

\[
\left( \frac{1}{u_{a_j} - u_{a_{j+1}}} + \frac{c_{a_j} c_{a_{j+1}}}{u_{a_j} + u_{a_{j+1}}} \right) \left( \frac{1}{u_{a_j} - u_{a_{j+1}}} - \frac{c_{a_j} c_{a_{j+1}}}{u_{a_j} + u_{a_{j+1}}} \right) = 1.
\]

Hence the expression (5.9) coincides with the element \( P_{\lambda'} \). \( \square \)

Lemma 5.10 can be reformulated for the column element \( Q_{\lambda'} \) since the substitutions

\[
(a_j, a_{j+1}) \mapsto \left( \frac{1}{u_{a_{j+1}} - u_{a_j}} + \frac{c_{a_j} c_{a_{j+1}}}{u_{a_{j+1}} + u_{a_j}} \right)
\]

performed in the above proof are valid for the element \( Q_{\lambda'} \) as well; here the symbols \( a_1, \ldots, a_m \) now denote the entries in a fixed column of \( \lambda' \). This result provides a collection of expressions preserving \( Q_{\lambda'} \) under left multiplication.

**Lemma 5.11.** The column element \( Q_{\lambda'} \) is invariant under left multiplication by the element

\[
\prod_{k=r+1}^{s-1} \left\{ 1 - \left( \frac{1}{u_{a_k} - u_{a_{k+1}}} + \frac{c_{a_k} c_{a_{k+1}}}{u_{a_k} + u_{a_{k+1}}} \right) (a_k, a_{k+1}) \right\}
\times \left( \frac{1}{u_{a_k} - u_{a_{k+1}}} - \frac{c_{a_k} c_{a_{k+1}}}{u_{a_k} + u_{a_{k+1}}} \right) (a_k, a_{k+1}) \tag{5.10}
\]

for any choice of integers \( 1 \leq r < s \leq m \).

We now present the principal result in this section; this uses Lemma 5.10 and Lemma 5.11 to establish that the elements \( P_{\lambda'} \) and \( Q_{\lambda'} \) are idempotents in \( M_n \) up to integer multipliers.
Theorem 5.12. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \text{SP}(n)$.

(a) The row element $P_{\lambda'} \in M_n$ is idempotent up to an integer scalar:
$P_{\lambda'}^2 = \lambda! P_{\lambda'}$, where the multiplier $\lambda!$ is defined as $\lambda_1! \lambda_2! \cdots \lambda_l!$.

(b) The column element $Q_{\lambda'} \in M_n$ is idempotent up to an integer; let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ denote the sequence of column heights in the shifted diagram of $\lambda$. Then $Q_{\lambda'}^2 = \mu! Q_{\lambda'}$.

Proof. (a) The row element $P_{\lambda'}$ is defined as the product $P_1 P_2 \cdots P_l$ where each component $P_i$ ($1 \leq i \leq l$) is a product of the $\phi$-terms associated with row $i$ in $\lambda'$. Thus the proof is reduced to verifying the equality $P_i \cdot P_{\lambda'} = \lambda_i! P_{\lambda'}$ for each $1 \leq i \leq l$. Let us fix the row index $i$ and denote the $m$ entries in this row by $a_1, a_2, \ldots, a_m$. Then the component $P_i$ has the alternative presentation (5.2):

$$P_i = \prod_{k=1}^{m-1} \left( \prod_{j=k+1}^{m} \phi(a_j, a_k) \right).$$

Furthermore, the action of the internal products on the element $P_{\lambda'}$ is given by

$$\left( \prod_{j=k+1}^{m} \phi(a_j, a_k) \right) \cdot P_{\lambda'} = (m-k+1) P_{\lambda'},$$

for each $1 \leq k \leq m-1$; here each equality is established by expanding the $m-k$ factors in the left hand side product from right to left applying Lemma 5.10 at each step.

(b) The parallel result for the column element $Q_{\lambda'}$ is verified in exactly the same manner. The symbols $a_1, a_2, \ldots, a_m$ denote here the entries in a column $c_j$ of $\lambda'$. Lemma 5.11 gives the action of each component $Q_j$ on $Q_{\lambda'}$ as scalar multiplication by the integer $\mu_j!$.

