A note on generalized hypergeometric functions, KZ solutions, and gluon amplitudes

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Abstract

Some aspects of Aomoto's generalized hypergeometric functions on Grassmannian spaces $Gr(k+1, n+1)$ are reviewed. Particularly, their integral representations in terms of twisted homology and cohomology are clarified with an example of the $Gr(2, 4)$ case which corresponds to Gauss' hypergeometric functions. The cases of $Gr(2, n+1)$ in general lead to $(n+1)$-point solutions of the Knizhnik–Zamolodchikov (KZ) equation. We further analyze the Schechtman–Varchenko integral representations of the KZ solutions in relation to the $Gr(k+1, n+1)$ cases. We show that holonomy operators of the so-called KZ connections can be interpreted as hypergeometric-type integrals. This result leads to an improved description of a recently proposed holonomy formalism for gluon amplitudes. We also present a (co)homology interpretation of Grassmannian formulations for scattering amplitudes in $\mathcal{N}=4$ super Yang–Mills theory.
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1. Introduction

Recently, much attention is paid to Grassmannian formulations of scattering amplitudes in $\mathcal{N}=4$ super Yang–Mills theory. The Grassmannian formulations are initially proposed (see, *e.g.*, [1–6]) to make it manifest that the $\mathcal{N}=4$ super Yang–Mills amplitudes are invariant under the dual superconformal symmetry [7] at tree level. For more recent developments, see, *e.g.*, [8–13].

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As discussed in [12,13], these Grassmannian formulations have revived interests in a purely mathematical subject, *i.e.*, generalized hypergeometric functions on Grassmannian spaces $Gr(k + 1, n + 1)$, which were introduced and developed by Gelfand [14] and independently by Aomoto [15] many years ago. In relation to physics, it has been known that solutions of the Knizhnik–Zamolodchikov (KZ) equation in conformal field theory are expressed in terms of the generalized hypergeometric functions [16–20]. One of the main goals of this note is to present a clear and systematic review on these particular topics in mathematical physics. Particularly, we revisit integral representations of the KZ solutions by Schechtman and Varchenko [17,18] and analyze them in terms of a bilinear construction of hypergeometric integrals, using twisted homology and cohomology. Along the way, we also consider in detail Gauss' original hypergeometric functions in Aomoto's framework so as to familiarize ourselves to the concept of twisted homology and cohomology.

Another goal of this note is to study and understand analytic aspects of the holonomy operator of the so-called KZ connection. The holonomy of the KZ connection is first introduced by Kohno [21] (see also Appendix 4 in [15]) as a monodromy representation of the KZ equation in a form of the iterated integral [22]. Inspired by Kohno's result and Nair's observation [23] on the maximally helicity violating (MHV) amplitudes of gluons (also called the Parke–Taylor amplitudes [24]) in supertwistor space, the author has recently proposed a novel framework of deriving gluon amplitudes [25] where an S-matrix functional for the gluon amplitudes is defined in terms of the holonomy operator of a certain KZ connection. This framework, what we call the holonomy formalism, is intimately related to braid groups and Yangian symmetries. As mentioned in [26], the holonomy formalism also suggests a natural origin of the dual conformal symmetries. Towards the end of this note we would provide more rigorous mathematical foundations of the holonomy formalism and present an improved description of it. Lastly, we also consider the more familiar Grassmannian formulations of gluon amplitudes in the same framework. Namely, we analyze integral representations of the Grassmannian formulations and present a (co)homology interpretation of those integrals.

This note is organized as follows. In the next section we review some formal results of Aomoto's generalized hypergeometric functions on $Gr(k + 1, n + 1)$, based on textbooks by Japanese mathematicians [15,27–29]. We present a review in a pedagogical fashion since these results are not familiar enough to many physicists. In section 3 we consider a particular case $Gr(2, n + 1)$ and present its general formulation. In section 4 we further study the case of $Gr(2, 4)$ which reduces to Gauss' hypergeometric function. Imposing permutation invariance among branch points, we here obtain new realizations of the hypergeometric differential equation in a form of a first order Fuchsian differential equation.

In section 5 we apply Aomoto's results to the KZ equation. We first focus on four-point KZ solutions and obtain them in a form of the hypergeometric integral. We then show that $(n + 1)$-point KZ solutions in general can be represented by generalized hypergeometric functions on $Gr(2, n + 1)$. We further consider the Schechtman–Varchenko integral representations of the KZ solutions in this context. The $(n + 1)$-point KZ solutions can also be represented by the hypergeometric-type integrals on $Gr(k + 1, n + 1)$ but we find that there exist ambiguities in the construction of such integrals for $k \geq 2$. In section 6 we review the construction of the holonomy operators of the KZ connections. We make a (co)homology interpretation of the holonomy operator and obtain a better understanding of analytic properties of the holonomy operator.

The holonomy operator gives a monodromy representation of the KZ equation, which turns out to be a linear representation of a braid group. This mathematical fact has been one of the essential ingredients in the holonomy formalism for gluon amplitudes. In section 7 we briefly

review this holonomy formalism and present an improved description of it. In section 8 we also consider the Grassmannian formulations of gluon amplitudes. We observe that these formulations can also be interpreted in terms of the hypergeometric-type integrals. Lastly, we present a brief conclusion.

2. Aomoto's generalized hypergeometric functions

Definition

Let Z be a $(k+1) \times (n+1)$ matrix

$$
Z = \begin{pmatrix} z_{00} & z_{01} & z_{02} & \cdots & z_{0n} \\ z_{10} & z_{11} & z_{12} & \cdots & z_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ z_{k0} & z_{k1} & z_{k2} & \cdots & z_{kn} \end{pmatrix} \tag{2.1}
$$

where $k < n$ and the matrix elements are complex, $z_{ij} \in \mathbf{C}$ ($0 \leq i \leq k$; $0 \leq j \leq n$). A function of Z, which we denote $F(Z)$, is defined as *a generalized hypergeometric function on Grassmannian space $Gr(k+1, n+1)$* when it satisfies the following relations:

$$
\sum_{j=0}^{n} z_{ij} \frac{\partial F}{\partial z_{pj}} = -\delta_{ip} F \quad (0 \leq i, p \leq k) \tag{2.2}
$$

$$
\sum_{i=0}^{k} z_{ij} \frac{\partial F}{\partial z_{ij}} = \alpha_j F \quad (0 \leq j \leq n) \tag{2.3}
$$

$$
\frac{\partial^2 F}{\partial z_{ip} \partial z_{jq}} = \frac{\partial^2 F}{\partial z_{iq} \partial z_{jp}} \quad (0 \leq i, j \leq k; \ 0 \leq p, q \leq n) \tag{2.4}
$$

where the parameters α_j obey the non-integer conditions

$$
\alpha_j \notin \mathbf{Z} \quad (0 \leq j \leq n) \tag{2.5}
$$

$$
\sum_{j=0}^{n} \alpha_j = -(k+1) \tag{2.6}
$$

Integral representation of $F(Z)$ and twisted cohomology

The essence of Aomoto's generalized hypergeometric function [15] is that, by use of the so-called twisted de Rham cohomology,[1] $F(Z)$ can be written in a form of integral:

$$
F(Z) = \int_{\Delta} \Phi \omega \tag{2.7}
$$

where

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[1] The *twisted* de Rham cohomology is a version of the ordinary de Rham cohomology into which multivalued functions, such as Φ in (2.8), are incorporated. For mathematical rigor on this, see Section 2 in [15].

$$\Phi = \prod_{j=0}^{n} l_j(\tau)^{\alpha_j} \tag{2.8}$$

$$l_j(\tau) = \tau_0 z_{0j} + \tau_1 z_{1j} + \cdots + \tau_k z_{kj} \quad (0 \le j \le n) \tag{2.9}$$

$$\omega = \sum_{i=0}^{k} (-1)^i \tau_i d\tau_0 \wedge d\tau_1 \wedge \cdots \wedge d\tau_{i-1} \wedge d\tau_{i+1} \wedge \cdots \wedge d\tau_k \tag{2.10}$$

The complex variables $\tau = (\tau_0, \tau_1, \cdots, \tau_k)$ are homogeneous coordinates of the complex projective space $\mathbf{CP}^k$, *i.e.*, $\mathbf{C}^{k+1} - \{0, 0, \cdots, 0\}$. The multivalued function Φ is then defined in a space

$$X = \mathbf{CP}^k - \bigcup_{j=0}^{n} \mathcal{H}_j \tag{2.11}$$

where

$$\mathcal{H}_j = \{\tau \in \mathbf{CP}^k \,;\, l_j(\tau) = 0\} \tag{2.12}$$

We now consider the meaning of the integral path Δ. Since the integrand $\Phi\omega$ is a multivalued k-form, simple choice of Δ as a k-chain on X is not enough. *Upon the choice of Δ, we need to implicitly specify branches of Φ on Δ as well, otherwise we can not properly define the integral.* In what follows we assume these implicit conditions.

Before considering further properties of Δ, we here notice that ω has an ambiguity in the evaluation of the integral (2.7). Suppose α is an arbitrary $(k-1)$-form defined in X. Then an integral over the exact k-form $d(\Phi\alpha)$ vanishes:

$$0 = \int_{\Delta} d(\Phi\alpha) = \int_{\Delta} \Phi\left(d\alpha + \frac{d\Phi}{\Phi} \wedge \alpha\right) = \int_{\Delta} \Phi \nabla\alpha \tag{2.13}$$

where ∇ can be interpreted as a covariant (exterior) derivative

$$\nabla = d + d\log\Phi \wedge = d + \sum_{j=0}^{n} \alpha_j \frac{dl_j}{l_j} \wedge \tag{2.14}$$

This means that $\omega' = \omega + \nabla\alpha$ is equivalent to ω in the definition of the integral (2.7). Namely, ω and ω' form an equivalent class, $\omega \sim \omega'$. This equivalent class is called the cohomology class.

To study this cohomology class, we consider the differential equation

$$\nabla f = df + \sum_{j=0}^{n} \alpha_j \frac{dl_j}{l_j} f = 0 \tag{2.15}$$

General solutions are locally determined by

$$f = \lambda \prod_{j=0}^{n} l_j(\tau)^{-\alpha_j} \quad (\lambda \in \mathbf{C}^\times) \tag{2.16}$$

These local solutions are thus basically given by $1/\Phi$. The idea of locality is essential since even if $1/\Phi$ is multivalued within a local patch it can be treated as a single-valued function. Analytic continuation of these solutions forms a fundamental homotopy group of a closed path

in X (or $1/X$ to be precise but it can be regarded as X by flipping the non-integer powers α_j in (2.8)). The representation of this fundamental group is called the *monodromy* representation. The monodromy representation determines the local system of the differential equation (2.15). The general solution f or $1/\Phi$ gives a rank-1 local system in this sense.[2] We denote this rank-1 local system by $\mathcal{L}$. The above cohomology class is then defined as an element of the k-th cohomology group of X over $\mathcal{L}$, *i.e.*,

$$[\omega] \in H^k(X, \mathcal{L}) \tag{2.17}$$

This cohomology group $H^k(X, \mathcal{L})$ is also called *twisted* cohomology group.

