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Orthonormal basis of the octonionic analytic functions ☆

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ABSTRACT

By confirming a conjecture proposed in Li and Peng (2001) [1], we obtain the orthonormal basis for the octonionic analytic functions.

1. Introduction

If a complex function $f(z)$ is holomorphic (analytic) in the annular domain ($0 \leqslant R_1 < |z| < R_2 \leqslant \infty$), then the function f can be expanded into a unique Laurent series

$$f(z) = \sum_{-\infty}^{+\infty} a_n z^n.$$

Restricted in $|z| = 1$, $\{z^n\}$ become the orthonormal basis of the compact group $\{e^{i\theta} \mid 0 \leqslant \theta \leqslant 2\pi\}$. In hypercomplex function theory, including quaternionic analysis and Clifford analysis, the above facts have their high-dimensional analogues [2]. It is the fact that quaternionic analytic functions exist orthonormal basis helps us to calculate the Cauchy–Szegö kernel in the quaternionic Heisenberg type group [3]. In order to get the Cauchy–Szegö kernel in the octonionic Heisenberg type group, we need the existence of orthonormal basis for the octonionic analytic functions. But, because of the non-associativity of octonions, the existence of octonionic orthonormal basis is still keeping in open.

In this paper, by confirming a conjecture proposed in [1], we obtain the orthonormal basis for the octonionic analytic function. As an application, we give the explicit formulas to calculate the coefficients in the octonionic Laurent series obtained in [4].

2. Preliminaries

As we know, there are only four normed division algebras [5–7]: the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$, with the relations $\mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H} \subseteq \mathbb{O}$. In other words, for any $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, if we define a product "xy" such that $xy \in \mathbb{R}^n$ and $|xy| = |x||y|$, where $|x| = \sqrt{\sum_1^n x_i^2}$, then the only four values of n are $1, 2, 4, 8$.

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Quaternions $\mathbb{H}$ is not commutative and octonions $\mathbb{O}$ is neither commutative nor associative. Unlike $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, the non-associative octonions cannot be embedded into the associative Clifford algebras. Let $e_0, e_1, \ldots, e_7$ be the basis of $\mathbb{O}$ and denote the set $\mathscr{W}$ by

$$\mathscr{W} = \{(1,2,3),(1,4,5),(2,4,6),(3,4,7),(2,5,7),(6,1,7),(5,3,6)\},$$

the multiplication rules between the basis are given as follows [7,8]:

$$e_0^2 = e_0, \qquad e_i e_0 = e_0 e_i = e_i, \qquad e_i^2 = -1, \quad i = 1,2,\ldots,7,$$

and for any triple $(\alpha,\beta,\gamma) \in \mathscr{W}$,

$$e_\alpha e_\beta = e_\gamma = -e_\beta e_\alpha, \qquad e_\beta e_\gamma = e_\alpha = -e_\gamma e_\beta, \qquad e_\gamma e_\alpha = e_\beta = -e_\alpha e_\gamma.$$

For each $x = \sum_0^7 x_i e_i \in \mathbb{O}$ ($x_i \in \mathbb{R}$, $i = 0,1,\ldots,7$), $\mathrm{Sc}\,x = x_0$ is called the scalar part of x and $\underline{x} = \sum_1^7 x_i e_i$ is termed its vector part. The norm of x is $|x| = (\sum_0^7 x_i^2)^{\frac{1}{2}}$ and its conjugate is defined by $\bar{x} = \sum_0^7 x_i \bar{e}_i = x_0 - \underline{x}$. We have $x\bar{x} = \bar{x}x = \sum_0^7 x_i^2$, $\overline{xy} = \bar{y}\bar{x}$ ($x, y \in \mathbb{O}$), $x^{-1} = \frac{\bar{x}}{|x|^2}$ ($x \neq 0$).

Let $x = \sum_0^7 x_i e_i = x_0 + \underline{x}$, $y = \sum_0^7 y_i e_i = y_0 + \underline{y}$ ($x_i, y_i \in \mathbb{R}$, $i = 0,1,\ldots,7$), then

$$xy = x_0 y_0 - \underline{x} \cdot \underline{y} + x_0 \underline{y} + y_0 \underline{x} + \underline{x} \times \underline{y},$$

where $\underline{x} \cdot \underline{y} := \sum_1^7 x_i y_i$ is the inner product of vectors $\underline{x}$, $\underline{y}$ and

$$\begin{aligned}\underline{x} \times \underline{y} := {} & e_1(A_{23} + A_{45} - A_{67}) + e_2(-A_{13} + A_{46} + A_{57}) + e_3(A_{12} + A_{47} - A_{56}) + e_4(-A_{15} - A_{26} - A_{37}) \\ & + e_5(A_{14} - A_{27} + A_{36}) + e_6(A_{17} + A_{24} - A_{35}) + e_7(-A_{16} + A_{25} + A_{34}), \\ A_{ij} = {} & \det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}, \quad i, j = 1,2,\ldots,7.\end{aligned}$$

For any $x, y \in \mathbb{O}$, we have (see [9]):

$$(\underline{x} \times \underline{y}) \cdot \underline{x} = 0, \qquad (\underline{x} \times \underline{y}) \cdot \underline{y} = 0, \qquad \underline{x} \parallel \underline{y} \iff \underline{x} \times \underline{y} = 0, \qquad \underline{x} \times \underline{y} = -\underline{y} \times \underline{x}.$$