The classical elements $p_\lambda, q_\lambda \in \mathbb{C}[S_n]$ described in the opening section obey precisely the same property relative to the (non-shifted) Young diagram of $\lambda$; cf. Lemma 1.1. Thus Theorem 5.12 can be regarded as the projective version of this well-known result; this justifies our claim that $P_{\lambda'}, Q_{\lambda'} \in M_n$ are analogues of the expressions introduced by Young.

6. A DECOMPOSITION FOR THE PROJECTIVE SYMMETRIZER

This final section continues the investigation into the projective symmetrizer $v_\lambda$ by proposing a decomposition for this element analogous to the $p_\lambda q_\lambda$-form exhibited in the classical case. By extending the techniques of Section 5, we will establish that $v_\lambda$ is divisible by the elements $P_{\lambda'}$ and...
on the left and right, respectively. This property implies that the projective symmetrizer has a structure comparable to the Young symmetrizer. The principal result in this paper is the structure theorem (Theorem 6.5): given any strict partition \( \lambda \in \text{SP}(n) \), the symmetrizer \( v_\lambda \) can be expressed in a standard form \((n_\lambda/n!) \cdot P_{\lambda} \cdot x_\lambda \cdot Q_{\lambda}^*\) for some additional factor \( x_\lambda \in M_n \).

The projective symmetrizer \( v_\lambda \) is, by definition, a uniquely defined element in the algebra \( M_n \); in contrast, the intermediate factor \( x_\lambda \) is not uniquely determined. The expression for the intermediate term given through our general approach will be referred to as the natural expression. In the special case \( \lambda = (n) \), the factor \( x_{(n)} \) may be taken as the identity element (Proposition 6.6); more generally, an expression for \( x_\lambda \) is non-trivial. The theory concerning the intermediate factor in this standard decomposition will be developed further in the subsequent papers.

Section 4 concluded with a result suggesting that the symmetrizer \( v_\lambda \) is divisible by certain elements in \( M_n \). The present section resumes this investigation into the structure of the projective symmetrizer; in particular, the next result extends Theorem 4.12 by explicitly demonstrating divisibility of the symmetrizer \( v_\lambda \) by elements with an appropriate form.

**Corollary 6.1.**

(a) Suppose that the entries \( k \) and \( k + 1 \) are adjacent within some row of the tableau \( \Lambda' \). Then \( \phi(k, 1, k) \cdot v_\lambda = 2 \cdot v_\lambda \).

(b) Suppose that \( k < l \) are adjacent entries in some column of \( \Lambda' \). Then \( v_\lambda \cdot \phi(l, 1, k) = 2 \cdot v_\lambda \).

**Proof.** The results follow from Theorem 4.12 via an elementary property of the \( \phi \)-terms: for any relevant pair of symbols \( j, k \), we have the equality \( \phi(j, 1, k) = 2 \). In (a), multiplying the identity by \( v_\lambda \) on the right yields

\[
\phi(k, 1, k) \cdot v_\lambda + \phi(k + 1, k) \cdot v_\lambda = 2 \cdot v_\lambda
\]

and the first component on the left hand side vanishes by Theorem 4.12(a). The result in (b) is established in a similar manner using Theorem 4.12(b).

Corollary 6.1 shows that the projective symmetrizer \( v_\lambda \) is divisible on the left by any factor in the row element \( P_{\lambda} \) associated with adjacent colinear positions in \( \Lambda' \) and simultaneously on the right by each adjacent factor in the column element \( Q_{\lambda}^* \); this result can be strengthened to obtain divisibility by the complete products \( P_{\lambda} \) and \( Q_{\lambda}^* \). We will use the same analytical technique as that employed in the preceding section: the substitution of row and column permutations by “equivalent” elements in the Clifford algebra. Our principal theorem will be obtained as a special case of some general results concerning the action of the projective elements \( P_{\lambda} \) and