Twisted homology and twisted cycles

Having defined the cohomology group $H^k(X, \mathcal{L})$, we can now define the dual of it, *i.e.*, the k-th homology group $H_k(X, \mathcal{L}^\vee)$, known as the twisted homology group, where $\mathcal{L}^\vee$ is the rank-1 dual local system given by Φ. A differential equation corresponding to $\mathcal{L}^\vee$ can be written as

$$\nabla^\vee g = dg - \sum_{j=0}^{n} \alpha_j \frac{dl_j}{l_j} g = 0 \tag{2.18}$$

We can easily check that the general solutions are given by Φ:

$$g = \lambda \prod_{j=0}^{n} l_j(\tau)^{\alpha_j} = \lambda \Phi \quad (\lambda \in \mathbf{C}^\times) \tag{2.19}$$

As before, an element of $H_k(X, \mathcal{L}^\vee)$ gives an equivalent class called a homology class.

In the following, we show that the integral path Δ forms an equivalent class and see that it coincides with the above homology class. Applying Stokes' theorem to (2.13), we find

$$0 = \int_\Delta \Phi \nabla \alpha = \int_{\partial \Delta} \Phi \alpha \tag{2.20}$$

where α is an arbitrary $(k-1)$-form as before. The boundary operator ∂ is in principle determined from Φ (with information on branches). Denoting $C_p(X, \mathcal{L}^\vee)$ a p-dimensional chain group on X over $\mathcal{L}^\vee$, we can express the boundary operator as $\partial : C_p(X, \mathcal{L}^\vee) \longrightarrow C_{p-1}(X, \mathcal{L}^\vee)$. Since the relation (2.20) holds for an arbitrary α, we find that the k-chain Δ vanishes by the action of ∂:

$$\partial \Delta = 0 \tag{2.21}$$

The k-chain Δ satisfying above is generically called the k-cycle. In the current framework it is also called the *twisted cycle*. Since the boundary operator satisfies $\partial^2 = 0$, the k-cycle has a redundancy in it. Namely, $\Delta' = \Delta + \partial C_{(+1)}$ also becomes the k-cycle where $C_{(+1)}$ is an arbitrary $(k+1)$-chain or an element of $C_{k+1}(X, \mathcal{L}^\vee)$. Thus Δ and Δ' form an equivalent class, $\Delta \sim \Delta'$, and this is exactly the homology class defined by $H_k(X, \mathcal{L}^\vee)$, *i.e.*,

$$[\Delta] \in H_k(X, \mathcal{L}^\vee) \tag{2.22}$$

To summarize, the generalized hypergeometric function (2.7) is determined by the following bilinear form

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[2] It is 'rank-1' because each factor $l_j(\tau)$ in the local solutions (2.16) is first order in the elements of τ.

$$H_k(X, \mathcal{L}^\vee) \times H^k(X, \mathcal{L}) \longrightarrow \mathbf{C} \tag{2.23}$$

$$([\Delta], [\omega]) \longrightarrow \int_\Delta \Phi\omega \tag{2.24}$$

Differential equations of $F(Z)$

The condition $l_j(\tau) = 0$ in (2.12) defines a hyperplane in $(k+1)$-dimensional spaces. To avoid redundancy in configuration of hyperplanes, we assume the set of hyperplanes are non-degenerate, that is, we consider the hyperplanes in *general position*. This can be realized by demanding that any $(k+1)$-dimensional minor determinants of the $(k+1) \times (n+1)$ matrix Z are nonzero. We then redefine X in (2.11) as

$$X = \{Z \in Mat_{k+1,n+1}(\mathbf{C}) | \text{ any } (k+1)\text{-dim minor determinants of } Z \text{ are nonzero}\} \tag{2.25}$$

In what follows we implicitly demand this condition in Z. The configuration of $n+1$ hyperplanes in $\mathbf{CP}^k$ is determined by this matrix Z.

Apart from the concept of hyperplanes, we can also interpret that the above Z provides $n+1$ *distinct points* in $\mathbf{CP}^k$. Since a homogeneous coordinate of $\mathbf{CP}^k$ is given by $\mathbf{C}^{k+1} - \{0, 0, \cdots, 0\}$, we can consider each of the $n+1$ column vectors of Z as a point in $\mathbf{CP}^k$; the j-th column representing the j-th homogeneous coordinates of $\mathbf{CP}^k$ ($j = 0, 1, \cdots, n$).

The scale transformation, under which the $\mathbf{CP}^k$ homogeneous coordinates are invariant, is realized by an action of $H_{n+1} = \{\text{diag}(h_0, h_1, \cdots h_n) | h_j \in \mathbf{C}^\times\}$ from right on Z. The general linear transformation of the homogeneous coordinates, on the other hand, can be realized by an action of $GL(k+1, \mathbf{C})$ from left. These transformations are then given by

$$\text{Linear transformation:} \quad Z \to Z' = gZ \tag{2.26}$$

$$\text{Scale transformation:} \quad Z \to Z' = Zh \tag{2.27}$$

where $g \in GL(k+1, \mathbf{C})$ and $h \in H_{n+1}$. Under these transformations the integral $F(Z)$ in (2.7) behaves as

$$F(gZ) = (\det g)^{-1} F(Z) \tag{2.28}$$

$$F(Zh) = F(Z) \prod_{j=0}^n h_j^{\alpha_j} \tag{2.29}$$

We now briefly show that the above relations lead to the defining equations of the generalized hypergeometric functions in (2.2) and (2.3), respectively. Let $\mathbf{1}_n$ be the n-dimensional identity matrix $\mathbf{1}_n = \text{diag}(1, 1, \cdots, 1)$, and $E_{ij}^{(n)}$ be an $n \times n$ matrix in which only the (i, j)-element is 1 and the others are zero. We consider g in a particular form of

$$g = \mathbf{1}_{k+1} + \epsilon E_{pi}^{(k+1)} \tag{2.30}$$

where ϵ is a parameter. Then gZ remains the same as Z except the p-th row which is replaced by $(z_{p0} + \epsilon z_{i0}, z_{p1} + \epsilon z_{i1}, \cdots, z_{pn} + \epsilon z_{in})$. Then the derivative of $F(gZ)$ with respect to ϵ is expressed as

$$\frac{\partial}{\partial \epsilon} F(gZ) = \sum_{j=0}^n z_{ij} \frac{\partial}{\partial z_{pj}} F(gZ) \tag{2.31}$$

On the other hand, using

$$\det g = \begin{cases} 1 & (i \neq p) \\ \epsilon & (i = p) \end{cases} \tag{2.32}$$

and (2.28), we find

$$\frac{\partial}{\partial \epsilon} F(gZ) = \begin{cases} 0 & (i \neq p) \\ -\frac{1}{\epsilon^2} F(Z) & (i = p) \end{cases} \tag{2.33}$$

Evaluating the derivative at $\epsilon = 0$ and $\epsilon = 1$ for $i \neq p$ and $i = p$, respectively, we then indeed find that (2.28) leads to the differential equation (2.2).

Similarly, parametrizing h as

$$h = \mathrm{diag}(h_0, \cdots, h_{j-1}, (1+\epsilon)h_j, h_{j+1}, \cdots, h_n) \tag{2.34}$$

with $0 \leq j \leq n$, we find that Zh has only one ϵ-dependent column corresponding to the j-th column, $\big(z_{0j}(1+\epsilon)h_j, z_{1j}(1+\epsilon)h_j, \cdots, z_{kj}(1+\epsilon)h_j\big)^T$. The derivative of $F(Zh)$ with respect to ϵ is then expressed as

$$\frac{\partial}{\partial \epsilon} F(Zh) = \sum_{i=0}^{k} z_{ij} \frac{\partial}{\partial z_{ij}} F(Zh) = \sum_{i=0}^{k} z_{ij} \frac{\partial}{\partial z_{ij}} F(Z)(1+\epsilon)^{\alpha_j} \prod_{l=0}^{n} h_l^{\alpha_l} \tag{2.35}$$

where in the last step we use the relation from (2.29):

$$F(Zh) = F(Z)(1+\epsilon)^{\alpha_j} \prod_{l=0}^{n} h_l^{\alpha_l} \tag{2.36}$$

The same derivative can then be expressed as

$$\frac{\partial}{\partial \epsilon} F(Zh) = \alpha_j F(Z)(1+\epsilon)^{\alpha_j - 1} \prod_{l=0}^{k} h_l^{\alpha_l} \tag{2.37}$$

Setting $\epsilon = 0$, we can therefore derive the equation (2.3).

The other equation (2.4) for $F(Z)$ follows from the definition of Φ. From (2.8) and (2.9) we find that Φ satisfies

$$\frac{\partial \Phi}{\partial z_{ip}} = \frac{\alpha_i \tau_p}{l_i(\tau)} \Phi \tag{2.38}$$

This relation leads to

$$\frac{\partial^2 \Phi}{\partial z_{ip} \partial z_{jq}} = \frac{\alpha_i \alpha_j \tau_p \tau_q}{l_i(\tau) l_j(\tau)} \Phi = \frac{\partial^2 \Phi}{\partial z_{iq} \partial z_{jp}} \tag{2.39}$$

which automatically derives the equation (2.4).

The integral $F(Z)$ in (2.7) therefore indeed satisfies the defining equations (2.2)–(2.4) of the generalized hypergeometric functions on $Gr(k+1, n+1)$. *The Grassmannian space $Gr(k+1, n+1)$ is defined as a set of $(k+1)$-dimensional linear subspaces in $(n+1)$-dimensional complex vector space $\mathbf{C}^{n+1}$.* It is defined as

$$Gr(k+1, n+1) = \widetilde{Z}/GL(k+1, \mathbf{C}) \tag{2.40}$$

where $\widetilde{Z}$ is $(k+1) \times (n+1)$ complex matrices with $rank\widetilde{Z} = k+1$. Consider some matrix M and assume that there exists a nonzero r-dimensional minor determinant of M. Then the rank of M

is in general defined by the largest number of such r's. Thus $\widetilde{Z}$ is not exactly same as Z defined in (2.25). $\widetilde{Z}$ is more relaxed since it allows some $(k+1)$-dimensional minor determinants vanish, that is, $Z \subseteq \widetilde{Z}$. In this sense $F(Z)$ is conventionally called the generalized hypergeometric functions on $Gr(k+1, n+1)$ and we follow this convention in the present note.

