



## Spin-orbit force from lattice QCD



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## ARTICLE INFO

## Article history:

Received 2 May 2013

Received in revised form 5 February 2014

Accepted 20 May 2014

Available online 27 May 2014

Editor: J.-P. Blaizot

## Keywords:

Lattice QCD

Nuclear force

Spin-orbit potential

Scattering phase shift

## ABSTRACT

We present a first attempt to determine nucleon–nucleon potentials in the parity-odd sector, which appear in the  $^1P_1$ ,  $^3P_0$ ,  $^3P_1$ ,  $^3P_2$ – $^3F_2$  channels, in  $N_f = 2$  lattice QCD simulations. These potentials are constructed from the Nambu–Bethe–Salpeter wave functions for  $J^P = 0^-, 1^-$  and  $2^-$ , which correspond to the  $A_1^-, T_1^-$  and  $T_2^- \oplus E^-$  representation of the cubic group, respectively. We have found a large and attractive spin-orbit potential  $V_{LS}(r)$  in the isospin-triplet channel, which is qualitatively consistent with the phenomenological determination from the experimental scattering phase shifts. The potentials obtained from lattice QCD are used to calculate the scattering phase shifts in the  $^1P_1$ ,  $^3P_0$ ,  $^3P_1$  and  $^3P_2$ – $^3F_2$  channels. The strong attractive spin-orbit force and a weak repulsive central force in spin-triplet  $P$ -wave channels lead to an attraction in the  $^3P_2$  channel, which is related to the  $P$ -wave neutron pairing in neutron stars.

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## 1. Introduction

A study of the nuclear force in QCD is a first step toward the understanding of hadronic properties beyond single hadrons. In addition, the nuclear force plays a key role in describing properties of atomic nuclei and neutron stars [1,2]. In lattice QCD, the standard method to study hadronic interactions is the finite volume method [3] by calculating the scattering phase shift [4–9]. Recently, a new method to extract hadronic interactions from (lattice) QCD has been proposed, where non-local potential can be defined through the Nambu–Bethe–Salpeter (NBS) wave function. The method has been successfully applied to the nuclear forces [10–12]. It has been extended to various other systems such as hyperon–nucleon (YN), hyperon–hyperon (YY), meson–baryon, and the three-nucleons [13]. In Ref. [14], an explicit comparison between the finite volume method and the potential method is made in the case of  $\pi\pi$  scattering phase shifts, where a good agreement is obtained.

In the case of the nucleon–nucleon (NN) system, for example, non-local potentials are defined through the Schrödinger equation for the NBS wave function. Below the pion production thresh-

old, the non-local NN potential can be expanded by the number of derivatives with respect to its non-locality of the relative coordinate. The leading order (LO) terms are the spin-independent central potential  $V_0(r)$ , the spin-dependent central potential  $V_\sigma(r)$  and the tensor potential  $V_T(r)$ , while the next-to-leading order (NLO) term is the spin-orbit potential  $V_{LS}(r)$ . Up to the NLO, there are altogether 8 independent local potentials,  $V_{0,\sigma,T,LS}^{I=0,1}$  where  $I$  denotes the total isospin [11]. In the previous studies [10–12],  $V_{C,S=0}^{I=1} \equiv V_0^{I=1} - 3V_\sigma^{I=1}$ ,  $V_{C,S=1}^{I=0} \equiv V_0^{I=0} + V_\sigma^{I=0}$  and  $V_T^{I=0}$  have been determined from the NBS wave functions in  $S$  and  $D$  waves (the parity-even sector) at various lattice parameters. For the complete determination, however, we must employ the NBS wave functions in  $P$  and  $F$  waves (the parity-odd sector) with non-zero relative momentum of the NN system, where relevant channels are  $^1P_1$ ,  $^3P_0$ ,  $^3P_1$  and  $^3P_2$ – $^3F_2$ . The corresponding potentials are given by

$$V(r; ^1P_1) = V_{C,S=0}^{I=0}(r) \equiv V_0^{I=0}(r) - 3V_\sigma^{I=0}(r) \quad (1)$$

$$V(r; ^3P_0) = V_{C,S=1}^{I=1}(r) - 4V_T^{I=1}(r) - 2V_{LS}^{I=1}(r) \quad (2)$$

$$V(r; ^3P_1) = V_{C,S=1}^{I=1}(r) + 2V_T^{I=1}(r) - V_{LS}^{I=1}(r) \quad (3)$$

$$V(r; ^3(P, F)_2) = \begin{pmatrix} V(r; ^3P_2) & \frac{6\sqrt{6}}{5}V_T^{I=1}(r) \\ \frac{6\sqrt{6}}{5}V_T^{I=1}(r) & V(r; ^3F_2) \end{pmatrix}$$

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with  $V_{C,S=1}^{l=1}(r) = V_0^{l=1}(r) + V_\sigma^{l=1}(r)$ ,  $V(r; {}^3P_2) = V_{C,S=1}^{l=1}(r) - \frac{2}{5}V_T^{l=1}(r) + V_{LS}^{l=1}(r)$  and  $V(r; {}^3F_2) = V_{C,S=1}^{l=1}(r) - \frac{8}{5}V_T^{l=1}(r) - 4V_{LS}^{l=1}(r)$ . The LS force has close relation to the spin-orbit splittings of the nuclear spectra and the nuclear magic numbers [15]. Large neutron-neutron attraction due to the LS force in the  ${}^3P_2$ - ${}^3F_2$  channel leads to the  $P$ -wave superfluidity in the stellar environment such as the neutron star interiors [16–18]. It also affects the cooling properties of neutron stars [19].