\( Q_{\lambda} \) on the left and right, respectively. This property implies that the projective symmetrizer has a structure comparable to the Young symmetrizer. The principal result in this paper is the structure theorem (Theorem 6.5): given any strict partition \( \lambda \in \text{SP}(n) \), the symmetrizer \( v_\lambda \) can be expressed in a standard form \((n_\lambda/n!) \cdot P_{\lambda} \cdot x_\lambda \cdot Q_{\lambda}^*\) for some additional factor \( x_\lambda \in M_n \).
$Q_\Lambda'$ on certain subspaces in the algebra $M_n$. These subspaces are defined via special operators on $M_n$ described below.

Let us fix the strict partition $\lambda \in SP(n)$. Given any adjacent colinear entries $k$ and $k + 1$ in $\Lambda'$, introduce the mapping $x_{k,k+1} : M_n \to M_n$ defined by

$$x_{k,k+1}(m) = \left( (k + 1) - \frac{1}{u_{k+1} - u_k} - \frac{c_k c_{k+1}}{u_{k+1} + u_k} \right) \cdot m \quad \text{for all } m \in M_n,$$

that is, the operator $x_{k,k+1}$ represents left multiplication by the element $(k + 1) \cdot \phi(k, k + 1)$. Similarly, for any pair $k < l$ of adjacent cocolumnar entries in $\Lambda'$, define $y_{k,l} : M_n \to M_n$ by

$$y_{k,l}(m) = m \cdot \left( (l) + \frac{1}{u_k - u_l} + \frac{c_k c_l}{u_k + u_l} \right) \quad \text{for all } m \in M_n.$$

The operator $y_{k,l}$ describes right multiplication on $M_n$ by the element $\phi(k, l) \cdot (k, l)$.

We will examine the kernels of these transformations; in particular, let us introduce the subspaces $M^R, M^C \subseteq M_n$ associated with the rows and columns in $\Lambda'$, respectively, as

- $M^R = \bigcap \ker(x_{k,k+1})$ with the intersection over all pairs of adjacent colinear entries;
- $M^C = \bigcap \ker(y_{k,l})$ with the intersection over all pairs of adjacent cocolumnar entries.

These subspaces are referred to as the row kernel and the column kernel, respectively. The following exposition is clarified by introducing separate notation for each of these spaces: we denote an arbitrary element in the row kernel by $m_R$; whereas an element in the column kernel will be written as $m_C$. The next result demonstrates that the left action of the row element $P^\Lambda_\alpha$ on the subspace $M^R$ is a scalar multiplication.

**Proposition 6.2.** Each element $m_R \in M^R$ is an eigenvector with the integer eigenvalue $\lambda^!$ for the operator describing left multiplication on $M_n$ by the row element $P^\Lambda_\alpha$.

**Proof.** Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. Then the element $P^\Lambda_\alpha$ is defined as a product $P_1 P_2 \cdots P_l$ with pairwise-commuting components $P_i$ determined by the rows in $\Lambda'$. It will be sufficient to verify the appropriate result on each row: that left multiplication by the component $P_i$ on $M^R$ coincides with multiplication by the integer $\lambda_i^!$ for each index $1 \leq i \leq l$. Let us fix the row index $i$. Expanding the product (5.1) shows that the element $P_i$ has the form $\sum_{\sigma \in R_i} c_\sigma \cdot \sigma$ for some $c_\sigma \in Z_n$; here $R_i$ denotes
the set of permutations on the symbols in row $i$. Thus the action of the component $P_i$ on an element $m_R$ can be written as

$$P_i \cdot m_R = \sum_{\sigma \in R_i} c_\sigma (\sigma \cdot m_R).$$

(6.1)