Non-projected formulation

In terms of the homogeneous coordinate $\tau = (\tau_0, \tau_1, \cdots, \tau_k)$, the homogeneous coordinates on $\mathbf{CP}^k$, coordinates on $\mathbf{C}^k$ can be parametrized as

$$t_1 = \frac{\tau_1}{\tau_0} , \ t_1 = \frac{\tau_1}{\tau_0} , \ \cdots , \ t_k = \frac{\tau_k}{\tau_0} \tag{2.41}$$

For simplicity, we now fix $(z_{00}, z_{10}, \cdots, z_{n0})^T$ at $(1, 0, \cdots, 0)^T$, i.e.,

$$Z = \begin{pmatrix} 1 & z_{01} & z_{02} & \cdots & z_{0n} \\ 0 & z_{11} & z_{12} & \cdots & z_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & z_{k1} & z_{k2} & \cdots & z_{kn} \end{pmatrix} \tag{2.42}$$

Then the integrand of $F(Z)$ can be expressed as

$$\Phi\omega = \tau_0^{\alpha_0} \prod_{j=1}^{n} \left(\tau_0 z_{0j} + \tau_1 z_{1j} + \cdots + \tau_k z_{kj} \right)^{\alpha_j}$$

$$\times \sum_{i=0}^{k} (-1)^i \tau_i d\tau_0 \wedge d\tau_1 \wedge \cdots \wedge d\tau_{i-1} \wedge d\tau_{i+1} \wedge \cdots \wedge d\tau_k$$

$$= \prod_{j=1}^{n} \left(z_{0j} + \frac{\tau_1}{\tau_0} z_{1j} + \cdots + \frac{\tau_k}{\tau_0} z_{kj} \right)^{\alpha_j} d\left(\frac{\tau_1}{\tau_0}\right) \wedge d\left(\frac{\tau_2}{\tau_0}\right) \wedge \cdots \wedge d\left(\frac{\tau_k}{\tau_0}\right)$$

$$= \widetilde{\Phi}\widetilde{\omega} \tag{2.43}$$

where we use (2.6) and define $\widetilde{\Phi}, \widetilde{\omega}$ by

$$\widetilde{\Phi} = \prod_{j=1}^{n} \widetilde{l}_j(t)^{\alpha_j} \tag{2.44}$$

$$\widetilde{l}_j(t) = z_{0j} + t_1 z_{1j} + t_2 z_{2j} + \cdots + t_k z_{kj} \quad (1 \le j \le n) \tag{2.45}$$

$$\widetilde{\omega} = dt_1 \wedge dt_2 \wedge \cdots \wedge dt_k \tag{2.46}$$

The exponents α_j $(j = 1, 2, \cdots, n)$ are also imposed to the non-integer conditions $\alpha_j \notin \mathbf{Z}$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n \notin \mathbf{Z}$. The multivalued function $\widetilde{\Phi}$ is now defined in the following space

$$\widetilde{X} = \mathbf{C}^k - \bigcup_{j=1}^{n} \widetilde{\mathcal{H}}_j \tag{2.47}$$

where

$$\widetilde{\mathcal{H}}_j = \{t \in \mathbf{C}^k \, ; \, \widetilde{l}_j(t) = 0\} \tag{2.48}$$

These are non-projected versions of (2.11) and (2.12).

As before, from $\widetilde{\Phi}$ we can define rank-1 local systems $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{L}}^{\vee}$ on $\widetilde{X}$, which lead to the k-th homology and cohomology groups, $H_k(\widetilde{X}, \widetilde{\mathcal{L}}^{\vee})$ and $H^k(\widetilde{X}, \widetilde{\mathcal{L}})$. Then the integral over $\widetilde{\Phi}\widetilde{\omega}$ is defined as

$$F(Z) = \int_{\widetilde{\Delta}} \widetilde{\Phi}\widetilde{\omega} \tag{2.49}$$

where $[\widetilde{\Delta}] = H_k(\widetilde{X}, \widetilde{\mathcal{L}}^{\vee})$ and $[\widetilde{\omega}] = H^k(\widetilde{X}, \widetilde{\mathcal{L}})$.

In regard to the cohomology group $H^k(\widetilde{X}, \widetilde{\mathcal{L}})$, Aomoto shows the following theorem[3]:

1. The dimension of $H^k(\widetilde{X}, \widetilde{\mathcal{L}})$ is given by $\binom{n-1}{k}$.
2. The basis of $H^k(\widetilde{X}, \widetilde{\mathcal{L}})$ can be formed by $d \log \widetilde{l}_{j_1} \wedge d \log \widetilde{l}_{j_1} \wedge \cdots \wedge d \log \widetilde{l}_{j_k}$ where $1 \leq j_1 < j_2 < \cdots < j_k \leq n-1$.

Correspondingly, the homology group $H_k(\widetilde{X}, \widetilde{\mathcal{L}}^{\vee})$ has dimension $\binom{n-1}{k}$ and its basis can be formed finite regions bounded by $\widetilde{\mathcal{H}}_j$. In terms of $\widetilde{l}_j$'s the basis of $H^k(\widetilde{X}, \widetilde{\mathcal{L}})$ can also be chosen as [28]:

$$\varphi_{j_1 j_2 \dots j_k} = d \log \frac{\widetilde{l}_{j_1+1}}{\widetilde{l}_{j_1}} \wedge d \log \frac{\widetilde{l}_{j_2+1}}{\widetilde{l}_{j_2}} \wedge \cdots \wedge d \log \frac{\widetilde{l}_{j_k+1}}{\widetilde{l}_{j_k}} \tag{2.50}$$

where $1 \leq j_1 < j_2 < \cdots < j_k \leq n-1$.

3. Generalized hypergeometric functions on $Gr(2, n+1)$

In this section we consider a particular case of $Gr(2, n+1)$. The corresponding configuration space is simply given by $n+1$ distinct points in $\mathbf{CP}^1$. This can be represented by a $2 \times (n+1)$ matrix Z any of whose 2-dimensional minor determinants are nonzero. Allowing the freedom of coordinate transformations $GL(2, \mathbf{C})$ from the right and scale transformations $H_2 = \mathrm{diag}(h_0, h_1)$, we can uniquely parametrize Z as

$$Z = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & -1 & -z_3 & \cdots & -z_n \end{pmatrix} \tag{3.1}$$

where $z_i \neq 0, 1, z_j$ ($i \neq j$, $3 \leq i, j \leq n$). Thus we can regard Z as

$$Z \simeq \{(z_3, z_4, \cdots, z_n) \in \mathbf{C}^{n-2} \, | \, z_i \neq 0, 1, z_j \, (i \neq j)\} \tag{3.2}$$

The three other points (z_0, z_1, z_2) can be fixed at $\{0, 1, \infty\}$. This agrees with the fact that the $GL(2, \mathbf{C})$ invariance fixes three points out of the $(n+1)$ distinct points in $\mathbf{CP}^1$.

In application of the previous section, we can carry out a systematic formulation of the generalized hypergeometric functions on $Gr(2, n+1)$ as follows. We begin with a multivalued function of a form

$$\Phi = 1^{\alpha_0} \cdot t^{\alpha_1} (1-t)^{\alpha_2} (1-z_3 t)^{\alpha_3} \cdots (1-z_n t)^{\alpha_n} = \prod_{j=1}^{n} l_j(t)^{\alpha_j} \tag{3.3}$$

---

[3] Theorem 9.6.2 in [15].

where

$$l_0(t) = 1\,,\ l_1(t) = t\,,\ l_2(t) = 1 - t\,,\ l_j(t) = 1 - z_j t\ \ (3 \le j \le n) \tag{3.4}$$

As in (2.8), (2.9), the exponents obey the non-integer conditions

$$\alpha_j \notin \mathbf{Z}\ (0 \le j \le n)\,,\quad \sum_{j=0}^{n} \alpha_j = -2 \tag{3.5}$$

As considered before, the latter condition applies to the expression (2.46), that is, when $F(Z)$ is expressed as $F(Z) = \int_\Delta \Phi dt$. The defining space of Φ is given by

$$X = \mathbf{CP}^1 - \{0, 1, 1/z_3, \cdots, 1/z_n, \infty\} \tag{3.6}$$

From Φ we can determine a rank-1 local system $\mathcal{L}$ on X and its dual local system $\mathcal{L}^\vee$. Applying the result in (2.50), the basis of the cohomology group $H^1(X, \mathcal{L})$ is then given by

$$d \log \frac{l_{j+1}}{l_j} \qquad (0 \le j \le n - 1) \tag{3.7}$$

In the present case the basis of the homology group $H_1(X, \mathcal{L}^\vee)$ can be specified by a set of paths connecting the branch points. For example, we can choose these by

$$\Delta_{\infty 0}\,,\ \Delta_{01}\,,\ \Delta_{1\frac{1}{z_3}}\,,\ \Delta_{\frac{1}{z_3}\frac{1}{z_4}}\,,\ \cdots,\ \Delta_{\frac{1}{z_{n-1}}\frac{1}{z_n}} \tag{3.8}$$

where Δ_{pq} denotes a path on $\mathbf{CP}^1$ connecting branch points p and q. To summarize, for an element $\Delta \in H_1(X, \mathcal{L}^\vee)$ associated with Φ of (3.3), we can define a set of generalized hypergeometric functions on $Gr(2, n + 1)$ as

$$f_j(Z) = \int_\Delta \Phi\, d \log \frac{l_{j+1}}{l_j} \tag{3.9}$$

where $0 \le j \le n - 1$. In the next section we consider the case of $n = 3$, the simplest case where only one variable exists, which corresponds to Gauss' hypergeometric function.

4. Reduction to Gauss' hypergeometric function

Basics of Gauss' hypergeometric function

We first review the basics of Gauss' hypergeometric function. In power series, it is defined as

$$F(a, b, c; z) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} z^n \tag{4.1}$$

where $|z| < 1$, $c \notin \mathbf{Z}_{\le 0}$ and

$$(a)_n = \begin{cases} 1 & (n = 1) \\ a(a+1)(a+2)\cdots(a+n-1) & (n \ge 1) \end{cases} \tag{4.2}$$

$F(a, b, c; z)$ satisfies the hypergeometric differential equation

$$\left[\frac{d^2}{dz^2} + \left(\frac{c}{z} + \frac{a + b + 1 - c}{z - 1} \right) \frac{d}{dz} + \frac{ab}{z(z - 1)} \right] F(a, b, c; z) = 0 \tag{4.3}$$

Euler's integral formula for $F(a, b, c; z)$ is written as

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,dt \tag{4.4}$$

where $|z| < 1$ and $0 < \Re(a) < \Re(c)$.[4] $\Gamma(a)$'s denote the Gamma functions

$$\Gamma(a) = \int_0^\infty e^{-t}t^{a-1}dt \quad (\Re(a) > 0) \tag{4.5}$$

The second order differential equation (4.3) has regular singularities at $z = 0, 1, \infty$. Two independent solutions around each singular point are expressed as

$$z = 0 \;:\; \begin{cases} f_1(z) = F(a, b, c; z) \\ f_2(z) = z^{1-c}F(a-c+1, b-c+1, 2-c; z) \end{cases} \tag{4.6}$$

$$z = 1 \;:\; \begin{cases} f_3(z) = F(a, b, a+b-c+1; 1-z) \\ f_4(z) = (1-z)^{c-a-b}F(c-a, c-a, c-a-b+1; 1-z) \end{cases} \tag{4.7}$$

$$z = \infty \;:\; \begin{cases} f_5(z) = z^{-a}F(a, a-c+1, a-b+1; 1/z) \\ f_6(z) = z^{-b}F(b-c+1, b, b-a+1; 1z) \end{cases} \tag{4.8}$$

where we assume $c \notin \mathbf{Z}$, $a+b-c \notin \mathbf{Z}$ and $a-b \notin \mathbf{Z}$ at $z = 0$, $z = 1$ and $z = \infty$, respectively.