The present paper is organized as follows. In Section 2, we give a brief review of our method to obtain NN potentials on the lattice. Section 3 is devoted to the discussion on the NBS wave functions with non-zero angular momenta. In Section 4, we present numerical results of the potentials. The scattering phases and mixing parameters in the  ${}^1P_1$ ,  ${}^3P_0$ ,  ${}^3P_1$ , and  ${}^3P_2$ - ${}^3F_2$  channels calculated from these potentials are also presented.

## 2. NN potential in QCD

Below the inelastic threshold of the NN system ( $W < 2m_N + m_\pi$ ), we can define the non-local but energy-independent potential as [11]

$$(k^2/m_N - H_0)\phi(\vec{r}; W) = \int d^3r' U(\vec{r}, \vec{r}')\phi(\vec{r}', W), \quad (4)$$

from the Nambu–Bethe–Salpeter (NBS) wave function in the center of mass (CM) frame defined by

$$\phi_{\alpha\beta}(\vec{r}; W) \equiv \langle 0 | p_\alpha(\vec{x}) n_\beta(\vec{y}) | B=2, W \rangle \quad (\vec{r} = \vec{x} - \vec{y}), \quad (5)$$

where  $H_0 \equiv -\nabla^2/m_N$  with the nucleon mass  $m_N$ ,  $p_\alpha(\vec{x})$ ,  $n_\beta(\vec{y})$  denote local composite nucleon operators with spinor indices  $\alpha, \beta$  in Dirac representation restricted to  $\alpha, \beta = 0, 1$ ,  $|B=2, W\rangle$  is the state with baryon-number  $B=2$  and vanishing total momentum with the total energy  $W$ . Due to the confinement of quarks and gluons,  $\phi(\vec{r}; W)$  for large  $|\vec{r}|$  reduces to the relative wave function of the non-interacting nucleons, so that  $W$  can be written as  $W = 2\sqrt{k^2 + m_N^2}$  with  $k \equiv |\vec{k}|$  being the asymptotic relative momentum between the nucleons. The identity  $U(\vec{r}, \vec{r}') = V(\vec{r}, \vec{\nabla})\delta(\vec{r} - \vec{r}')$  leads to the derivative expansion up to the NLO order:

$$V^l(\vec{r}, \vec{\nabla}) = V_0^l(r) + V_\sigma^l(r)\vec{\sigma}_1 \cdot \vec{\sigma}_2 + V_T^l(r)S_{12} + V_{LS}^l(r)\vec{L} \cdot \vec{S} + O(\nabla^2), \quad (6)$$

where  $S_{12} \equiv 3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r})/r^2 - \vec{\sigma}_1 \cdot \vec{\sigma}_2$ ,  $\vec{S} \equiv (\vec{\sigma}_1 + \vec{\sigma}_2)/2$  and  $\vec{L} \equiv i\vec{r} \times \vec{\nabla}$  denote the tensor, the total spin and the (relative) orbital angular momentum operators, respectively, and  $l = 0, 1$  is the total isospin of the two nucleons.

These NN potentials can be extracted by solving Eq. (4) with Eq. (6) for NBS wave functions projected to appropriate quantum numbers.

The NBS wave function in Eq. (5) and thus the non-local potential defined in Eq. (4) depend on the choice of nucleon operators  $p(x)$  and  $n(x)$ . This is not surprising since the potential is not a physical observable and depends on how it is defined. On the other hand, it has been shown that the NBS wave function carries information of the scattering phase shift in its asymptotic behavior at large  $r$  [20–22,11]. This property holds for an arbitrary choice of operators to define the NBS wave function, as long as basic properties in quantum field theories such as locality and unitarity are satisfied. Therefore, by construction, non-local potentials in different definitions give same and correct phase shifts through the Schrödinger equation in Eq. (4). Despite that the above points have been already stated clearly and explicitly in our previous papers (see, e.g. [11]), similar remarks appear repeatedly in later literature (see e.g. [24]).

In our study, we take local interpolating operators for the nucleon as  $p(x) \equiv \epsilon_{abc}(u_a^T(x)C\gamma_5 d_b(x))u_c(x)$ , and  $n(x) \equiv \epsilon_{abc}(u_a^T(x) \times C\gamma_5 d_b(x))d_c(x)$ , to define the NBS wave function, where  $a, b, c$  denotes the color indices. In our actual calculation, the derivative expansion of the non-local potential in Eq. (6) is truncated at fixed order (the NLO in the present paper); then the resultant phase shifts are valid only in a certain energy interval which depends on the order of the truncation and the choice of the interpolating operator. The former dependence has been investigated for the NN case [23] and the  $\pi\pi$  case [14], while the latter dependence is still left for future studies.