Furthermore, the action of the individual row permutations can be described by exploiting the structure of the row kernel: by definition, any element $m_R \in M^R$ obeys the equality

$$(k, k+1) \cdot m_R = \left( \frac{1}{u_{k+1} - u_k} + \frac{c_k c_{k+1}}{u_{k+1} + u_k} \right) \cdot m_R$$

for any adjacent entries $k$ and $k + 1$ in row $i$. That is, each adjacent row transposition acts on the row kernel in precisely the same manner as some expression in the Clifford algebra. Moreover, this property enables each permutation $\sigma \in R_i$ in (6.1) to be replaced by an equivalent expression in the Clifford algebra. The action of the component $P_i$ on $M^R$ therefore coincides with the left action of some element $C_i \in Z_n$. It should be emphasised that this Clifford element $C_i$ is identical for each $m_R \in M^R$ since the substitutions are independent of the particular element in the row kernel. We will establish that $C_i$ has the integer value $\lambda_i!$ via the results of the preceding section.

The substitution of row permutations by Clifford elements has been performed explicitly in the proof of Theorem 5.12(a) for the row element $P_{3^r}$; in particular, we verified the equality $P_i \cdot P_{3^r} = \lambda_i! P_{3^r}$. Meanwhile, the row kernel contains the element $P_{3^r}$ as a consequence of Proposition 5.7(a). Combining these results gives

$$C_i \cdot P_{3^r} = \lambda_i! P_{3^r}.$$ 

(6.2)

Finally, since the row element $P_{3^r}$ has the expansion

$$P_{3^r} = 1 + \sum_{\sigma \in S_{3} \setminus \{1\}} c^p_{\sigma} \cdot \sigma \quad (c^p_{\sigma} \in Z_n),$$

then comparison of the coefficients for the identity permutation in (6.2) gives $C_i = \lambda_i!$. □

A parallel result exists for the column element $Q_{3^r}$.

**Proposition 6.3.** Let $\mu = (\mu_1, \mu_2, \ldots, \mu_\lambda)$ denote the sequence of column heights in the shifted diagram of $\lambda$. Then each element $m_C \in M^C$ is an eigenvector with eigenvalue $\mu_\lambda!$ for the operator describing right multiplication on $M_n$ by the element $Q_{3^r}$. 


Proof. This statement is verified in precisely the same manner as Proposition 6.2 using the results in Proposition 5.7(b) and Theorem 5.12(b). Restricting consideration to each column, the action of each component $Q_j$ in $Q_\lambda$ on $M^C$ reduces to scalar multiplication by $\mu_j!$.

Theorem 4.12 demonstrates that the projective symmetrizer $v_\lambda$ belongs to both the row space $M^R$ and the column space $M^C$. Therefore, as a corollary to Proposition 6.2 and Proposition 6.3, the following fundamental result has been established.

Theorem 6.4. For any $\lambda \in SP(n)$, the symmetrizer $v_\lambda$ is a simultaneous eigenvector for the operators on the algebra $M_n$ describing left multiplication by the row element $P_\lambda$ and right multiplication by the column element $Q_\lambda$; in particular

$$P_\lambda \cdot v_\lambda = \lambda! \cdot v_\lambda; \quad v_\lambda \cdot Q_\lambda = \mu! \cdot v_\lambda,$$

where $\mu$ denotes the sequence of column heights in the shifted diagram for $\lambda$.

Theorem 6.4 implies that there is a decomposition for the projective symmetrizer analogous to the structure exhibited by the classical Young symmetrizer; indeed, $v_\lambda$ has the property

$$v_\lambda = \frac{1}{\lambda! \mu!} P_\lambda \cdot v_\lambda \cdot Q_\lambda,$$

while Definition 4.9 then gives the canonical description

$$v_\lambda = \frac{n^\lambda}{n!} P_\lambda \left( \frac{1}{\lambda! \mu!} \cdot \psi_\lambda \cdot w_\lambda^{-1} \right) Q_\lambda.$$

Thus we have established our principal theorem:

Theorem 6.5. Given any $\lambda \in SP(n)$, the symmetrizer $v_\lambda$ has the form

$$v_\lambda = \frac{n^\lambda}{n!} \cdot P_\lambda \cdot x_\lambda Q_\lambda \quad (6.3)$$

for some element $x_\lambda \in M_n$.