Reduction to Gauss' hypergeometric function 1: from defining equations

From (4.4) we find the relevant 2×4 matrix in a form of

$$Z = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -z \end{pmatrix} \tag{4.9}$$

The set of equations (2.2)–(2.4) then reduce to the followings:

$$(\partial_{00} + \partial_{02} + \partial_{03})F = -F \tag{4.10}$$

$$(\partial_{11} - \partial_{12} + z\partial_z)F = -F \tag{4.11}$$

$$(\partial_{10} + \partial_{12} - \partial_z)F = 0 \tag{4.12}$$

$$(\partial_{01} + \partial_{02} - \partial_{03})F = 0 \tag{4.13}$$

$$\partial_{00}F = \alpha_0 F \tag{4.14}$$

$$\partial_{11}F = \alpha_1 F \tag{4.15}$$

$$(\partial_{02} - \partial_{12})F = \alpha_2 F \tag{4.16}$$

$$(\partial_{03} + z\partial_z)F = \alpha_3 F \tag{4.17}$$

$$-\partial_z\partial_{02}F = \partial_{12}\partial_{03}F \tag{4.18}$$

where $\partial_{ij} = \frac{\partial}{\partial z_{ij}}$ and $\partial_{13} = -\frac{\partial}{\partial z} = -\partial_z$. The last relation (4.18) arises from (2.4); we here write down the one that is nontrivial and involves ∂_z. Since the sum of (4.10) and (4.11) equals to the

---

[4] This condition can be relaxed to $a \notin \mathbf{Z}$, $c - a \notin \mathbf{Z}$ by use of the well-known Pochhammer contour in the integral (4.3).

sum of (4.14)–(4.17), we can easily find $\alpha_0 + \cdots + \alpha_3 = -2$ in accord with (3.5). The second order equation (4.18) is then expressed as

$$-\partial_z(\alpha_1 + \alpha_2 + 1 + z\partial_z)F = (\alpha_1 + 1 + z\partial_z)(\alpha_3 - z\partial_z)F \tag{4.19}$$

This can also be written as

$$\left[z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab \right] F = 0 \tag{4.20}$$

where

$$a = \alpha_1 + 1$$
$$b = -\alpha_3 \tag{4.21}$$
$$c = \alpha_1 + \alpha_2 + 2$$

We can easily check that (4.20) identifies with the hypergeometric differential equation (4.3).

As seen in (3.1), there exist multiple complex variables for $n > 3$. In these cases reduction of the defining equations (2.2)–(2.4) can be carried out in principle but, unfortunately, is not as straightforward as the case of $n = 3$.

Reduction to Gauss' hypergeometric function 2: Use of twisted cohomology

The hypergeometric equation (4.3) is a second order differential equation. Setting $f_1 = F$, $f_2 = \frac{z}{b}\frac{d}{dz}F$, we can express (4.3) in a form of a first order Fuchsian differential equation [15]:

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} \right)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{4.22}$$

where

$$A_0 = \begin{pmatrix} 0 & b \\ 0 & 1-c \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -a & c-a-b-1 \end{pmatrix} \tag{4.23}$$

Using the results (3.3)–(3.9), we now obtain other first order representations of the hypergeometric function.

Let us start with a *non-projected* multivalued function

$$\Phi = t^a(1-t)^{c-a}(1-zt)^{-b} \tag{4.24}$$

where

$$a, \, c-a, \, -b \notin \mathbf{Z} \tag{4.25}$$

Φ is defined on $X = \mathbf{CP}^1 - \{0, 1, 1/z, \infty\}$. From these we can determine a rank-1 local system $\mathcal{L}$ and its dual $\mathcal{L}^\vee$ on X. Then, using (3.7), we can obtain a basis of the cohomology group $H^1(X, \mathcal{L})$ given by the following set

$$\varphi_{\infty 0} = \frac{dt}{t} \tag{4.26}$$

$$\varphi_{01} = \frac{dt}{t(1-t)} \tag{4.27}$$

$$\varphi_{1\frac{1}{z}} = \frac{(z-1)dt}{(1-t)(1-zt)} \tag{4.28}$$

Similarly, from (3.8) a basis of the homology group $H_1(X, \mathcal{L}^\vee)$ is given by

$$\{\Delta_{\infty 0}, \ \Delta_{01}, \ \Delta_{1\frac{1}{z}}\} \qquad (4.29)$$

In terms of these we can express Gauss' hypergeometric function as

$$f_{01}(Z) = \int_{\Delta_{01}} \Phi \varphi_{01} = \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} dt \qquad (4.30)$$

The derivative of $f_{01}(Z) = f_{01}(z)$ with respect to z is written as

$$\frac{d}{dz} f_{01}(z) = \frac{d}{dz} \int_{\Delta_{01}} \Phi \varphi_{01} = \int_{\Delta_{01}} \Phi \nabla_z \varphi_{01} \qquad (4.31)$$

where

$$\nabla_z = \partial_z + \partial_z \log \Phi = \partial_z + \frac{bt}{1-zt} \qquad (4.32)$$

Thus the derivative comes down to the computation of $\nabla_z \varphi_{01}$; notice that the choice of a twisted cycle Δ is irrelevant as far as the derivative itself is concerned. In order to make sense of (4.31) we should require $\nabla_z \varphi_{01} \in H^1(X, \mathcal{L})$, that is, it should be represented by a linear combinations of (4.26)–(4.28). There is a caveat here, however. We know that an element of $H^1(X, \mathcal{L})$ forms an equivalent class as discussed earlier; see (2.13) and (2.14). In the present case ($k = 1$), α in (2.13) is a 0-form or a constant. So we can demand

$$d \log \Phi = a \frac{dt}{t} - (c-a) \frac{dt}{1-t} + b \frac{z\, dt}{1-zt} \equiv 0 \qquad (4.33)$$

in the computation of $\nabla_z \varphi_{01}$. This means that the number of the base elements can be reduced from 3 to 2. Namely, any elements of $H^1(X, \mathcal{L})$ can be expressed by a combinations of an arbitrary pair in (4.26)–(4.28) under the condition (4.33). This explains the numbering discrepancies between (2.50) and (3.7) and agrees with the general result in the previous section that the dimension of the cohomology group is given by $\binom{n-1}{k} = \binom{2}{1} = 2$.

Choosing the pair of $(\varphi_{01}, \varphi_{\infty 0})$, we find

$$\nabla_z \varphi_{\infty 0} = \frac{b\, dt}{1-zt}$$
$$\equiv \frac{1}{z}\left(-a\frac{dt}{t} + (c-a)\frac{dt}{1-t}\right)$$
$$= \frac{c-a}{z}\varphi_{01} - \frac{c}{z}\varphi_{\infty 0} \qquad (4.34)$$
$$\nabla_z \varphi_{01} = \nabla_z \left(\varphi_{\infty 0} + \frac{dt}{1-t}\right)$$
$$= \nabla_z \varphi_{\infty 0} + \frac{b}{1-z}\left(\frac{dt}{1-t} - \frac{dt}{1-zt}\right)$$
$$= \frac{z}{z-1}\nabla_z \varphi_{\infty 0} - \frac{b}{z-1}(\varphi_{01} - \varphi_{\infty 0})$$
$$\equiv \frac{c-a-b}{z-1}\varphi_{01} + \frac{b-c}{z-1}\varphi_{\infty 0} \qquad (4.35)$$

where notation $\equiv$ means the use of condition (4.33). Using (4.31), we obtain a first order differential equation

$$\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{\infty 0} \end{pmatrix} = \left(\frac{A_0^{(\infty 0)}}{z} + \frac{A_1^{(\infty 0)}}{z-1} \right) \begin{pmatrix} f_{01} \\ f_{\infty 0} \end{pmatrix} \tag{4.36}$$

where

$$A_0^{(\infty 0)} = \begin{pmatrix} 0 & 0 \\ c-a & -c \end{pmatrix}, \quad A_1^{(\infty 0)} = \begin{pmatrix} c-a-b & b-c \\ 0 & 0 \end{pmatrix} \tag{4.37}$$

Solving for f_{01}, we can easily confirm that (4.36) leads to Gauss' hypergeometric differential equation (4.3).

Similarly, for the choice of $(\varphi_{01}, \varphi_{1\frac{1}{z}})$ we find

$$\nabla_z \varphi_{01} = \frac{b}{z-1} \varphi_{1\frac{1}{z}} \tag{4.38}$$

$$\nabla_z \varphi_{1\frac{1}{z}} \equiv \nabla_z \left(-\frac{a}{b} \frac{z-1}{z} \varphi_{01} + \frac{c-a}{b} \frac{z-1}{z} \frac{dt}{(1-t)^2} \right)$$

$$\equiv -\frac{a}{z} \varphi_{01} + \left(-\frac{c+1}{z} + \frac{c-a-b+1}{z-1} \right) \varphi_{1\frac{1}{z}} \tag{4.39}$$

Notice that φ_{01} and $\varphi_{1\frac{1}{z}}$ have the same factor $(1-t)^{-1}$. This factor can be absorbed in the definition of Φ in (4.24). Thus, in applying the derivative formula (4.31), we should replace c by $c-1$. This leads to another first order differential equation

$$\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{1\frac{1}{z}} \end{pmatrix} = \left(\frac{A_0^{(1\frac{1}{z})}}{z} + \frac{A_1^{(1\frac{1}{z})}}{z-1} \right) \begin{pmatrix} f_{01} \\ f_{1\frac{1}{z}} \end{pmatrix} \tag{4.40}$$

where

$$A_0^{(1\frac{1}{z})} = \begin{pmatrix} 0 & 0 \\ -a & -c \end{pmatrix}, \quad A_1^{(1\frac{1}{z})} = \begin{pmatrix} 0 & b \\ 0 & c-a-b \end{pmatrix} \tag{4.41}$$

Solving for f_{01}, we can also check that (4.40) becomes Gauss' hypergeometric differential equation (4.3).