## 3. NBS wave functions with non-zero angular momenta

The NBS wave function for the ground state can be extracted from nucleon four-point functions at large  $t$  as

$$\begin{aligned} G_{\alpha\beta}(\vec{r}, t - t_0; \mathcal{J}) & \equiv \frac{1}{V} \sum_{\vec{y}} \langle 0 | T [p_\alpha(\vec{r} + \vec{y}, t) n_\beta(\vec{y}, t) \mathcal{J}(J^P, S; t_0)] | 0 \rangle, \\ & = \sum_m \phi(\vec{r}; W_m) \langle m | \mathcal{J}(J^P, S; 0) | 0 \rangle e^{-W_m(t-t_0)}, \end{aligned} \quad (7)$$

where two-nucleon source  $\mathcal{J}(J^P; t_0)$  located at the time-slice  $t = t_0$  is used to control the quantum numbers such as  $J^P$ , and  $W_m$  denotes the energy of the intermediate state  $|m\rangle$ . The summation over  $\vec{y}$  is performed to select states with the vanishing total spatial momentum.

To construct sources coupled to the parity-odd states, we employ wall sources with non-zero momentum given by

$$\mathcal{J}_{\alpha\beta}(f_i) \equiv \sum_{\vec{x}_1, \dots, \vec{x}_6} \bar{P}_\alpha(\vec{x}_1, \vec{x}_2, \vec{x}_3) \bar{N}_\beta(\vec{x}_4, \vec{x}_5, \vec{x}_6) f_i(\vec{x}_3 - \vec{x}_6), \quad (8)$$

with  $\bar{P}_\alpha(\vec{x}_1, \vec{x}_2, \vec{x}_3) \equiv \epsilon_{abc}(\bar{u}_a(\vec{x}_1)C\gamma_5 \bar{d}_b^T(\vec{x}_2))\bar{u}_{c,\alpha}(\vec{x}_3)$ , and  $\bar{N}_\alpha(\vec{x}_4, \vec{x}_5, \vec{x}_6) \equiv \epsilon_{abc}(\bar{u}_a(\vec{x}_4)C\gamma_5 \bar{d}_b^T(\vec{x}_5))\bar{d}_{c,\alpha}(\vec{x}_6)$ , where  $f_i(\vec{r})$  denotes a plane wave with the spatial momentum parallel or anti-parallel to a coordinate axis as

$$\begin{aligned} f_0(\vec{r}) & \equiv \exp(+i2\pi x/L), & f_1(\vec{r}) & \equiv \exp(+i2\pi y/L), \\ f_2(\vec{r}) & \equiv \exp(+i2\pi z/L), & f_3(\vec{r}) & \equiv \exp(-i2\pi x/L), \\ f_4(\vec{r}) & \equiv \exp(-i2\pi y/L), & f_5(\vec{r}) & \equiv \exp(-i2\pi z/L). \end{aligned}$$

An element  $g$  of cubic group  $\mathcal{O}$  acts on these six functions as permutation,  $f_i(g\vec{r}) = U_{ij}(g)f_j(\vec{r})$ , where  $U(g)$  is the representation matrix of the cubic group whose explicit form can be generated by the basic matrices,

$$\begin{aligned} U(C_{4y}) & \equiv \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ U(C_{4z}) & \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned} \quad (9)$$

with  $C_{4y}$  and  $C_{4z}$  being the rotations by +90 degrees around  $y$ -axis and  $z$ -axis, respectively, and  $U(g)$  for other  $g \in \mathcal{O}$  are

obtained by multiplying these two matrices in suitable orders. For instance, the representation matrix for  $C_{4x}$ , a rotation by +90 degrees around the  $x$ -axis, is obtained as  $U(C_{4x}) = U(C_{4y}) \times U(C_{4z})U^{-1}(C_{4y})$ . The spatial reflection  $R$  corresponds to  $f_0 \leftrightarrow f_3$ ,  $f_1 \leftrightarrow f_4$ , and  $f_2 \leftrightarrow f_5$ , which leads to  $6 \times 6$  representation matrix  $U(R) = \begin{pmatrix} 0_{3 \times 3} & I_{3 \times 3} \\ I_{3 \times 3} & 0_{3 \times 3} \end{pmatrix}$ . Analysis based on the group characters shows that  $U(g)$  is reduced to the direct sum of irreducible representations  $T_1^- \oplus A_1^+ \oplus E^+$ .

Together with the transformation property of the quark field  $\bar{q}(\vec{x}) \mapsto \bar{q}(g^{-1}\vec{x})S(g^{-1})$  where  $S(g)$  denotes the standard rotation matrix acting on upper two components in the Dirac representation, we have

$$\begin{aligned} \mathcal{J}_{\alpha\beta}(f_i) &\mapsto \sum_{\vec{x}_1, \dots, \vec{x}_6} \bar{P}_{\alpha'}(g^{-1}\vec{x}_1, \dots, g^{-1}\vec{x}_3) \bar{N}_{\beta'}(g^{-1}\vec{x}_4, \dots, g^{-1}\vec{x}_6) \\ &\quad \times f_i(\vec{x}_3 - \vec{x}_6) S_{\alpha'\alpha}(g^{-1}) S_{\beta'\beta}(g^{-1}) \\ &= U_{ij}(g) \mathcal{J}_{\alpha'\beta'}(f_j) S_{\alpha'\alpha}(g^{-1}) S_{\beta'\beta}(g^{-1}). \end{aligned} \quad (10)$$

Therefore our momentum wall source  $\mathcal{J}_{\alpha\beta}(f_i)$  covers  $T_1^- \oplus A_1^+ \oplus E^+$  ( $\simeq 1^- \oplus 0^+ \oplus 2^+$ ) for the spin singlet sector and  $(T_1^- \oplus A_1^+ \oplus E^+) \otimes T_1 = A_1^- \oplus T_1^- \oplus (T_2^- \oplus E^-) \oplus T_1^+ \oplus T_1^+ \oplus T_2^+$  ( $\simeq 0^- \oplus 1^- \oplus 2^- \oplus 1^+ \oplus 2^+$ ) for the spin triplet sector.<sup>1</sup>