For any $x, y, z \in \mathbb{O}$, the **associator** $[x,y,z]$ is defined by $[\mathbf{x},\mathbf{y},\mathbf{z}] = (\mathbf{xy})\mathbf{z} - \mathbf{x}(\mathbf{yz})$. Octonions obey some weakened associative laws, including the so-called Moufang identities [5,6]: for any $x, y, z, u, v \in \mathbb{O}$,

$$\begin{aligned}&[x,y,z] = [y,z,x] = [z,x,y], \qquad [x,y,z] = -[y,x,z], \qquad [x,x,y] = [\bar{x},x,y] = 0, \\ &(uvu)x = u\big(v(ux)\big), \qquad x(uvu) = \big((xu)v\big)u, \qquad u(xy)u = (ux)(uy).\end{aligned}$$

Proposition 2.1. *(See [8].) For any $i, j, k \in \{0,1,\ldots,7\}$, we have*

$$[e_i, e_j, e_k] = 0 \iff ijk = 0, \quad \text{or } (i-j)(j-k)(k-i) = 0, \quad \text{or } (e_i e_j)e_k = \pm 1.$$

Proposition 2.2. *(See [8].) Let e_i, e_j, e_k be three different elements of $\{e_1, e_2, \ldots, e_7\}$, $(e_i e_j)e_k \neq \pm 1$. Then $(e_i e_j)e_k = -e_i(e_j e_k)$.*

Proposition 2.3. *Let $x_1, x_2, \ldots, x_n \in \mathbb{O}$, $(l_1, \ldots, l_k)$ be k elements out of $\{1, \ldots, n\}$, repetitions being allowed. Let $(x_{l_1} x_{l_2} \cdots x_{l_k})_{\otimes_k}$ be the product of k octonions in a fixed associative order $\otimes_k$. Then $\sum_{\pi(l_1,\ldots,l_k)} (x_{l_1} x_{l_2} \cdots x_{l_k})_{\otimes_k}$ is independent of the associative order $\otimes_k$, where the sum runs over all distinguishable permutations of $(l_1, \ldots, l_k)$.*

Proof. To see this, notice that for any associative order $\otimes_k$, the sum

$$\sum_{\pi(l_1,\ldots,l_k)} (x_{l_1} x_{l_2} \cdots x_{l_k})_{\otimes_k}$$

is just the coefficient of $(s_1! s_2! \cdots s_n!)\lambda_1 \lambda_2 \cdots \lambda_k$ in the expression $(xx\cdots x)_{\otimes_k}$, where $x = \sum_{i=1}^k \lambda_i x_{l_i}$, and s_i is the appearing times of i in $(l_1, \ldots, l_k)$, $1 \leqslant i \leqslant n$. The result follows by the power-associativity of octonions (see [10,11]). □

Let Ω be an open set in $\mathbb{R}^8$. A function f in $C^1(\Omega, \mathbb{O})$ is said to be left (right) $\mathbb{O}$-analytic function in Ω when

$$Df = \sum_{i=0}^{7} e_i \frac{\partial f}{\partial x_i} = 0 \left(fD = \sum_{i=0}^{7} \frac{\partial f}{\partial x_i} e_i = 0 \right),$$

where the Dirac D-operator and its adjoint $\overline{D}$ are the first-order systems of differential operators in $C^1(\Omega, \mathbb{O})$ defined by $D = \sum_0^7 e_i \frac{\partial}{\partial x_i}$, $\overline{D} = \sum_0^7 \bar{e}_i \frac{\partial}{\partial x_i}$.

If f is a simultaneously left and right $\mathbb{O}$-analytic function, then f is called an $\mathbb{O}$-analytic function. If f is a left (right) $\mathbb{O}$-analytic function in $\mathbb{R}^8$, then f is called a left (right) $\mathbb{O}$-entire function. Since $D\overline{D} = \overline{D}D = \Delta_8 = \sum_0^7 \frac{\partial^2}{\partial x_i^2}$, the real-valued components of any left (right) $\mathbb{O}$-analytic function are always harmonic.

$\mu = (\mu_0, \mu_1, \dots, \mu_n)$ is called a Stein–Weiss conjugate harmonic system if they satisfy the following equations [12]:

$$\sum_{i=0}^{n} \frac{\partial \mu_i}{\partial x_i} = 0, \qquad \frac{\partial \mu_i}{\partial x_j} = \frac{\partial \mu_j}{\partial x_i}, \quad 0 \leqslant i < j \leqslant n.$$

It is easy to see that if $F(x_0, x_1, \dots, x_7) = (f_0, f_1, \dots, f_7)$ is a Stein–Weiss conjugate harmonic system in an open set Ω of $\mathbb{R}^8$, then there exists a real-valued harmonic function Φ in Ω such that F is the gradient of Φ. Thus $\overline{F} = f_0e_0 - f_1e_1 - \cdots - f_7e_7 = \overline{D}\Phi$ is an $\mathbb{O}$-analytic function. But inversely, this is not true [13].