The representations (4.23) and (4.37) are obtained by Aomoto–Kita [15] and Haraoka [28], respectively. The last one (4.41) is not known in the literature as far as the author notices. Along the lines of the above derivation, we can also obtain the Aomoto–Kita representation (4.23) as follows. We introduce a new one-form

$$\widetilde{\varphi}_{1\frac{1}{z}} = \frac{z}{z-1} \varphi_{1\frac{1}{z}} = \frac{z\,dt}{(1-t)(1-zt)} \tag{4.42}$$

The corresponding hypergeometric function is given by $\widetilde{f}_{1\frac{1}{z}} = \int_{\Delta_{01}} \Phi \widetilde{\varphi}_{1\frac{1}{z}}$. From (4.38) we can easily see $\nabla_z \varphi_{01} = \frac{b}{z} \widetilde{\varphi}_{1\frac{1}{z}}$. This is consistent with the condition $f_1 = F$, $f_2 = \frac{z}{b} \frac{d}{dz} F$ in (4.22). Since z is defined as $z \neq 0, 1$, $\frac{b}{z} \widetilde{\varphi}_{1\frac{1}{z}}$ and $\frac{b}{z-1} \varphi_{1\frac{1}{z}}$ are equally well defined one-forms. We can then choose the pair $(\varphi_{01}, \widetilde{\varphi}_{1\frac{1}{z}})$ as a possible basis of the cohomology group. The derivatives $\nabla_z \varphi_{01}, \nabla_z \widetilde{\varphi}_{1\frac{1}{z}}$ are calculated as

$$\nabla_z \varphi_{01} = \frac{b}{z} \widetilde{\varphi}_{1\frac{1}{z}} \tag{4.43}$$

$$\nabla_z \widetilde{\varphi}_{1\frac{1}{z}} \equiv \nabla_z \left(-\frac{a}{b} \varphi_{01} + \frac{c-a}{b} \frac{dt}{(1-t)^2} \right)$$

$$\equiv -\frac{a}{z-1} \varphi_{01} + \left(\frac{-c}{z} + \frac{c-a-b}{z-1} \right) \widetilde{\varphi}_{1\frac{1}{z}} \tag{4.44}$$

where we use the relations

$$\frac{t\,dt}{(1-zt)(1-t)} = \frac{1}{z-1} \left(\frac{dt}{1-zt} - \frac{dt}{1-t} \right) \tag{4.45}$$

$$\frac{dt}{(1-t)^2} \equiv \frac{1}{c-a} \left(a\varphi_{01} + b\widetilde{\varphi}_{1\frac{1}{z}} \right) \tag{4.46}$$

As before, φ_{01} and $\widetilde{\varphi}_{1\frac{1}{z}}$ have the same factor $(1-t)^{-1}$. Thus, replacing c by $c-1$, we obtain a first order differential equation

$$\frac{d}{dz} \begin{pmatrix} f_{01} \\ \widetilde{f}_{1\frac{1}{z}} \end{pmatrix} = \left(\frac{\widetilde{A}_0^{(1\frac{1}{z})}}{z} + \frac{\widetilde{A}_1^{(1\frac{1}{z})}}{z-1} \right) \begin{pmatrix} f_{01} \\ \widetilde{f}_{1\frac{1}{z}} \end{pmatrix} \tag{4.47}$$

where $\widetilde{f}_{1\frac{1}{z}} = \int_{\Delta_{01}} \Phi \widetilde{\varphi}_{1\frac{1}{z}}$ and

$$\widetilde{A}_0^{(1\frac{1}{z})} = \begin{pmatrix} 0 & b \\ 0 & 1-c \end{pmatrix}, \quad \widetilde{A}_1^{(1\frac{1}{z})} = \begin{pmatrix} 0 & 0 \\ -a & c-a-b-1 \end{pmatrix} \tag{4.48}$$

We therefore reproduce the Aomoto–Kita representation (4.22), (4.23) by a systematic construction of first order representations of the hypergeometric differential equation.

Lastly, we note that $\varphi_{\infty 0} = \frac{dt}{t}$ and $\varphi_{01} = \frac{dt}{t(1-t)}$ have the same factor t^{-1} but we can not absorb this factor into Φ. This is because we can not obtain dt as a base element of $H^1(X, \mathcal{L})$ which is generically given in a form of $d \log \frac{l_{j+1}}{l_j}$ as discussed in (3.7).

Reduction to Gauss' hypergeometric function 3: Permutation invariance

The choice of twisted cycles or Δ's is irrelevant in the above derivations of the first order Fuchsian differential equations. The hypergeometric differential equation is therefore satisfied by a more general integral form, rather than (4.30), *i.e.*,

$$f_{01}^{(\Delta_{pq})}(z) = \int_{\Delta_{pq}} \Phi \varphi_{01} = \int_p^q t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,dt \tag{4.49}$$

where (p, q) represents an arbitrary pair among the four branch points $p, q \in \{0, 1, 1/z, \infty\}$. This means that we can impose *permutation invariance* on the branch points. Δ_{pq} is then given by the following set of twisted cycles:

$$\Delta_{pq} = \{\Delta_{\infty 0}, \Delta_{01}, \Delta_{1\frac{1}{z}}, \Delta_{1\infty}, \Delta_{\frac{1}{z}\infty}, \Delta_{0\frac{1}{z}}\} \tag{4.50}$$

so that the number of elements becomes $\binom{4}{2} = 6$. Correspondingly, the base elements of the cohomology group also include

$$\varphi_{1\infty} = \frac{dt}{1-t} \tag{4.51}$$

$$\varphi_{\frac{1}{z}\infty} = \frac{z\,dt}{1-zt} \tag{4.52}$$

$$\varphi_{0\frac{1}{z}} = \frac{dt}{t(1-zt)} \tag{4.53}$$

besides (4.26)–(4.28). It is known that $f_{01}^{(\Delta_{pq})}(z)$ are related to the local solutions $f_i(z)$ ($i = 1, 2, \cdots, 6$) in (4.6)–(4.8) by

$$f_{01}^{(\Delta_{01})}(z) = B(a, c-a) f_1(z) \tag{4.54}$$

$$f_{01}^{(\Delta_{\frac{1}{z}\infty})}(z) = e^{i\pi(a+b-c+1)} B(b-c+1, 1-b) f_2(z) \tag{4.55}$$

$$f_{01}^{(\Delta_{\infty 0})}(z) = e^{i\pi(1-a)} B(a, b-c+1) f_3(z) \tag{4.56}$$

$$f_{01}^{(\Delta_{1\frac{1}{z}})}(z) = e^{i\pi(a-c+1)} B(c-a, 1-b) f_4(z) \tag{4.57}$$

$$f_{01}^{(\Delta_{1\frac{1}{z}})}(z) = B(a, 1-b) f_5(z) \tag{4.58}$$

$$f_{01}^{(\Delta_{1\infty})}(z) = e^{-i\pi(a+b-c+1)} B(b-c+1, c-a) f_6(z) \tag{4.59}$$

where $B(a, b)$ is the beta function

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}dt \quad (\Re(a) > 0, \, \Re(b) > 0) \tag{4.60}$$

(For derivations and details of these relations, see [28].)

The relevant configuration space represented by Z is given by $Gr(2, 4)/\mathcal{S}_4$ where $\mathcal{S}_4$ denotes the rank-4 symmetry group. The permutation invariance can also be confirmed by deriving another set of the first order differential equations with the choice of φ_{01} and one of (4.51)–(4.53). This is what we will present in the following.

For the choice of $(\varphi_{01}, \varphi_{1\infty})$ we find

$$\nabla_z \varphi_{1\infty} = \frac{bt}{1-zt} \frac{dt}{(1-t)}$$
$$\equiv -\frac{a}{z(z-1)} \varphi_{01} + \left(\frac{c}{z(z-1)} - \frac{b}{z-1} \right) \varphi_{1\infty} \tag{4.61}$$

$$\nabla_z \varphi_{01} = \nabla_z \frac{dt}{t} + \nabla_z \varphi_{1\infty}$$
$$\equiv -\frac{a}{z-1} \varphi_{01} + \frac{c-b}{z-1} \varphi_{1\infty} \tag{4.62}$$

The corresponding differential equation is then expressed as

$$\frac{d}{dz} \begin{pmatrix} f_{01} \\ f_{1\infty} \end{pmatrix} = \left(\frac{A_0^{(1\infty)}}{z} + \frac{A_1^{(1\infty)}}{z-1} \right) \begin{pmatrix} f_{01} \\ f_{1\infty} \end{pmatrix} \tag{4.63}$$

where

$$A_0^{(1\infty)} = \begin{pmatrix} 0 & 0 \\ a & -c \end{pmatrix}, \quad A_1^{(1\infty)} = \begin{pmatrix} -a & c-b \\ -a & c-b \end{pmatrix} \tag{4.64}$$

Solving for f_{01}, we can check that (4.63) indeed becomes Gauss' hypergeometric differential equation (4.3).

Similarly, for $(\varphi_{01}, \varphi_{\frac{1}{z}\infty})$ we find

$$
\begin{aligned}
\nabla_z \varphi_{01} &= \frac{b\,dt}{(1-zt)(1-t)} \\
&\equiv -\frac{1}{z-1}\frac{ab}{c}\varphi_{01} - \frac{1}{z-1}\frac{b}{c}(b-c)\varphi_{\frac{1}{z}\infty}
\end{aligned} \tag{4.65}
$$

$$
\begin{aligned}
\nabla_z \varphi_{\frac{1}{z}\infty} &\equiv \nabla_z\left(-\frac{a}{b}\varphi_{01} + \frac{c}{b}\frac{dt}{1-t}\right) \\
&\equiv \frac{1}{z-1}\frac{a}{c}(a-c)\varphi_{01} + \left(\frac{1}{z-1}\frac{1}{c}(b-c)(a-c) - \frac{c}{z}\right)\varphi_{\frac{1}{z}\infty}
\end{aligned} \tag{4.66}
$$

The first order differential equation is then expressed as

$$
\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{\frac{1}{z}\infty} \end{pmatrix} = \left(\frac{A_0^{(\frac{1}{z}\infty)}}{z} + \frac{A_1^{(\frac{1}{z}\infty)}}{z-1}\right)\begin{pmatrix} f_{01} \\ f_{\frac{1}{z}\infty} \end{pmatrix} \tag{4.67}
$$

where

$$
A_0^{(\frac{1}{z}\infty)} = \begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix}, \quad A_1^{(\frac{1}{z}\infty)} = \begin{pmatrix} -\frac{ab}{c} & -\frac{b}{c}(b-c) \\ \frac{a}{c}(a-c) & \frac{1}{c}(b-c)(a-c) \end{pmatrix} \tag{4.68}
$$

We can check that (4.67) reduces to the hypergeometric differential equation for f_{01}.

Lastly, for $(\varphi_{01}, \varphi_{0\frac{1}{z}})$ we find

$$
\begin{aligned}
\nabla_z \varphi_{01} &= \frac{b\,dt}{(1-zt)(1-t)} \\
&= -\frac{b}{z-1}\left(\varphi_{01} - \varphi_{0\frac{1}{z}}\right)
\end{aligned} \tag{4.69}
$$

$$
\begin{aligned}
\nabla_z \varphi_{0\frac{1}{z}} &\equiv \frac{b\,dt}{(1-zt)^2} \\
&\equiv -\frac{c-a}{z(z-1)}\varphi_{01} + \frac{c-az}{z(z-1)}\varphi_{0\frac{1}{z}}
\end{aligned} \tag{4.70}
$$

The corresponding differential equation becomes

$$
\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{0\frac{1}{z}} \end{pmatrix} = \left(\frac{A_0^{(0\frac{1}{z})}}{z} + \frac{A_1^{(0\frac{1}{z})}}{z-1}\right)\begin{pmatrix} f_{01} \\ f_{0\frac{1}{z}} \end{pmatrix} \tag{4.71}
$$

where

$$
A_0^{(0\frac{1}{z})} = \begin{pmatrix} 0 & 0 \\ c-a & -c \end{pmatrix}, \quad A_1^{(0\frac{1}{z})} = \begin{pmatrix} -b & b \\ -c+a & c-a \end{pmatrix} \tag{4.72}
$$

We can check that (4.71) reduces to the hypergeometric differential equation for f_{01} as well.