We introduce several projections to fix quantum numbers of the source operator  $\mathcal{J}_{\alpha\beta}(f_i)$ . The projection for the total angular momentum  $J$  is given by

$$\begin{aligned} \mathcal{P}^{(J)}[\mathcal{J}_{\alpha\beta}(f_i)] &\equiv \frac{d^{(J)}}{24} \sum_{g \in \mathcal{O}} \chi^{(J)}(g^{-1}) \cdot U_{ij}(g) \mathcal{J}_{\alpha'\beta'}(f_j) \\ &\quad \times S_{\alpha'\alpha}(g^{-1}) S_{\beta'\beta}(g^{-1}) \\ &= P_{\alpha\beta i; \alpha' \beta' j}^{(J)} \cdot \mathcal{J}_{\alpha'\beta'}(f_j), \end{aligned} \quad (11)$$

where  $d^{(J)}$  and  $\chi^{(J)}(g)$  denote the dimension and the character of the irreducible representation  $J$  of the cubic group, and the  $24 \times 24$  matrix  $P_{\alpha\beta i; \alpha' \beta' j}^{(J)}$  is the projection matrix onto the irreducible representation  $J$ . Similar considerations give the projection matrices for the parity

$$P_{\alpha\beta i; \alpha' \beta' j}^{(P=\pm)} \equiv \frac{1}{2} (\delta_{ij} \pm U_{ij}(R)) \cdot \delta_{\alpha\alpha'} \delta_{\beta\beta'}, \quad (12)$$

as well as the total spin  $S$

$$\begin{aligned} P_{\alpha\beta i; \alpha' \beta' j}^{(S=0)} &\equiv \frac{1}{4} (1 - \vec{\sigma}_1 \cdot \vec{\sigma}_2)_{\alpha\beta; \alpha' \beta'} \cdot \delta_{ij}, \\ P_{\alpha\beta i; \alpha' \beta' j}^{(S=1)} &\equiv \frac{1}{4} (3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2)_{\alpha\beta; \alpha' \beta'} \cdot \delta_{ij}. \end{aligned} \quad (13)$$

The projection matrices for  $J_z$  are defined based on the  $C_4$  subgroup of the cubic group which consists of multiples of  $C_{4z}$  as

$$\begin{aligned} P_{\alpha\beta i; \alpha' \beta' j}^{(J_z=M)} &\equiv \frac{1}{4} \sum_{n=0,1,2,3} e^{i(\pi/2)Mn} \cdot U_{ij}((C_{4z})^n) S_{\alpha'\alpha}((C_{4z})^{-n}) \\ &\quad \times S_{\beta'\beta}((C_{4z})^{-n}). \end{aligned} \quad (14)$$

Since a product of these projection matrices<sup>2</sup>

$$P_{\alpha\beta i; \alpha' \beta' j}^{(J, J_z, P, S)} \equiv (P^{(J)} P^{(J_z)} P^{(P)} P^{(S)})_{\alpha\beta i; \alpha' \beta' j} \quad (15)$$

<sup>1</sup> See Appendix A in Ref. [23] for a decomposition of a product of two irreducible representations of the cubic group and for the relation of irreducible representations between the cubic group and  $SO(3)$ .

<sup>2</sup> The result does not depend on the order of multiplications, since these projection matrices are mutually commutative with each other.

has the property  $(P^{(J, J_z, P, S)})^2 = P^{(J, J_z, P, S)}$ , the eigenvalues of  $P^{(J, J_z, P, S)}$  are either 0 or 1. We diagonalize  $P^{(J, J_z, P, S)}$  to obtain its eigenvectors  $\eta_{\alpha\beta i}^{(J, J_z, P, S); n}$  with eigenvalue 1 as  $P_{\alpha\beta i; \alpha' \beta' j}^{(J, J_z, P, S)} \eta_{\alpha' \beta' j}^{(J, J_z, P, S); n} = \eta_{\alpha\beta i}^{(J, J_z, P, S); n}$ , which is used to perform the projection of our two-nucleon source as

$$\mathcal{J}(J^P, J_z, S) \equiv \mathcal{J}_{\alpha\beta}(f_i) \eta_{\alpha\beta i}^{(J, J_z, P, S); n}, \quad (16)$$

where  $n$  is used to describe a possible degeneracy of a given set of quantum numbers  $J, J_z, P, S$ . By construction, a state  $\mathcal{J}(J^P, J_z, S)|0\rangle$  has conserved quantum numbers  $J^P, J_z$  and  $S$  in the two-nucleon sector.

## 4. Numerical results

### 4.1. Lattice setup

In this study, we employ  $N_f = 2$  full QCD configurations generated by the CP-PACS Collaboration on a  $16^3 \times 32$  lattice with the RG improved gauge action (Iwasaki action) at  $\beta = 1.95$  and with the  $\mathcal{O}(a)$ -improved Wilson quark (clover) action at  $\kappa = 0.1375$  and  $C_{SW} = 1.53$ , which gives the lattice spacing  $a = 0.1555(17)$  fm, the spatial extension  $L = 16a = 2.489(27)$  fm, the pion mass  $m_\pi \simeq 1133$  MeV and the nucleon mass  $m_N \simeq 2158$  MeV [25]. The Dirichlet boundary condition along the temporal direction is employed to generate quark propagators to avoid contamination from backward propagation of nucleons with negative parity.