This result reveals a structure for the projective analogue $v_\lambda$ comparable with the $p_\lambda q_\lambda$-form for the classical symmetrizer $e_\lambda \in \mathbb{C}[S_n]$ described in Section 1; contrary to the classical case, however, there is a requirement for an additional intermediate factor $x_\lambda$ in the decomposition of the projective symmetrizer. The subsequent papers in this series will be concerned with the further analysis of this intermediate element.

To conclude the present paper, a result is presented for the special class of one-part partitions $\lambda = (n)$; in this exceptional case, the decomposition (6.3) for $v_{(n)}$ precisely mirrors the classical structure since $x_{(n)}$ can be taken as the identity element in the algebra $M_n$. 

Proposition 6.6. The intermediate factor is trivial for any one-part partition \( \lambda = (n) \).

Proof. The row tableau
\[
\Lambda' = \begin{array}{c}
1 \\
2 \\
\vdots \\
n
\end{array}
\]
determines the row and column elements as
\[
P_\Lambda = \prod_{k=2}^{n} \left( \prod_{j=1}^{k-1} \phi(k, j) \right), \quad Q_\Lambda = 1.
\]
The result will be obtained by directly verifying that \( x_{(n)} = 1 \) satisfies the decomposition (6.3). Equivalently, this reduces to establishing the equality
\[
\psi_\Lambda \cdot w_\Lambda^{-1} = P_\Lambda, \quad (6.4)
\]
where \( w_\Lambda \in S_n \) is the involution \( (1\ n)(2\ n-1) \cdots \), while (4.4) gives
\[
\psi_\Lambda = \prod_{k=2}^{n} \left( \prod_{j=1}^{k-1} \psi_{k-j}(u_k, u_j) \right).
\]
The equality (6.4) is proved by induction on the integer \( n \): the result is obvious for \( n = 2 \) since \( \psi_1(\sqrt{2}, 0) \cdot (1\ 2) = \phi(2, 1) \). Consider now the general case \( \lambda = (n) \). Let \( \Omega' \) denote the row tableau for the partition \( \omega = (n - 1) \) obtained by deleting the entry \( n \) in \( \Lambda' \). By the induction hypothesis, the left hand side in (6.4) can be expressed as
\[
P_{\Omega'} w_{\Omega'} \prod_{j=1}^{n-1} \psi_{n-j}(u_n, u_j) \cdot w_{\Lambda'}^{-1}.
\]
Moreover, the element \( w_\Lambda \in S_n \) can be obtained from \( w_{\Omega'} \) (regarded as a permutation in \( S_n \)) via the equality \( w_\Lambda = w_{\Omega'} \cdot (1\ n)(2\ n) \cdots (n - 1\ n) \); thus the above expression becomes
\[
P_{\Omega'} \prod_{j=1}^{n-1} \left( w_{\Omega'} \psi_{n-j}(u_n, u_j) \right) w_{\Lambda'}^{-1} \cdot (n\ n-1\ \cdots\ 2\ 1).
\]
The conjugation of the factors by \( w_{\Omega'} \) is described explicitly by the formulae
\[
w_{\Omega'} \psi_{n-1}(u_n, u_1) w_{\Omega'}^{-1} = (1\ n) + \frac{1}{u_n - u_1} - \frac{c_1 c_n}{u_n + u_1}
\]
and
\[
w_{\Omega'} \psi_{n-j}(u_n, u_j) w_{\Omega'}^{-1} = (j - 1\ j) + \frac{1}{u_n - u_j} - \frac{c_j c_{j-1}}{u_n + u_j}
\]
for each \( 2 \leq j \leq n - 1 \). The equality (6.4) then follows via some elementary identities between permutations and Clifford algebra elements. \( \blacksquare \)
The result in Proposition 6.6 is exceptional: an intermediate expression is non-trivial for any remaining strict partition of \( n \). For instance, in the final paper of this series, it will be shown that the simplest expression giving an intermediate factor for the partition \((2, 1)\in SP(3)\) is

\[
\hat{x}_{(2,1)} = 1 + \frac{\sqrt{2}}{4}(c_1c_2 + c_2c_3)(1 \ 3).
\]

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