As in the case of (4.42), it is tempting to think of $\widetilde{\varphi}_{\frac{1}{z}\infty} = \frac{z-1}{z}\varphi_{\frac{1}{z}\infty} = \frac{(z-1)\,dt}{1-zt}$. But, with φ_{01} and $\widetilde{\varphi}_{\frac{1}{z}\infty}$, it is not feasible to obtain a first order differential equation in the form of (4.67) which leads to the hypergeometric differential equation. This is because, if expanded in φ_{01} and $\widetilde{\varphi}_{\frac{1}{z}\infty}$, the z-dependence of the derivatives $\nabla_z\varphi_{01}$ and $\nabla_z\widetilde{\varphi}_{\frac{1}{z}\infty}$, can not be written in terms of $\frac{1}{z}$ or $\frac{1}{z-1}$.

Summary

In this section we carry out a systematic derivation of first order representations of the hypergeometric differential equation by use of twisted cohomology as the simplest reduction of Aomoto's generalized hypergeometric function. The first order equations are generically expressed as

$$\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{pq} \end{pmatrix} = \left(\frac{A_0^{(pq)}}{z} + \frac{A_1^{(pq)}}{z-1} \right)\begin{pmatrix} f_{01} \\ f_{pq} \end{pmatrix} = A_{01}^{(pq)}\begin{pmatrix} f_{01} \\ f_{pq} \end{pmatrix} \tag{4.73}$$

where (pq) denotes a pair of four branch points $\{0, 1, 1/z, \infty\}$ in $\Phi = t^a(1-t)^{c-a}(1-zt)^{-b}$. A list of the (2×2) matrices $A_{01}^{(pq)}$ obtained in this section is given by the following:

$$A_{01}^{(\infty 0)} = \begin{pmatrix} \frac{c-a-b}{z-1} & \frac{b-c}{z-1} \\ \frac{c-a}{z} & -\frac{c}{z} \end{pmatrix} \tag{4.74}$$

$$A_{01}^{(1\frac{1}{z})} = \begin{pmatrix} 0 & \frac{b}{z-1} \\ -\frac{a}{z} & -\frac{c}{z} + \frac{c-a-b}{z-1} \end{pmatrix} \tag{4.75}$$

$$\widetilde{A}_{01}^{(1\frac{1}{z})} = \begin{pmatrix} 0 & \frac{b}{z} \\ -\frac{a}{z-1} & -\frac{c-1}{z} + \frac{c-a-b-1}{z-1} \end{pmatrix} \tag{4.76}$$

$$A_{01}^{(1\infty)} = \begin{pmatrix} -\frac{a}{z-1} & \frac{c-b}{z-1} \\ \frac{a}{z} - \frac{a}{z-1} & -\frac{c}{z} + \frac{c-b}{z-1} \end{pmatrix} \tag{4.77}$$

$$A_{01}^{(\frac{1}{z}\infty)} = \begin{pmatrix} -\frac{1}{z-1}\frac{ab}{c} & -\frac{1}{z-1}\frac{b}{c}(b-c) \\ \frac{1}{z-1}\frac{a}{c}(a-c) & -\frac{c}{z} + \frac{1}{z-1}\frac{1}{c}(b-c)(a-c) \end{pmatrix} \tag{4.78}$$

$$A_{01}^{(0\frac{1}{z})} = \begin{pmatrix} -\frac{b}{z-1} & \frac{b}{z-1} \\ \frac{c-a}{z} - \frac{c-a}{z-1} & -\frac{c-1}{z} + \frac{c-a}{z-1} \end{pmatrix} \tag{4.79}$$

where we include the Aomoto–Kita representation $\widetilde{A}_{01}^{(1\frac{1}{z})}$. As far as the author notices, these expressions except (4.74), (4.76) are new for the description of the hypergeometric differential equation. A common feature among these matrices is that the determinant is identical:

$$\det A_{01}^{(pq)} = \frac{ab}{z(z-1)} \tag{4.80}$$

In terms of the first order differential equation (4.73), this means that the action of the derivative on the basis $\begin{pmatrix} f_{01} \\ f_{pq} \end{pmatrix}$ of the cohomology group $H^1(X, \mathcal{L})$ can be represented by a generator of the $SL(2, \mathbf{C})$ algebra. In other words, a change of the bases is governed by the $SL(2, \mathbf{C})$ symmetry. The $SL(2, \mathbf{C})$ invariance corresponds to the global conformal symmetry for holomorphic functions on $\mathbf{CP}^1$. In the present case we start from the holomorphic multivalued function Φ in

(4.24) which is defined on $X = \mathbf{CP}^1 - \{0, 1, 1/z, \infty\}$. The result (4.80) is thus natural in concept but nontrivial in practice because the equivalence condition $d \log \Phi \equiv 0$ in (4.33) is implicitly embedded into the expressions (4.74)–(4.79).

5. Integral representations of the KZ solutions

Having investigated thoroughly reduction of Aomoto's generalized hypergeometric function to Gauss' original hypergeometric function, we now consider a physical problem in relation to the above results. As mentioned in the introduction, solutions of the Knizhnik–Zamolodchikov (KZ) equation are known to be related to the generalized hypergeometric functions. In this section we shed light on this relation along the lines of arguments on cohomology and homology.

The KZ equation

We first review basic properties of the Knizhnik–Zamolodchikov (KZ) equation, following the description in [21]. The KZ equation is defined by

$$\frac{\partial \Psi}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i}^{n} \frac{\Omega_{ij} \Psi}{z_i - z_j} \quad (1 \le i, j \le n) \tag{5.1}$$

where $\Psi = \Psi(z_1, z_2, \cdots, z_n)$ is a function of n complex variables, z_i $(i = 1, 2, \cdots, n)$, corresponding to a correlation function in the Wess–Zumino–Witten (WZW) model. κ is parametrized as

$$\kappa = l + h^\vee \tag{5.2}$$

where l is the level number and $h^\vee$ is the dual Coexter number of the Lie algebra $\mathfrak{g}$ associated to the WZW model. The function Ψ is defined on a space

$$X_n = \mathbf{C}^n - \bigcup_{i < j} \mathcal{H}_{ij} \tag{5.3}$$

where $\mathcal{H}_{ij}$ is denotes a hyperplane in $\mathbf{C}^n$ defined by $z_i - z_j = 0$:

$$\mathcal{H}_{ij} = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \, ; \ z_i - z_j = 0 \ (i \neq j)\} \tag{5.4}$$

Quantum theoretically, the function Ψ should be evaluated as a vacuum expectation value of operators acting on the Hilbert space

$$V^{\otimes n} = V_1 \otimes V_2 \otimes \cdots \otimes V_n \tag{5.5}$$

where V_i $(i = 1, 2, \cdots, n)$ denotes a Fock space for a particle labeled by i. Ω_{ij}'s in the KZ equation are bialgebraic operators acting on the (i, j) entries of the Hilbert space $V^{\otimes n}$. These operators satisfy *the infinitesimal braid relations*:

$$\left[\Omega_{ij}, \Omega_{kl} \right] = 0 \quad (i, j, k, l \text{ are distinct}) \tag{5.6}$$

$$\left[\Omega_{ij} + \Omega_{jk}, \Omega_{ik} \right] = 0 \quad (i, j, k \text{ are distinct}) \tag{5.7}$$

It is well-known that the KZ equation is invariant under the global $SL(2, \mathbf{C})$ symmetry or the conformal transformations. It is then natural to consider each variable z_i on $\mathbf{CP}^1$ rather than on $\mathbf{C}$. In practice, this means that we can add an extra variable $z_0 = \infty$ in the definitions of Ψ, that is, $\Psi(z_1, z_2, \cdots, z_n) \longrightarrow \Psi(z_0, z_1, z_2, \cdots, z_n)$. In general, the solutions of the KZ equation (5.1)

give $(n + 1)$-point correlation functions on $\mathbf{CP}^1$ in the WZW model, which we call $(n + 1)$-point KZ solutions. Fixing $(z_1, z_2) = (0, 1)$, we can then identify the configuration space (5.3) as the $2 \times (n + 1)$ matrix Z in (3.1) defined for the generalized hypergeometric function on $Gr(2, n+1)$.

In order to relate Ψ to the generalized hypergeometric functions, we need to determine a multivalued function analogous to Φ in (2.8). From the defining space (5.3) we find that the relevant multivalued function is given by

$$\Phi_0 = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{1}{\kappa} \Omega_{ij}} \tag{5.8}$$

In order to interpret Φ_0 as a function, we need to specify the Lie algebra $\mathfrak{g}$ for Ω_{ij} and relevant actions to the vacuum state. *We shall not specify these algebraic properties here and interpret Ω_{ij} as $\langle \Omega_{ij} \rangle$, the vacuum expectation value of Ω_{ij}, for the moment.* We will specify the algebraic structure to $SL(2, \mathbf{C})$ in the next section. In analogy to (2.15) or (2.18), we now consider the following covariant derivatives

$$D = d - d \log \Phi_0 = d - \frac{1}{\kappa} \sum_{1 \le i < j \le n} \Omega_{ij} \frac{d(z_i - z_j)}{z_i - z_j} \tag{5.9}$$

General solutions of $Df = 0$ are given by

$$f = \lambda \Phi_0 = \lambda \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{1}{\kappa} \Omega_{ij}} \qquad (\lambda \in \mathbf{C}^\times) \tag{5.10}$$

As before, analytic continuation of these solutions forms a fundamental homotopy group of a closed path in X_n. The representation of $\Pi_1(X_n)$ gives the monodromy representation of the differential equation $Df = 0$.

We now notice that in a differential form the KZ equation (5.1) can be expressed as

$$D\Psi = (d - \Omega)\Psi = 0 \tag{5.11}$$

where

$$\Omega = \frac{1}{\kappa} \sum_{1 \le i < j \le n} \Omega_{ij} \, \omega_{ij} \tag{5.12}$$

$$\omega_{ij} = d \log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j} \tag{5.13}$$

Notice that from the result in section 2 we find that ω_{ij}'s form a basis of the cohomology group of X_n. These satisfy the identity

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0 \qquad (i < j < k) \tag{5.14}$$

From (5.6), (5.7) and (5.14) we can show the flatness of Ω, *i.e.*,

$$d\Omega - \Omega \wedge \Omega = 0 \tag{5.15}$$

Ω is called the KZ connection one-form.

Imposing permutation invariance on X_n, we can in fact express the defining space as

$$\mathcal{C} = \frac{X_n}{\mathcal{S}_n} \tag{5.16}$$

where $\mathcal{S}_n$ is the rank-n symmetric group. The fundamental homotopy group of $\mathcal{C}$ is given by the braid group

$$\Pi_1(\mathcal{C}) = \mathcal{B}_n \tag{5.17}$$

The braid group $\mathcal{B}_n$ has generators, $b_1, b_2, \cdots, b_{n-1}$, and they satisfy the following relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \quad (|i-j|=1) \tag{5.18}$$

$$b_i b_j = b_j b_i \quad\quad (|i-j|>1) \tag{5.19}$$

where we identify b_n with b_1. One of the main results in the KZ equation is that the monodromy representation of the KZ equation can be given by the braid group.