With the projection defined in the previous section, we obtain the NBS wave functions for  $T_1^-$  in the spin singlet sector and for  $A_1^-, T_1^-, (E^- \oplus T_2^-)$  in the spin triplet sector. In order to improve statistics, we perform the measurement on 32 source points by temporally shifting the location of the source, in addition to averages over the charge-conjugation/time-reversal transformations. We further reduce noises by the average over the cubic group, as will be discussed later.

To construct the potentials, we use the time-dependent method [12] with a slight modification to cope with the deviation of relativistic dispersion relation due to heavy quark mass, as explained in the following subsection. The nearest neighbor derivative is used to define the discretized Laplacian, while the symmetric derivative is employed to define the operator  $\bar{L}$ . To define  $S_{12}$  and  $\bar{L}$  on the periodic lattice, we take the origin of  $\vec{r}$  to be the nearest periodic copy of the origin. On the spatial boundaries, i.e.,  $x = \pm L/2$  or  $y = \pm L/2$  or  $z = \pm L/2$ , however, these operators are still ill-defined. We therefore exclude data on these boundaries in our analysis. To extract potentials, we employ  $t - t_0 = 8$ , which is determined from  $t$  dependencies of potentials and phase shifts.

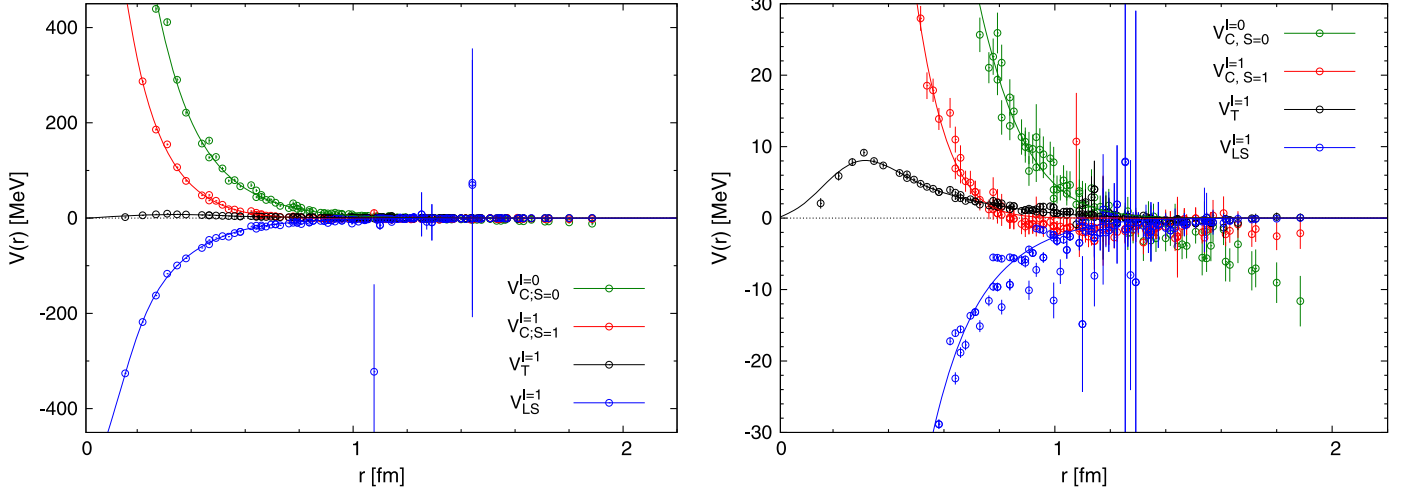
### 4.2. Modified time-dependent method

In Ref. [12], the time-dependent method has been proposed to extract the potential directly from the four-point functions. The method indeed gives more accurate and stable results, since it does not rely on the ground-state saturation at large  $t$ . We therefore employ this method also in this paper.

For a coarse lattice, however, the heavy quark may violate the relativistic dispersion relation of a single nucleon as  $E^2 \simeq m_N^2 + \alpha k^2$  with  $\alpha \neq 1$  [26]. In such a case, the formula in Ref. [12] receives a slight modification as

$$\begin{aligned} &\left\{ \frac{1}{\alpha} \left( \frac{1}{4m_N} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} \right) - H_0 \right\} \mathcal{R}(\vec{r}, t; \mathcal{J}) \\ &= \int d^3 r' U(\vec{r}, \vec{r}') \mathcal{R}(\vec{r}', t; \mathcal{J}), \end{aligned} \quad (17)$$

where  $\mathcal{R}(\vec{r}, t; \mathcal{J}) \equiv G(\vec{r}, t; \mathcal{J}) / (e^{-m_N t})^2$ .



**Fig. 1.** Central ( $S = 0$  and  $1$ ), tensor and spin-orbit potentials in parity-odd sector obtained by lattice QCD (left), and their enlargements (right). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In our simulation, the dispersion relation for the nucleon can be fitted well with  $\alpha = 0.88(1)$  ( $\chi^2/\text{d.o.f.} = 2.6$ ) at  $m_N = 2152(3)$  MeV, showing no sign of higher order contributions in  $k^2$  for  $k^2 \leq 1.25$  [GeV $^2$ ] ( $ka \leq \sqrt{5} \times 2\pi/L$ ) within statistical errors.