Proposition 2.4. *(See [8].) If f is left (right) $\mathbb{O}$-analytic in an open set $\Omega \subset \mathbb{R}^8$ and vanishes in the open set $\mathscr{E} \subset \Omega \cap \{x_0 = a_0\} \neq \emptyset$, then f is identically zero in Ω.*

For any $(l_1, \dots, l_k) \in \{1, 2, \dots, 7\}^k$, the polynomials $V_{l_1 \cdots l_k}$ of order k are defined by

$$V_{l_1 \cdots l_k}(x) = \frac{1}{k!} \sum_{\pi(l_1, \dots, l_k)} \big(\cdots\big((z_{l_1} z_{l_2}) z_{l_3}\big) \cdots\big) z_{l_k},$$

where the sum runs over all distinguishable permutations of $(l_1, \dots, l_k)$ and $z_{l_j} = x_{l_j} e_0 - x_0 e_{l_j}$, $j = 1, \dots, k$. $V_{l_1 \cdots l_k}$ are called the **inner spherical analytic functions of order** k [1], note that the polynomials $V_{l_1 \cdots l_k}$'s are the suitable substitutions of z^k's in complex analysis. For more references about octonions and octonionic analysis, we refer the reader to [1,4,8,13–15].

3. Proof of a conjecture

In [1], the authors proved that the polynomials $V_{l_1 \cdots l_k}(x)$ are all $\mathbb{O}$-analytic functions, and proposed the following conjecture:

Let $(l_1, \dots, l_k) \in \{1, 2, \dots, 7\}^k$ and $(s_1, \dots, s_t) \in \{1, 2, \dots, 7\}^t$. $M \in \mathscr{M}_8(\mathbb{R}^8)$ (by $\mathscr{M}_8(\Omega)$ we mean a set that consisting of M, where M is an 8-dimensional compact and oriented differentiable manifold with boundary ∂M contained in some open subset Ω of $\mathbb{R}^8$), for each $x \in \partial M$, let $n(x) = \sum_0^7 n_j e_j$ be the outer unit normal to ∂M at x, dS_x is the scalar element of surface area on ∂M, and $d\sigma = n\,dS$. Then

$$\int_{\partial M} (V_{l_1 \cdots l_k}\, d\sigma) V_{s_1 \cdots s_t} = \int_{\partial M} V_{l_1 \cdots l_k} (d\sigma\, V_{s_1 \cdots s_t}) = 0. \tag{3.1}$$

Now, we prepare to confirm this conjecture by proving the following theorems.

Theorem 3.1. *Let Ω be an open set of $\mathbb{R}^8$ and $M \in \mathscr{M}_8(\Omega)$.*

(i) *If f is a right $\mathbb{O}$-analytic function in Ω and $\bar{g}$ is a Stein–Weiss conjugate harmonic system in Ω, then $\int_{\partial M} (f\, d\sigma) g = 0$;*
(ii) *If g is a left $\mathbb{O}$-analytic function in Ω and $\overline{f}$ is a Stein–Weiss conjugate harmonic system in Ω, then $\int_{\partial M} f(d\sigma\, g) = 0$.*

Proof. Let $f = \sum_0^7 e_i f_i$, $d\sigma = \sum_0^7 (-1)^j e_j\, d\widehat{x}_j$ and $g = \sum_0^7 e_k g_k$. Then by Stokes's Theorem we have

$$\begin{aligned}
\int_{\partial M} (f\, d\sigma) g &= \int_{\partial M} \left(\sum_{i=0}^{7} e_i f_i \sum_{j=0}^{7} (-1)^j e_j\, d\widehat{x}_j \right) \left(\sum_{k=0}^{7} e_k g_k \right) \\
&= \int_{\partial M} \sum_{i,j,k=0}^{7} (-1)^j (e_i e_j) e_k f_i g_k\, d\widehat{x}_j \\
&= \int_{M} \sum_{i,j,k=0}^{7} (e_i e_j) e_k \frac{\partial}{\partial x_j} (f_i g_k)\, dx \\
&= \int_{M} \sum_{i,j,k=0}^{7} (e_i e_j) e_k \left(\frac{\partial f_i}{\partial x_j} g_k + f_i \frac{\partial g_k}{\partial x_j} \right) dx
\end{aligned}$$

$$= \int_M \sum_{i,j,k=0}^{7} (e_i e_j) e_k \frac{\partial f_i}{\partial x_j} g_k \, dx + \int_M \sum_{i,j,k=0}^{7} \big(e_i (e_j e_k) + [e_i, e_j, e_k]\big) f_i \frac{\partial g_k}{\partial x_j} \, dx$$

$$= \int_M (fD) g \, dx + \int_M f (Dg) \, dx + \int_M \sum_{j,k=0}^{7} [f, e_j, e_k] \frac{\partial g_k}{\partial x_j} \, dx. \tag{3.2}$$

If f is a right $\mathbb{O}$-analytic function and $\overline{g}$ is a Stein–Weiss conjugate harmonic system in Ω, then $fD = 0$, $Dg = 0$. Thus (3.2) leads to

$$\int_{\partial M} (f \, d\sigma) g = \int_M \sum_{j,k=0}^{7} [f, e_j, e_k] \frac{\partial g_k}{\partial x_j} \, dx.$$