Relation to 4-point KZ solutions

For $n = 3$ we can check that the KZ equation (5.1) has a solution of the form [21]:

$$\Psi(z_1, z_2, z_3) = (z_3 - z_1)^{\frac{1}{\kappa}(\Omega_{12} + \Omega_{13} + \Omega_{23})} G\left(\frac{z_2 - z_1}{z_3 - z_1}\right) \tag{5.20}$$

where $G(z)$ satisfies the differential equation

$$\frac{d}{dz} G(z) = \frac{1}{\kappa}\left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1}\right) G(z) \tag{5.21}$$

This equation is equivalent in structure to the first order differential equation (4.73). The solution $G(z)$ thus corresponds to Gauss' hypergeometric function if Ω_{12} and Ω_{23} are represented by 2×2 matrices. In what follows we shall clarify this statement by constructing hypergeometric-type integral representations of the 4-point KZ solutions.

As before, $G(z)$ has singular points at $0, 1, \infty$. Thus, evaluated on the Riemann sphere $\mathbf{CP}^1$, the solution (5.20) can be interpreted as a 4-point solution. Thanks to the $SL(2, \mathbf{C})$ invariance, without losing generality, we can fix the three points $(0, 1, \infty)$. For example, choosing $(z_0, z_1, z_2, z_3) \to (\infty, 0, z, 1)$, we find

$$\Psi(z_0, z_1, z_2, z_3) \to \Psi(\infty, 0, z, 1) = G(z) \tag{5.22}$$

where we omit the z-independent factor $(z_3 - z_1)^{\frac{1}{\kappa}(\Omega_{12}+\Omega_{13}+\Omega_{23})} \to e^{i2\pi m \frac{1}{\kappa}(\Omega_{12}+\Omega_{13}+\Omega_{23})}$ with $m \in \mathbf{Z}$. When evaluated as a vacuum expectation value (vev), this factor enters nontrivially into the solution Ψ. If the exponent is evaluated as an integer we can not properly define Φ or Φ_0 since these function can be multiplied by $1 = e^{i2\pi m}$ for any times. Thus we can naturally demand non-integer conditions for the vev of these exponents:

$$\left\langle \frac{1}{\kappa}\Omega_{12}\right\rangle, \left\langle \frac{1}{\kappa}\Omega_{13}\right\rangle, \left\langle \frac{1}{\kappa}\Omega_{23}\right\rangle, \left\langle \frac{1}{\kappa}(\Omega_{12}+\Omega_{13}+\Omega_{23})\right\rangle \notin \mathbf{Z} \tag{5.23}$$

These conditions are analogous to the non-integer conditions we have considered for the exponents α_j, say, in (2.44). As mentioned earlier, we will omit the brackets in the following expressions. In terms of the covariant derivative (5.9), we find

$$D_{z_2} = \partial_{z_2} - \partial_{z_2} \log \Phi_0 \rightarrow D_z = \partial_z - \frac{1}{\kappa}\left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1}\right) \tag{5.24}$$

In this case the KZ equation $D_z \Psi = 0$ directly reduces to the differential equation (5.21). In general, the reduction is not so simple but, as expected, we may relate the KZ equation $D_z \Phi = 0$

to the equation (5.21). For example, take the previous choice $(z_0, z_1, z_2, z_3) \to (\infty, 0, 1, 1/z)$, which leads to the following parametrization:

$$\Psi(z_0, z_1, z_2, z_3) \to \Psi(\infty, 0, 1, 1/z) = z^{-\frac{1}{\kappa}(\Omega_{12} + \Omega_{13} + \Omega_{23})} G(z) \tag{5.25}$$

$$\Phi_0 = (-1)^{\frac{1}{\kappa}\Omega_{12}} (1 - 1/z)^{\frac{1}{\kappa}\Omega_{23}} (-1/z)^{\frac{1}{\kappa}\Omega_{13}} \tag{5.26}$$

$$D_{1/z_3} = \partial_{1/z_3} - \partial_{1/z_3} \log \Phi_0 \to D_z = \partial_z - \frac{1}{\kappa} \left(-\frac{\Omega_{23} + \Omega_{13}}{z} + \frac{\Omega_{23}}{z - 1} \right) \tag{5.27}$$

$$D_z \Psi = z^{-\frac{1}{\kappa}(\Omega_{12} + \Omega_{13} + \Omega_{23})} \left[D_z - \frac{1}{\kappa} \left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z - 1} \right) \right] G(z) \tag{5.28}$$

$$d \log \Phi_0 = \left(-\frac{\Omega_{23} + \Omega_{13}}{z} + \frac{\Omega_{23}}{z - 1} \right) dz \equiv 0 \tag{5.29}$$

The last relation arises from the equivalent relations that are associated to the cohomology group $H^1(X_3, \mathcal{L}_0)$ where $\mathcal{L}_0$ is a rank-1 local system determined by Φ_0. Considering in one-form, we can express $D_z \Psi dz$ as

$$D_z \Psi dz \equiv z^{-\frac{1}{\kappa}(\Omega_{12} + \Omega_{13} + \Omega_{23})} \left[\partial_z - \frac{1}{\kappa} \left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z - 1} \right) \right] G(z) dz \tag{5.30}$$

The KZ equation $D_z \Psi dz = 0$ then reduces to the differential equation for $G(z)$ in (5.21). In the above analysis the concept of cohomology plays an essential role to replace the covariant derivative D_z by the ordinary derivative ∂_z in (5.28) and (5.30). In the language of gauge theory the replacement can be implemented by taking a pure gauge of the KZ connection one-form.

Essence of the integral representation lies in the choice of multivalued function. The choice of Φ_0 in (5.26) is, however, not appropriate to derive Gauss' hypergeometric function as Φ_0 does not contain the integral parameter t. To obtain Gauss' hypergeometric function, we need to incorporate the following multivalued function

$$\Phi_t = t^{\alpha_1} (1 - t)^{\alpha_2} (1 - zt)^{\alpha_3} \tag{5.31}$$

which is defined on $X = \mathbf{CP}^1 - \{\infty, 0, 1, 1/z\}$. Note that the choice of X_3 automatically determines X. As discussed earlier, X_n is represented by distinct $(n + 1)$ points in $\mathbf{CP}^1$. The physical configuration space $\mathcal{C} = X_n/\mathcal{S}_n$ imposes permutation (or bosonic) invariance on these so that it gives rise to distinct *ordered* $(n + 1)$ points in $\mathbf{CP}^1$.

Another issue on Φ_0 is that it is an algebraic function because of the exponent $\frac{1}{\kappa}\Omega_{ij}$. A rank-1 local system determined by Φ_0 is therefore an algebraic one. To incorporate Φ_t into Φ_0 we then need to demand that the exponents α_i of Φ_t be algebraic. To be concrete, we set

$$\alpha_i = -\frac{1}{\kappa}\Omega_i \qquad (i = 1, 2, 3) \tag{5.32}$$

where Ω_i acts on the Fock space V_i of $V^{\otimes n}$ in (5.5). We assume non-integer conditions on these, $\langle \alpha_i \rangle \notin \mathbf{Z}$, as well. The multivalued function of interest becomes

$$\begin{aligned} \Phi &= \Phi_0 \Phi_t \\ &= (-1)^{\frac{1}{\kappa}\Omega_{12}} (1 - 1/z)^{\frac{1}{\kappa}\Omega_{23}} (-1/z)^{\frac{1}{\kappa}\Omega_{13}} t^{-\frac{1}{\kappa}\Omega_1} (1 - t)^{-\frac{1}{\kappa}\Omega_2} (1 - zt)^{-\frac{1}{\kappa}\Omega_3} \end{aligned} \tag{5.33}$$

As usually Φ determines a rank-1 local system which we denote $\mathcal{L}_\mathfrak{g}$ to indicate the algebraic nature of Φ. The covariant derivative associated with Φ is expressed as

$$\nabla_z = \partial_z - \partial_z \log \Phi_0 - \partial_z \log \Phi_t$$
$$= D_z + \frac{\Omega_3}{\kappa} \frac{t}{1 - zt} \tag{5.34}$$

where D_z is the same as (5.27):

$$D_z = \partial_z - \frac{1}{\kappa} \left(-\frac{\Omega_{23} + \Omega_{13}}{z} + \frac{\Omega_{13}}{z - 1} \right) \tag{5.35}$$

The covariant derivative (5.34) is essentially the same as the one in (4.32) except that the derivative ∂_z is now covariantized as D_z. We should emphasize that, owing to the equivalent relation (5.29), the action of D_z on a one-form $F(z)dz$ reduces to $D_z F(z)dz \equiv \partial_z F(z)dz$ where $F(z)$ is an arbitrary function (or 0-form) of z. The negative sign in the exponents (5.32) is chosen such that (5.34) and (4.32) are compatible.

A basis of the cohomology group $H^1(X, \mathcal{L}_{\mathfrak{g}})$ can be given by a pair $(\varphi_{01}, \varphi_{pq})$ where $\varphi_{pq} = \{\varphi_{\infty 0}, \varphi_{1\frac{1}{z}}, \varphi_{1\infty}, \varphi_{\frac{1}{z}\infty}, \varphi_{0\frac{1}{z}}\}$; concrete expressions of these elements are shown in (4.26)–(4.28) and (4.51)–(4.53). Similarly, elements of the homology group $H_1(X, \mathcal{L}_{\mathfrak{g}}^\vee)$ can also be obtained from (4.50).

For simplicity, we now choose the pair $(\varphi_{01}, \varphi_{\infty 0})$. The hypergeometric-type integrals of interest are expressed as

$$F_{01}(z) = \int_\Delta \Phi_0 \Phi_t \varphi_{01} \tag{5.36}$$

$$F_{\infty 0}(z) = \int_\Delta \Phi_0 \Phi_t \varphi_{\infty 0} \tag{5.37}$$

where $\varphi_{01} = \frac{dt}{t(1-t)}$, $\varphi_{\infty 0} = \frac{dt}{t}$ and $\Delta \in H_1(X, \mathcal{L}_{\mathfrak{g}}^\vee)$. The covariant derivative D_z of F_{01} with respect to z is calculated as

$$D_z F_{01}(z) = \int_\Delta \Phi_0 \Phi_t \nabla_z \varphi_{01} \tag{5.38}$$

where $\nabla_z = D_z + \frac{\Omega_3}{\kappa} \frac{t}{1-zt}$. The calculation of $\nabla_z \varphi_{01}$ should be executed under the equivalent condition

$$d \log \Phi_t = -\frac{1}{\kappa} \left(\Omega_1 \frac{dt}{t} - \Omega_2 \frac{dt}{1-t} - \Omega_3 \frac{zdt}{1-zt} \right) \equiv 0 \tag{5.39}$$