#### 4.3. Extractions of potentials

The potential for the spin-singlet sector at NLO can be easily extracted from the equation

$$V_{C,S=0}^{I=0}(r) \langle \mathcal{R}(\vec{r}, t; \mathcal{J}), \mathcal{R}(\vec{r}, t; \mathcal{J}) \rangle = \langle \mathcal{R}(\vec{r}, t; \mathcal{J}), (D_t - H_0) \mathcal{R}(\vec{r}, t; \mathcal{J}) \rangle \quad (18)$$

for  $\mathcal{J} = \mathcal{J}(T_1^-)$ , dominated by  $^1P_1$ , where  $\alpha D_t = \frac{1}{4m_N} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t}$ , and we define an inner product with an average over the cubic group as  $\langle F(\vec{r}), H(\vec{r}) \rangle \equiv \sum_{g \in \mathcal{O}} F_{\beta\alpha}^*(g\vec{r}) H_{\alpha\beta}(g\vec{r})$ , which reduces statistical noises of potentials. Note that here and in the following we use the fact that local potentials,  $V_{C,S}^I$ ,  $V_T^I$  and  $V_{LS}^I$ , are invariant under the rotation  $g$  in the cubic group. The result for  $V_{C,S=0}^{I=0}(r)$  is plotted in Fig. 1 by green circles, which shows a strong repulsion at short distances.

For the spin-triplet sector, three unknown functions up to NLO can be determined from the equation

$$V_{C,S=1}^{I=1}(r) F_C^{\mathcal{J}}(r) + V_T^{I=1}(r) F_T^{\mathcal{J}}(r) + V_{LS}^{I=1}(r) F_{LS}^{\mathcal{J}}(r) = K^{\mathcal{J}}(r) \quad (19)$$

for three different sources,  $\mathcal{J} = \mathcal{J}(A_1^-)$ ,  $\mathcal{J}(T_1^-)$ ,  $\mathcal{J}(E^-)$  (or  $\mathcal{J}(T_2^-)$ ), dominated by  $^3P_0$ ,  $^3P_1$  and  $^3P_2$ - $^3F_2$ , respectively, where

$$\begin{aligned} F_C^{\mathcal{J}}(r) &\equiv \langle \mathcal{R}(\vec{r}, t; \mathcal{J}), \mathcal{R}(\vec{r}, t; \mathcal{J}) \rangle, \\ F_T^{\mathcal{J}}(r) &\equiv \langle \mathcal{R}(\vec{r}, t; \mathcal{J}), S_{12} \mathcal{R}(\vec{r}, t; \mathcal{J}) \rangle, \\ F_{LS}^{\mathcal{J}}(r) &\equiv \langle \mathcal{R}(\vec{r}, t; \mathcal{J}), \vec{L} \cdot \vec{S} \mathcal{R}(\vec{r}, t; \mathcal{J}) \rangle, \\ K^{\mathcal{J}}(r) &\equiv \langle \mathcal{R}(\vec{r}, t; \mathcal{J}), (D_t - H_0) \mathcal{R}(\vec{r}, t; \mathcal{J}) \rangle. \end{aligned}$$

In Fig. 1, we also plot  $V_{C,S=1}^{I=1}(r)$  (red),  $V_T^{I=1}(r)$  (black) and  $V_{LS}^{I=1}(r)$  (blue), obtained from  $A_1^-$ ,  $T_1^-$ ,  $E^-$  sources. (The result obtained from  $A_1^-$ ,  $T_1^-$ ,  $T_2^-$  sources instead does not show a significant difference.) We observe that (i) the central potential  $V_{C,S=1}^{I=1}(r)$  is repulsive, (ii) the tensor potential  $V_T^{I=1}(r)$  is positive and weak compared to  $V_{C,S=1}^{I=1}(r)$  and  $V_{LS}^{I=1}(r)$ , and (iii) the spin-orbit potential

$V_{LS}^{I=1}(r)$  is negative and strong. These features agree qualitatively well with those of the phenomenological potential in Ref. [27].

For both spin-singlet and spin-triplet central potentials, there may be a very weak attractive pocket of less than a few MeV at medium distance ( $r \simeq 1$  fm). However, considering the statistical and systematic errors, its existence should be carefully examined in future studies.

We make a technical comment. We sometimes observe large condition numbers for Eq. (19) (with three sources) near the spatial boundaries, which gives rise to points with large statistical errors at  $r \simeq 1$ – $1.5$  fm in Fig. 1.