In view of Proposition 2.1, we have $[f, e_j, e_k] = 0$ when $j = 0$ or $k = 0$ or $j = k$. Note that $[f, e_k, e_j] = -[f, e_j, e_k]$ $(1 \leqslant j \neq k \leqslant 7)$ and

$$\frac{\partial g_j}{\partial x_k} = \frac{\partial g_k}{\partial x_j}, \quad j, k = 1, 2, \ldots, 7,$$

we have

$$\int_{\partial M} (f \, d\sigma) g = \int_M \sum_{\substack{j,k=1 \\ j \neq k}}^{7} [f, e_j, e_k] \frac{\partial g_k}{\partial x_j} \, dx$$

$$= \int_M \sum_{1 \leqslant j < k \leqslant 7} \left([f, e_j, e_k] \frac{\partial g_k}{\partial x_j} + [f, e_k, e_j] \frac{\partial g_j}{\partial x_k} \right) dx$$

$$= \int_M \sum_{1 \leqslant j < k \leqslant 7} [f, e_j, e_k] \left(\frac{\partial g_k}{\partial x_j} - \frac{\partial g_j}{\partial x_k} \right) dx = 0.$$

The proof of (ii) is similar to (i). □

Theorem 3.2. *Let Ω be an open set in $\mathbb{R}^8$ and $f \in C^1(\Omega, \mathbb{O})$. Then*

$$\int_{\partial M} (\lambda \, d\sigma) f = 0 \qquad \left(\int_{\partial M} f (d\sigma \, \lambda) = 0 \right)$$

for any $M \in \mathscr{M}_8(\Omega)$ and any constant $\lambda \in \mathbb{O}$ if and only if $\overline{f}$ is a Stein–Weiss conjugate harmonic system in Ω.

Remark. This theorem shows the conditions in Theorem 3.1 are reasonable.

Lemma 3.3. *(See [16].) Let Ω be an open set in $\mathbb{R}^8$ and $f \in C^1(\Omega, \mathbb{O})$. Then $f\lambda$ (λf) is a left (right) $\mathbb{O}$-analytic function for any $\lambda \in \mathbb{O}$ if and only if $\overline{f}$ is a Stein–Weiss conjugate harmonic system.*

Lemma 3.4 *(Morera type). Let Ω be an open set in $\mathbb{R}^8$ and $f \in C^1(\Omega, \mathbb{O})$. Then for any $M \in \mathscr{M}_8(\Omega)$,*

$$\int_{\partial M} d\sigma \, f = 0 \qquad \left(\int_{\partial M} f \, d\sigma = 0 \right),$$

if and only if f is a left (right) $\mathbb{O}$-analytic function in Ω.

Proof. We only prove the necessity. If there exists a point $x_0 \in \Omega$ such that $Df(x_0) \neq 0$, we assume without loss of generality that $\mathrm{Sc}(Df(x_0)) > 0$. Then there exists $M_0 \in \mathscr{M}_8(\Omega)$ such that $\mathrm{Sc}(Df(x)) > 0$ for any $x \in M_0$, thus we have

$$\mathrm{Sc}\left(\int_{\partial M_0} d\sigma_x \, f(x) \right) = \int_{M_0} \mathrm{Sc}\big(Df(x)\big) \, dx > 0.$$

This is a contradiction. □

Proof of Theorem 3.2. By Theorem 3.1, the sufficiency is obvious. Inversely, we take $\lambda = 1$ in $\int_{\partial M}(\lambda d\sigma)f$, and M is arbitrarily, Lemma 3.4 gives $Df = 0$. Combining this with (3.2), for any constant λ and for any $M \in \mathscr{M}_8(\Omega)$, we have

$$\int_{\partial M}(\lambda\, d\sigma)f = \int_M \sum_{j,k=0}^{7}[\lambda, e_j, e_k]\frac{\partial f_k}{\partial x_j}\,dx = \int_M \sum_{j,k=0}^{7}\big((e_je_k)\lambda - e_j(e_k\lambda)\big)\frac{\partial f_k}{\partial x_j}\,dx$$
$$= \int_M \big((Df)\lambda - D(f\lambda)\big)\,dx = -\int_M D(f\lambda)\,dx = 0.$$

Hence $D(f\lambda) = 0$ in Ω for any constant λ and then Lemma 3.3 shows that $\overline{f}$ is a Stein–Weiss conjugate harmonic system. □

Theorem 3.5. *All the $\overline{V}_{l_1\cdots l_k}(x)$ are Stein–Weiss conjugate harmonic systems in $\mathbb{R}^8$.*

Proof. We claim that

$$V_{l_1\cdots l_k}(x) = \frac{1}{s!}\sum_{n=0}^{[\frac{k}{2}]}(-1)^n\overline{D}\left(\frac{x_0^{2n+1}}{(2n+1)!}\Delta_7^n x_1^{s_1}\cdots x_7^{s_7}\right), \tag{3.3}$$

where s_i $(i = 1, 2, \ldots, 7)$ is the appearing times of i in $(l_1, \cdots, l_k)$, $s! = s_1!s_2!\cdots s_7!$ and $\Delta_7 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_7^2}$ is the Laplace operator in $\mathbb{R}^7$.