#### 4.4. Scattering phase shifts and effective potentials

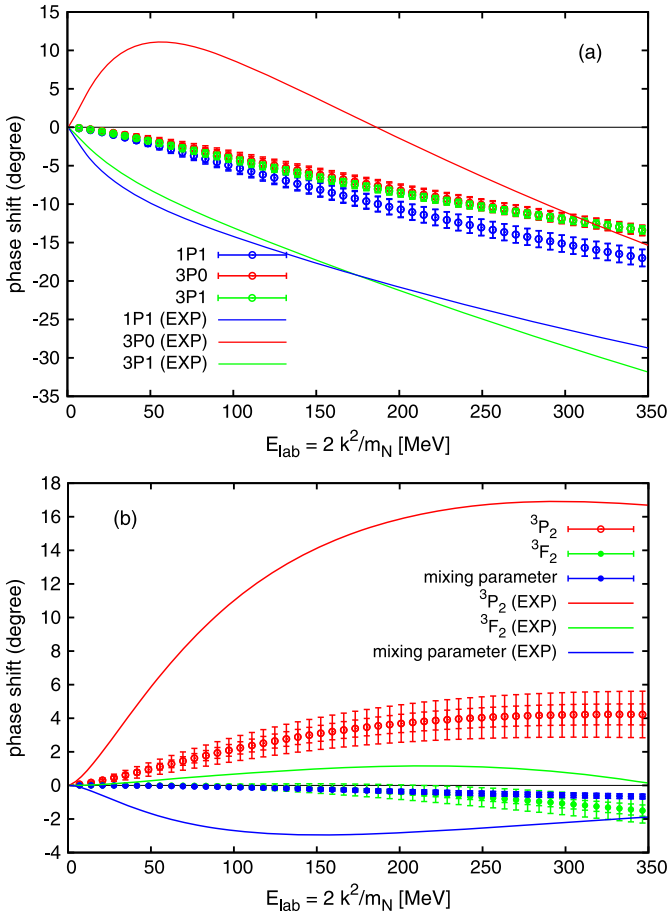
For quantitative studies of the interactions, it is desirable to calculate not only the potential but also scattering phase shifts, since the potential is not a physical observable as mentioned above. In this section, we therefore investigate a nature of interactions, by calculating scattering phase shifts from the obtained potentials. In particular, we study whether the LS potential of Fig. 1 leads to attractive behaviors in the scattering phase shifts in the  $^3P_2$  channel.

We calculate the scattering phase shifts by solving the Schrödinger equation with the above potentials, parameterized with multi-Gaussian forms,  $v(r) \equiv \sum_{i=1}^{N_{\text{gauss}}} a_i \exp(-v_i(r/b)^2)$  with  $N_{\text{gauss}} = 3$  for the central and spin-orbit potentials, whereas  $v(r) \equiv a_1(r/b) \exp(-v_1(r/b)^2) + a_2(r/b)^3 \exp(-v_2(r/b)^2)$  for the tensor potential to mimic the short distance behavior, as shown in Fig. 1. Here, a scaling parameter  $b \equiv 0.1555$  fm is introduced to simplify the notation. The uncorrelated fits are performed reasonably. The resultant fit parameters and  $\chi^2/\text{d.o.f.}$  are given in Table 1.

The scattering observables are obtained from the long distance behaviors of linearly independent regular solutions, and are shown in Fig. 2. The inner error is statistical, while the outer one is statistical and systematic combined in quadrature. Here, to estimate the systematic error, we take into account the uncertainty arising from the truncation of the derivative expansion and from the choice of fitting functions for the potentials. To estimate systematic errors associated with the truncation of the derivative expansion, we calculate phase shifts also at  $t - t_0 = 7$ , and take differences of central values between  $t - t_0 = 8$  and  $7$  as systematic errors. A dependence of phase shifts on a choice of fitting functions for the potentials is estimated by changing the fitting function to a Yukawa-type. It turns out that the former dominates the systematic error except

**Table 1**  
Fit parameters and  $\chi^2/\text{d.o.f.}$

Channel	$a_1$ [MeV]	$a_2$ [MeV]	$a_3$ [MeV]	$\nu_1$	$\nu_2$	$\nu_3$	$\chi^2/\text{d.o.f.}$
$V_{C;S=0}^{I=0}$	2173 (268)	762 (62)	236 (65)	11 (2)	2.1 (0.3)	0.6 (0.1)	1.7 (0.8)
$V_{C;S=1}^{I=1}$	421 (122)	233 (74)	397 (16)	11.5 (0.4)	1.3 (0.2)	3.9 (0.1)	1.1 (1.0)
$V_T^{I=1}$	711 (11)	16 (5)	–	2.6 (0.2)	0.5 (0.1)	–	0.8 (0.5)
$V_{LS}^{I=1}$	–45 (17)	–181 (5)	–315 (12)	0.4 (0.1)	1.4 (0.2)	5.3 (0.3)	3.6 (0.7)

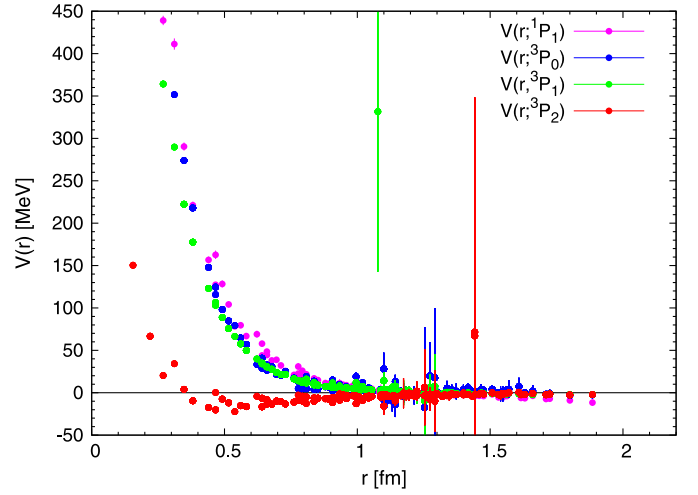


**Fig. 2.** The scattering phase shifts from Schrödinger equations by using the potentials obtained at  $m_\pi \simeq 1133$  MeV from lattice QCD. (a) Phase shifts in  $^1P_1$ ,  $^3P_0$  and  $^3P_1$  together with the experimental ones for comparison. (b) The phase shifts and mixing parameter in the  $^3P_2$ – $^3F_2$  channel together with the experimental ones. (Stapp's convention is adopted [30].) The inner error is statistical, while the outer one is statistical and systematic combined in quadrature.

that the latter dominates in the  $^3F_2$  channel. Although the magnitude of the phase shifts obtained from our potentials are smaller than the experimental ones, general trends are well reproduced except for the  $^3P_0$  case at low energies. The missing attraction in the  $^3P_0$  channel is likely due to the weak tensor force  $V_T$  caused by the large pion mass. Among others, the most interesting feature in Fig. 2 is the attraction in the  $^3P_2$  channel, which is directly related to the pairing correlation of the neutrons inside the neutron stars.

To obtain an intuitive understanding of the behavior of these phase shifts, we plot the potentials of the  $^1P_1$ ,  $^3P_0$ ,  $^3P_1$  and  $^3P_2$  channels in Fig. 3, as defined in Eqs. (1)–(3) and below. Indeed, one can see that  $V(r; ^3P_2)$  has a weak repulsive core surrounded by an attractive well; the attraction is driven by the strongly attractive LS force,  $V_{LS}^{I=1}(r)$  in Fig. 1.

We give a comment on the reliable energy region of these phase shifts. Through the time-dependent method, these phase



**Fig. 3.** The potentials for the  $^1P_1$ ,  $^3P_0$ ,  $^3P_1$  and  $^3P_2$  channels given in Eqs. (1)–(3) and below.

shift are obtained based on the NBS wave functions at the energy points  $E_{\text{lab}} \simeq 2(2\pi/L)^2 \bar{n}^2/m_N \simeq 230, 460, 690, \dots$  MeV with  $\bar{n} \in \mathbb{Z}^3$ . Except for these energy points and the point for  $E_{\text{lab}} = 0$  where the value of the phase shift vanishes by definition, the phase shifts are obtained by assuming that the derivative expansion is converged so that the truncated potentials in Eq. (6) do not depend on the energy. Because the contribution from  $E_{\text{lab}} \simeq 230$  MeV gradually dominates the intermediate states in  $\mathcal{R}(\vec{r}, t; \mathcal{J})$  as  $t$  increases, the most reliable energy region of the phase shift is around  $E_{\text{lab}} \simeq 230$  MeV. Reliability for  $E_{\text{lab}} < 230$  MeV can be explicitly examined by enlarging the spatial volume. Reliability for  $E_{\text{lab}} > 230$  MeV can be examined by changing relative weight of excited states, which can be done either by studying the  $t$  dependence of the truncated potentials or by changing the two-nucleon source operator. Note that these arguments apply to any choices of interpolating fields.

## 5. Conclusion

We have made a first attempt to determine NN potentials up to NLO in the parity-odd sector, which appears in the  $^1P_1$ ,  $^3P_0$ ,  $^3P_1$ ,  $^3P_2$ – $^3F_2$  channels. Using  $N_f = 2$  CP-PACS gauge configurations on a  $16^3 \times 32$  lattice ( $a \simeq 0.16$  fm and  $m_\pi \simeq 1100$  MeV), not only the central and the tensor potentials but also the spin–orbit potential have been derived for the first time. These potentials are constructed from NBS wave functions for  $J^P = 0^-, 1^-, 2^-$ , which are generated by using the momentum wall sources with projections based on the representation theory of the cubic group.

We have observed that the qualitative behavior of the resultant potentials agree with those of phenomenological potentials: For the spin-singlet sector, the central potential  $V_{C;S=0}^{I=0}(r)$  is repulsive with a strong repulsive core at short distance. For the spin-triplet sector, (i) the central potential  $V_{C;S=1}^{I=1}(r)$  is also repulsive with a repulsive core at short distance, (ii) the tensor potential  $V_T^{I=1}(r)$  is

positive and quite weak, and (iii) the spin-orbit potential  $V_{LS}^{I=1}(r)$  is negative and strong at short distance.

We have then calculated scattering observables in the  $^1P_1$ ,  $^3P_0$ ,  $^3P_1$  and  $^3P_2$ - $^3F_2$  channels, by solving Schrödinger equations with these potentials. It is interesting enough that we obtain, from the first principle lattice QCD approach, attractive phase shift driven by the strongly attractive LS force in the  $^3P_2$  channel, which has been known experimentally and has various implications in atomic nuclei and dense matter, though the magnitude of the phase shift is still small due to the large quark mass in our calculation. This, together with the missing attraction in the  $^3P_0$  channel, indicates an importance to carry out simulations at and around the physical quark mass.

### Acknowledgements

Numerical calculations in this report are performed on the University of Tsukuba Supercomputer system (T2K). This work is supported in part by the JSPS Grant-in-Aid for Scientific Research (23540321, 24740144, 24740146), the JSPS Grant-in-Aid for Scientific Research on Innovative Areas (Nos. 2004: 20105001, 20105003) and SPIRE (Strategic Program for Innovative Research). We are also grateful for authors and maintainer of CPS++ [28], a modified version of which is used for this work. We thank the CP-PACS Collaboration and ILDG/JLDG for providing the 2-flavor QCD gauge configurations [25,29].

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