Indeed, let $x_0 = 0$, then the both sides of (3.3) equal to $\frac{1}{s!}x_1^{s_1}\cdots x_7^{s_7}$. Also,

$$D\left(\frac{1}{s!}\sum_{n=0}^{[\frac{k}{2}]}(-1)^n\overline{D}\left(\frac{x_0^{2n+1}}{(2n+1)!}\Delta_7^n x_1^{s_1}\cdots x_7^{s_7}\right)\right)$$
$$= \frac{1}{s!}\sum_{n=0}^{[\frac{k}{2}]}(-1)^n\Delta_8\left(\frac{x_0^{2n+1}}{(2n+1)!}\Delta_7^n x_1^{s_1}\cdots x_7^{s_7}\right)$$
$$= \frac{1}{s!}\sum_{n=1}^{[\frac{k}{2}]}(-1)^n\frac{x_0^{2n-1}}{(2n-1)!}\Delta_7^n x_1^{s_1}\cdots x_7^{s_7} + \frac{1}{s!}\sum_{n=0}^{[\frac{k}{2}]-1}(-1)^n\frac{x_0^{2n+1}}{(2n+1)!}\Delta_7^{n+1}x_1^{s_1}\cdots x_7^{s_7}$$
$$= 0.$$

Since all the $V_{l_1\cdots l_k}(x)$ are $\mathbb{O}$-analytic functions in $\mathbb{R}^8$ [1], by Proposition 2.4 we have (3.3).

Equality (3.3) shows that $\overline{V}_{l_1\cdots l_k}(x)$ is just the gradient of real-valued harmonic function

$$\frac{1}{s!}\sum_{n=0}^{[\frac{k}{2}]}(-1)^n\frac{x_0^{2n+1}}{(2n+1)!}\Delta_7^n x_1^{s_1}\cdots x_7^{s_7},$$

hence all the polynomials $\overline{V}_{l_1\ldots l_k}(x)$ are Stein–Weiss conjugate harmonic systems in $\mathbb{R}^8$. □

Remarks.

(1) Combining Theorems 3.1 and 3.5 we confirm the conjecture.
(2) In the case of associative algebras, such as quaternion and Clifford algebra, the proofs of the conjecture and Theorem 3.1 are all trivial. In fact (see [2]), there only requires that $f(x)$ is a right Clifford monogenic function and $g(x)$ is a left Clifford monogenic function in an open subset Ω of $\mathbb{R}^{m+1}$, then $\int_{\partial M} f\, d\sigma\, g = 0$ for any $M \in \mathscr{M}_{m+1}(\Omega)$. We also notice that the set of left (right) Clifford monogenic functions is a right (left) Clifford module. But in the octonionic case, this is not true, and Lemma 3.3 is a substitute for that.

4. Orthogonality of the octonionic analytic functions

For any $(l_1, \ldots, l_k) \in \{1, 2, \ldots, 7\}^k$, the **outer spherical analytic functions of order** k are defined by

$$W_{l_1\cdots l_k}(x) = (-1)^k\partial_{x_{l_1}}\cdots\partial_{x_{l_k}}E(x),$$

where $E(x) = \frac{1}{\omega_8}\frac{\bar{x}}{|x|^8}$ and $\omega_8 = \frac{\pi^4}{3}$ is the area of unit sphere S^7 in $\mathbb{R}^8$.

Now, we study the orthogonality relations of the polynomials $V_{l_1\cdots l_k}(x)$ and $W_{l_1\cdots l_k}(x)$. It is easy to see that $\overline{E}$ is a Stein–Weiss conjugate harmonic system in $\mathbb{R}^8_0 = \mathbb{R}^8 \setminus \{0\}$. Hence, all the $\overline{W}_{l_1\cdots l_k}$ are Stein–Weiss conjugate harmonic systems in $\mathbb{R}^8_0$. In the rest of this section we suppose $M \in \mathscr{M}_8(\mathbb{R}^8)$ and $0 \in \mathring{M}$.

Theorem 4.1. *Let* $(l_1, \ldots, l_k) \in \{1, 2, \ldots, 7\}^k$ *and* $(s_1, \ldots, s_t) \in \{1, 2, \ldots, 7\}^t$. *Then*

$$\int_{\partial M} (W_{l_1\cdots l_k}\, d\sigma) W_{s_1\cdots s_t} = \int_{\partial M} W_{l_1\cdots l_k}(d\sigma\, W_{s_1\cdots s_t}) = 0.$$

Proof. Choose a sufficiently large R such that $M \subset B(0, R)$. Since all the $\overline{W}_{l_1\cdots l_k}$ are Stein–Weiss conjugate harmonic systems in $B(0, R) \setminus \mathring{M}$, from Theorem 3.1 it follows that

$$\int_{\partial B(0,R)+\partial M^-} \big(W_{l_1\cdots l_k}(x)\, d\sigma_x\big) W_{s_1\cdots s_t}(x) = 0.$$

We thus get

$$\begin{aligned}\int_{\partial M} \big(W_{l_1\cdots l_k}(x)\, d\sigma_x\big) W_{s_1\cdots s_t}(x) &= \lim_{R\to\infty} \int_{\partial B(0,R)} \big(W_{l_1\cdots l_k}(x)\, d\sigma_x\big) W_{s_1\cdots s_t}(x) \\ &= \lim_{R\to\infty} \int_{S^7} R^{-(k+7)} \big(W_{l_1\cdots l_k}(\omega) R^7\, d\sigma_\omega\big) R^{-(t+7)} W_{s_1\cdots s_t}(\omega) \\ &= \lim_{R\to\infty} \frac{1}{R^{k+t+7}} \int_{S^7} \big(W_{l_1\cdots l_k}(\omega)\, d\sigma_\omega\big) W_{s_1\cdots s_t}(\omega) = 0.\end{aligned}$$

Similarly, we have $\int_{\partial M} W_{l_1\cdots l_k}(d\sigma\, W_{s_1\cdots s_t}) = 0$. □

Theorem 4.2 *(Orthogonality). Let* $(l_1, \ldots, l_k)$ *and* $(s_1, \ldots, s_t)$ *be as above. Denote*

$$\begin{aligned}\mathcal{J}_1 &= \int_{\partial M} (V_{l_1\cdots l_k}\, d\sigma) W_{s_1\cdots s_t}, \\ \mathcal{J}_2 &= \int_{\partial M} V_{l_1\cdots l_k}(d\sigma\, W_{s_1\cdots s_t}), \\ \mathcal{J}_3 &= \int_{\partial M} (W_{s_1\cdots s_t}\, d\sigma) V_{l_1\cdots l_k}, \\ \mathcal{J}_4 &= \int_{\partial M} W_{s_1\cdots s_t}(d\sigma\, V_{l_1\cdots l_k}).\end{aligned}$$

Then for $i = 1, 2, 3, 4$, *we have*

$$\mathcal{J}_i = \delta^{s_1\ldots s_t}_{l_1\ldots l_k} = \begin{cases} 1, & \text{if } (l_1, \ldots, l_k) = (s_1, \ldots, s_t); \\ 0, & \text{otherwise}. \end{cases} \tag{4.1}$$

Proof. The starting point is the known series expansion for the potential (see [17])

$$\frac{1}{|y-x|^6} = \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} \langle x, \nabla_y \rangle^t \frac{1}{|y|^6}, \tag{4.2}$$

which converges, together with all possible series of derivatives, normally in $|x| < R < |y|$ for each $R > 0$. Since $|y|^{-6}$ is the fundamental solution of the Laplacian Δ_8, each of the terms in (4.2) is a spherical harmonic in x, and is harmonic in $y \in \mathbb{R}^8_0$. The action of the operator $\overline{D}_x$ on (4.2) leads to the following expansion of the Cauchy kernel in $|x| < R < |y|$ for each $R > 0$:

$$E(y-x) = \sum_{t=0}^{\infty} \left(\frac{(-1)^t}{t!} \sum_{s_1=0}^{7} \cdots \sum_{s_t=0}^{7} x_{s_1} \cdots x_{s_t} \partial_{y_{s_1}} \cdots \partial_{y_{s_t}} E(y) \right). \tag{4.3}$$

Denote

$$\mathfrak{E}_{t,y}(x) = \frac{(-1)^t}{t!} \sum_{s_1=0}^{7} \cdots \sum_{s_t=0}^{7} x_{s_1} \cdots x_{s_t} \partial_{y_{s_1}} \cdots \partial_{y_{s_t}} E(y),$$

then $\overline{\mathfrak{E}}_{t,y}(x)$ is a Stein–Weiss conjugate harmonic system of order t in $|x| < |y|$.

Since $V_{l_1 \cdots l_k}(x)$ is an $\mathbb{O}$-analytic function in $\mathbb{R}^8$, by Cauchy's Integral Formula (see [14]) we have

$$V_{l_1 \cdots l_k}(x) = \int_{S^7} E(\omega - x)\big(d\sigma_\omega V_{l_1 \cdots l_k}(\omega)\big) \quad \text{for all } x \in \mathring{B}(0,1).$$

Combing this with (4.3) we obtain

$$V_{l_1 \cdots l_k}(x) = \sum_{t=0}^{\infty} \int_{S^7} \mathfrak{E}_{t,\omega}(x)\big(d\sigma_\omega V_{l_1 \cdots l_k}(\omega)\big),$$

which converges normally in $x \in \mathring{B}(0,1)$.

But $\overline{\mathfrak{E}}_{t,\omega}(x)$ is a Stein–Weiss conjugate harmonic system in $|x| < 1$, by Lemma 3.3 it follows that

$$\int_{S^7} \mathfrak{E}_{t,\omega}(x)\big(d\sigma_\omega V_{l_1 \cdots l_k}(\omega)\big), \tag{4.4}$$

is a left $\mathbb{O}$-analytic function in $\mathring{B}(0,1)$, so (4.4) has the following Taylor expansion in $|x| < 1$ (see [1, Lemma 2]):

$$\int_{S^7} \mathfrak{E}_{t,\omega}(x)\big(d\sigma_\omega V_{l_1 \cdots l_k}(\omega)\big) = \sum_{(s_1,\ldots,s_t)} V_{s_1 \cdots s_t}(x) \int_{S^7} W_{s_1 \cdots s_t}(\omega)\big(d\sigma_\omega V_{l_1 \cdots l_k}(\omega)\big),$$

where $(s_1, \ldots, s_t)$ runs over all possible combinations of t elements out of $\{1, \ldots, 7\}$, repetitions being allowed.

Thus we have

$$V_{l_1 \cdots l_k}(x) = \sum_{t=0}^{\infty} \sum_{(s_1,\ldots,s_t)} V_{s_1 \cdots s_t}(x) \int_{S^7} W_{s_1 \cdots s_t}(\omega)\big(d\sigma_\omega V_{l_1 \cdots l_k}(\omega)\big).$$

By the uniqueness of the Taylor expansion of $V_{l_1 \cdots l_k}(x)$ we have

$$\int_{S^7} W_{s_1 \cdots s_t}(\omega)\big(d\sigma_\omega V_{l_1 \cdots l_k}(\omega)\big) = \begin{cases} 1, & \text{if } (l_1, \ldots, l_k) = (s_1, \ldots, s_t); \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

In the following, we will show that $\mathcal{J}_4$ has another expression. By definition we have

$$\begin{aligned} W_{s_1 \cdots s_t}(x) &= \frac{(-1)^t}{\omega_8} \partial_{x_{s_1}} \cdots \partial_{x_{s_t}} \frac{\bar{x}}{|x|^8} \\ &= \frac{(-1)^{t+1}}{6\omega_8} \overline{D}\left(\partial_{x_{s_1}} \cdots \partial_{x_{s_t}} \frac{1}{|x|^6}\right) \\ &= \overline{D}\left(\frac{H_{s_1 \cdots s_t}(x)}{|x|^{2t+6}}\right), \end{aligned}$$

where $g_{s_1 \cdots s_t}(x) = |x|^{-2t-6} H_{s_1 \cdots s_t}(x)$ is a real-valued homogeneous harmonic function of order $-(t+6)$ in $\mathbb{R}^8_0$. Now

$$\Delta_8 g_{s_1 \cdots s_t}(x) = \frac{\Delta_8 H_{s_1 \cdots s_t}(x)}{|x|^{2t+6}}, \quad x \in \mathbb{R}^8_0,$$

which means that $H_{s_1 \cdots s_t}(x)$ is a real-valued spherical harmonic of order t. Furthermore

$$W_{s_1 \cdots s_t}(x) = \frac{1}{|x|^{2t+8}}\big(|x|^2 \overline{D} H_{s_1 \cdots s_t}(x) - (2t+6)\bar{x} H_{s_1 \cdots s_t}(x)\big),$$

by Theorem 3.1 we have

$$\begin{aligned}
\mathcal{J}_4 &= \int_{\partial M} W_{s_1\cdots s_t}(d\sigma V_{l_1\cdots l_k}) \\
&= \int_{S^7} W_{s_1\cdots s_t}(\omega)\big(d\sigma_\omega V_{l_1\cdots l_k}(\omega)\big) \\
&= \int_{S^7} \big(\overline{D}H_{s_1\cdots s_t}(\omega) - (2t+6)\bar{\omega}H_{s_1\cdots s_t}(\omega)\big)\big(d\sigma_\omega V_{l_1\cdots l_k}(\omega)\big) \\
&= \int_{S^7} \overline{D}H_{s_1\cdots s_t}(\omega)\big(d\sigma_\omega V_{l_1\cdots l_k}(\omega)\big) - (2t+6)\int_{S^7} \bar{\omega}H_{s_1\cdots s_t}(\omega)\big(d\sigma_\omega V_{l_1\cdots l_k}(\omega)\big) \\
&= -(2t+6)\int_{S^7} \bar{\omega}H_{s_1\cdots s_t}(\omega)\big(\omega dS_\omega V_{l_1\cdots l_k}(\omega)\big) \\
&= -(2t+6)\int_{S^7} (\bar{\omega}\omega)H_{s_1\cdots s_t}(\omega)V_{l_1\cdots l_k}(\omega)\,dS_\omega \\
&= -(2t+6)\int_{S^7} H_{s_1\cdots s_t}(\omega)V_{l_1\cdots l_k}(\omega)\,dS_\omega.
\end{aligned} \tag{4.6}$$

The fifth and sixth steps in (4.6) hold is due to $DH_{s_1\cdots s_t}(x)$ is a Stein–Weiss conjugate harmonic system in $\mathbb{R}^8$ and the properties of associator, respectively.

A similar deduction yields

$$\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}_3 = -(2t+6)\int_{S^7} H_{s_1\cdots s_t}(\omega)V_{l_1\cdots l_k}(\omega)\,dS_\omega = \mathcal{J}_4.$$

Combining this with (4.5) we have (4.1). □

5. Application

Let $f(x)$ be left $\mathbb{O}$-analytic in the annular domain $\mathscr{D} = \mathring{B}(0,R_2) \setminus \bar{B}(0,R_1)$ $(0 \leqslant R_1 < R_2)$. Then the function f may be expanded into a unique Laurent series in $\mathscr{D}$ [4]:

$$f(x) = \sum_{k=0}^{\infty} P_k f(x) + \sum_{k=0}^{\infty} Q_k f(x),$$

where

$$P_k f(x) = \sum_{(l_1,\ldots,l_k)} V_{l_1\cdots l_k}(x)\lambda_{l_1\cdots l_k} \tag{5.1}$$

and

$$Q_k f(x) = \sum_{(l_1,\ldots,l_k)} W_{l_1\cdots l_k}(x)\mu_{l_1\cdots l_k}. \tag{5.2}$$

But the representations of coefficients in (5.1) and (5.2) are not satisfying, because the authors did not determine the analyticity of $Q_k f(x) = \sum_{(l_1,\ldots,l_k)} W_{l_1\cdots l_k}(x)\mu_{l_1\cdots l_k}$.

Since $\overline{W}_{l_1\cdots l_k}(x)$ is a Stein–Weiss conjugate harmonic system in $\mathbb{R}^8_0$, by Lemma 3.3, we now conclude that $Q_k f(x)$ is a left $\mathbb{O}$-analytic function in $\mathbb{R}^8_0$ for each $k \in \mathbb{N}$. Furthermore, by using our new results, we can determine the coefficients in the Laurent series.

Theorem 5.1. *The coefficients in* (5.1) *and* (5.2) *are given by*

$$\lambda_{l_1\cdots l_k} = \int_{\partial B(0,R)} W_{l_1\cdots l_k}(y)\big(d\sigma_y f(y)\big) \tag{5.3}$$

and

$$\mu_{l_1\cdots l_k} = \int_{\partial B(0,R)} V_{l_1\cdots l_k}(y)\big(d\sigma_y f(y)\big), \tag{5.4}$$

respectively, where the radius R being arbitrary chosen in (R_1, R_2).

Proof. Let $(s_1, \ldots, s_t) \in \{1, 2, \ldots, 7\}^t$. Since $f(x)$ is left $\mathbb{O}$-analytic in $\mathscr{D}$, the integral

$$\int_{\partial B(0,R)} W_{s_1\cdots s_t}(y)\big(d\sigma_y f(y)\big)$$

exists for any $R_1 < R < R_2$. Thus we have

$$\begin{aligned}\int_{\partial B(0,R)} W_{s_1\cdots s_t}(y)\big(d\sigma_y f(y)\big) &= \int_{\partial B(0,R)} W_{s_1\cdots s_t}(y)\left(d\sigma_y\left(\sum_{k=0}^{\infty} P_k f(x) + \sum_{k=0}^{\infty} Q_k f(x)\right)\right)\\ &= \sum_{k=0}^{\infty}\sum_{(l_1,\ldots,l_k)}\int_{\partial B(0,R)} W_{s_1\cdots s_t}(y)\big(d\sigma_y\big(V_{l_1\cdots l_k}(y)\lambda_{l_1\cdots l_k}\big)\big)\\ &\quad+\sum_{k=0}^{\infty}\sum_{(l_1,\ldots,l_k)}\int_{\partial B(0,R)} W_{s_1\cdots s_t}(y)\big(d\sigma_y\big(W_{l_1\cdots l_k}(y)\mu_{l_1\cdots l_k}\big)\big).\end{aligned} \tag{5.5}$$

Note that $V_{l_1\cdots l_k}(y)\lambda_{l_1\cdots l_k}$ is a left $\mathbb{O}$-analytic function in $\mathbb{R}^8$, a similar technique as (4.6) yields

$$\int_{\partial B(0,R)} W_{s_1\cdots s_t}(y)\big(d\sigma_y\big(V_{l_1\cdots l_k}(y)\lambda_{l_1\cdots l_k}\big)\big) = -(2t+6)\int_{S^7} H_{s_1\cdots s_t}(\omega)V_{l_1\cdots l_k}(\omega)\,dS_\omega\lambda_{l_1\cdots l_k}.$$

Combining this with (4.6) and Theorem 4.2 we have

$$\int_{\partial B(0,R)} W_{s_1\cdots s_t}(y)\big(d\sigma_y\big(V_{l_1\cdots l_k}(y)\lambda_{l_1\cdots l_k}\big)\big) = \lambda_{l_1\cdots l_k}\delta_{l_1\ldots l_k}^{s_1\ldots s_t}. \tag{5.6}$$

From Theorem 3.1 it follows that

$$\begin{aligned}\int_{\partial B(0,R)} W_{s_1\cdots s_t}(y)\big(d\sigma_y\big(W_{l_1\cdots l_k}(y)\mu_{l_1\cdots l_k}\big)\big) &= \lim_{R\to\infty}\int_{\partial B(0,R)} W_{s_1\cdots s_t}(y)\big(d\sigma_y\big(W_{l_1\cdots l_k}(y)\mu_{l_1\cdots l_k}\big)\big)\\ &= \lim_{R\to\infty}\frac{1}{R^{k+t+7}}\int_{S^7} W_{s_1\cdots s_t}(\omega)\big(d\sigma_\omega\big(W_{l_1\cdots l_k}(\omega)\mu_{l_1\cdots l_k}\big)\big)\\ &= 0.\end{aligned} \tag{5.7}$$

Combine (5.5), (5.6) and (5.7) we get (5.3). Similarly, we can get (5.4). □

Remarks.

(1) According to the Laurent series [4], together with Theorems 4.2 and 5.1, we conclude that the set $\{W_{s_1\cdots s_t}, V_{l_1\cdots l_k}\}$ becomes an orthonormal basis for the left (right) octonionic analytic functions, where $(l_1, \ldots, l_k) \in \{1, 2, \ldots, 7\}^k$ and $(s_1, \ldots, s_t) \in \{1, 2, \ldots, 7\}^t$.

(2) Recently, by using some spherical harmonic functions, quite deferent with us, S. Bock and K. Gürlebeck construct a mutually orthogonal complete system in $L_2(S, \mathbb{R})$. We refer the reader to [18].

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