The calculations of $\nabla_z \varphi_{01}$ and $\nabla_z \varphi_{\infty 0}$ are therefore exactly the same as those of (4.34) and (4.35), respectively, except that the exponents are now replaced by $(a, c - a, -b) \to (\frac{1}{\kappa}\Omega_1, \frac{1}{\kappa}\Omega_2, \frac{1}{\kappa}\Omega_3)$. To be concrete, we obtain the following result:

$$D_z \begin{bmatrix} F_{01}(z) \\ F_{\infty 0}(z) \end{bmatrix} = \int_\Delta \Phi_0 \Phi_t \nabla_z \begin{bmatrix} \varphi_{01} \\ \varphi_{\infty 0} \end{bmatrix} = \frac{1}{\kappa} \left(\frac{B_0^{(\infty 0)}}{z} + \frac{B_1^{(\infty 0)}}{z - 1} \right) \begin{bmatrix} F_{01}(z) \\ F_{\infty 0}(z) \end{bmatrix} \tag{5.40}$$

where

$$B_0^{(\infty 0)} = \begin{pmatrix} 0 & 0 \\ \Omega_2 & -(\Omega_1 + \Omega_2) \end{pmatrix}, \quad B_1^{(\infty 0)} = \begin{pmatrix} \Omega_2 + \Omega_3 & -(\Omega_1 + \Omega_2 + \Omega_3) \\ 0 & 0 \end{pmatrix} \tag{5.41}$$

Note that we have used notation Ω_i for the vacuum expectation value $\langle \Omega_i \rangle$ in the above expressions. The differential equation (5.40) is first order in the covariant derivative D_z. Using the results in the previous section, we then find that $F_{01}(z)$ satisfies the following *covariantized* hypergeometric differential equation

$$\left[D_z^2 + \left(\frac{c}{z} + \frac{a+b+1-c}{z-1} \right) D_z + \frac{ab}{z(z-1)} \right] F_{01}(z) = 0 \tag{5.42}$$

where

$$a = \frac{1}{\kappa}\Omega_1, \quad b = -\frac{1}{\kappa}\Omega_3, \quad c = \frac{1}{\kappa}(\Omega_1 + \Omega_2) \tag{5.43}$$

Notice that, due to the equivalent relation $D_z F_{01}(z)dz \equiv \partial_z F_{01}(z)dz$, the covariantized equation (5.42) becomes the ordinary hypergeometric differential equation (4.3) when evaluated in the integrand over the physical configuration space $\mathcal{C}$ for $n=3$.

An integral representation of the 4-point KZ solution can be constructed as

$$\Psi(\infty, 0, 1, 1/z) = z^{-\frac{1}{\kappa}(\Omega_{12}+\Omega_{13}+\Omega_{23})} \int_\Delta \Phi_0 \Phi_t \begin{bmatrix} \varphi_{01} \\ \varphi_{\infty 0} \end{bmatrix}$$

$$= z^{-\frac{1}{\kappa}(\Omega_{12}+\Omega_{13}+\Omega_{23})} \begin{bmatrix} F_{01}(z) \\ F_{\infty 0}(z) \end{bmatrix} \tag{5.44}$$

From this representation we can easily compute $D_z \Psi$ as

$$D_z \Psi = z^{-\frac{1}{\kappa}(\Omega_{12}+\Omega_{13}+\Omega_{23})} \left[D_z - \frac{1}{\kappa} \left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1} \right) \right] \begin{bmatrix} F_{01}(z) \\ F_{\infty 0}(z) \end{bmatrix}$$

$$= z^{-\frac{1}{\kappa}(\Omega_{12}+\Omega_{13}+\Omega_{23})} \frac{1}{\kappa} \left[\left(\frac{B_0^{(\infty 0)}}{z} + \frac{B_1^{(\infty 0)}}{z-1} \right) - \left(\frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z-1} \right) \right] \begin{bmatrix} F_{01}(z) \\ F_{\infty 0}(z) \end{bmatrix} \tag{5.45}$$

Thus we confirm that Ψ in (5.44) satisfies the KZ equation $D_z \Psi = 0$ when Ω_{12} and Ω_{23} are represented by the 2×2 matrices $B_0^{(\infty 0)}$ and $B_1^{(\infty 0)}$, respectively. As mentioned earlier, the basis of the cohomology group $H^1(X, \mathcal{L}_{\mathfrak{g}})$ can be given by a pair $(\varphi_{01}, \varphi_{pq})$ where $\varphi_{pq} = \{ \varphi_{\infty 0}, \varphi_{1\frac{1}{z}}, \varphi_{1\infty}, \varphi_{\frac{1}{z}\infty}, \varphi_{0\frac{1}{z}} \}$. Accordingly, we can construct $B_0^{(pq)}$, $B_1^{(pq)}$ from $A_0^{(pq)}$, $A_1^{(pq)}$ in the previous section, with the replacements in (5.43). In general, Ω_{12} and Ω_{23} should be related to these $B_0^{(pq)}$ and $B_1^{(pq)}$, respectively.

So far, we argue the 4-point KZ solutions in terms of the parametrization $(z_0, z_1, z_2, z_3) = (\infty, 0, 1, 1/z)$ so that we can utilize the results in the previous section. Now that the integral representation of the solutions become clear, we consider the simplest parametrization $(z_0, z_1, z_2, z_3) = (\infty, 0, z, 1)$ again. As discussed in (5.10), general solutions of the KZ equation is expressed as $\lambda \Phi_0$ ($\lambda \in \mathbf{C}^\times$) for any n. Thus the essential part of solving the KZ equation is to find λ, in the present case, in a form of integrals. From the above analyses we find that for $n=3$ we have two independent solutions and these can be expressed in terms of the hypergeometric-type integrals. The key ingredient of the integral is the elements of cohomology and homology groups $H^1(X, \mathcal{L}_{\mathfrak{g}})$, $H_1(X, \mathcal{L}_{\mathfrak{g}}^\vee)$ where $X = \mathbf{CP}^1 - \{\infty, 0, z, 1\}$ and the rank-1 local system $\mathcal{L}_{\mathfrak{g}}$ and its dual $\mathcal{L}_{\mathfrak{g}}^\vee$ are determined by Φ_t. We now *rewrite* Φ_t as

$$\Phi_t = t^{-\frac{1}{\kappa}\Omega_1} (t-z)^{-\frac{1}{\kappa}\Omega_2} (t-1)^{-\frac{1}{\kappa}\Omega_3} \tag{5.46}$$

The element of the homology group $\Delta = H_1(X, \mathcal{L}_{\mathfrak{g}}^{\vee})$ or the twisted cycle defines the integral path over X. The possible twisted cycles are given by Δ_{pq} where (p, q) denotes distinct pairs of $\{\infty, 0, z, 1\}$. The basis of the cohomology group, on the other hand, defines a one-form to be integrated apart from the factor of the multivalued function $\Phi = \Phi_0 \Phi_t$. Owing to the equivalent relation $d \log \Phi_t \equiv 0$, the basis consists of two elements of $H_1(X, \mathcal{L}_{\mathfrak{g}}^{\vee})$. As considered earlier, there are six different choices for the bases. From (5.46) we can easily find the one of these can be given by $\{\frac{dt}{t}, \frac{dt}{t-z}\}$. The two independent 4-point KZ solutions can then be expressed as

$$\Psi(\infty, 0, z, 1) = \int_{\Delta} \Phi_0 \Phi_t \begin{pmatrix} \frac{dt}{t} \\ \frac{dt}{t-z} \end{pmatrix} \tag{5.47}$$

where $\Phi_0 = (-z)^{\frac{1}{\kappa}\Omega_{12}}(-1)^{\frac{1}{\kappa}\Omega_{13}}(z-1)^{\frac{1}{\kappa}\Omega_{23}}$ and Φ_t is given by (5.46). As discussed below (5.22), the solution allows a phase factor arising from $(z_3 - z_1)^{\frac{1}{\kappa}(\Omega_{12}+\Omega_{13}+\Omega_{23})}$.

To summarize, the 4-point KZ solutions can be given by the following generalized hypergeometric functions on $Gr(4, 2)$:

$$F_j(z_0, z_1, z_2, z_3) = \int_{\Delta} \Phi_0 \Phi_t \, \varphi_j \tag{5.48}$$

$$\Phi_0 = \prod_{1 \le i < j \le 3} (z_i - z_j)^{\frac{1}{\kappa}\Omega_{ij}} \tag{5.49}$$

$$\Phi_t = \prod_{i=1}^{3} (t - z_i)^{-\frac{1}{\kappa}\Omega_i} \tag{5.50}$$

where $\Delta \in H_1(X, \mathcal{L}_{\mathfrak{g}}^{\vee})$ and $\varphi_j \in H^1(X, \mathcal{L}_{\mathfrak{g}})$. From our study on the generalized hypergeometric functions on $Gr(4, 2)$, we find the elements φ_j are given by the following set

$$\varphi_j = \left\{ \frac{dt}{t - z_1}, \frac{dt}{t - z_2}, \frac{dt}{t - z_3}, \frac{(z_1 - z_2)dt}{(t - z_1)(t - z_2)}, \frac{(z_1 - z_3)dt}{(t - z_1)(t - z_3)}, \frac{(z_2 - z_3)dt}{(t - z_2)(t - z_3)} \right\} \tag{5.51}$$

The 4-point KZ solutions have two independent solutions. These are obtained by choosing two elements from the above. The simplest choice may be $\{\frac{dt}{t-z_1}, \frac{dt}{t-z_2}\}$. This corresponds to the solutions in (5.47).

Relation to $(n + 1)$-point KZ solutions

At the present stage, it is straightforward to generalize the above results to $(n + 1)$-point solutions of the KZ equation. These are obtained as generalized hypergeometric functions on $Gr(2, n + 1)$. Following the representation (5.48)–(5.51), we can write down the $(n + 1)$-point KZ solutions as

$$F_j(z_0, z_1, z_2, \cdots, z_n) = \int_{\Delta} \Phi_0 \Phi_t \, \varphi_j \tag{5.52}$$

$$\Phi_0 = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{1}{\kappa}\Omega_{ij}} \tag{5.53}$$

$$\Phi_t = \prod_{i=1}^{n} (t - z_i)^{-\frac{1}{\kappa}\Omega_i} \tag{5.54}$$

where Φ_t is now defined on $X = \mathbf{CP}^1 - \{z_0, z_1, \cdots, z_n\}$. The construction of F_j is essentially the same as the one considered in (3.9). From Φ_t we can determine the homology group $H_1(X, \mathcal{L}_{\mathfrak{g}}^\vee)$ and the cohomology group $H^1(X, \mathcal{L}_{\mathfrak{g}})$. The twisted cycle Δ and the one-form φ_j are elements of these, respectively, *i.e.*, $\Delta \in H_1(X, \mathcal{L}_{\mathfrak{g}}^\vee)$ and $\varphi_j \in H^1(X, \mathcal{L}_{\mathfrak{g}})$. The number of elements for the basis of the cohomology group is $n - 1$ and such a basis can be chosen as

$$\varphi_j = d \log \frac{t - z_{j+1}}{t - z_j} \qquad (1 \le j \le n - 1) \tag{5.55}$$

There are $n - 1$ solutions and these correspond to $n - 1$ independent solutions of the $(n+1)$-point KZ equation. Namely, we can express the $(n + 1)$-point KZ solution as

$$\Psi = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_{n-1} \end{pmatrix} \tag{5.56}$$

As in the $n = 3$ case, it is known that this Ψ satisfy the differential equation $d\Psi = B\Psi$ where B is called the Gauss–Manin connection and represented by an $(n + 1) \times (n + 1)$ matrix. This Gauss–Manin connection is associated to the definition of the generalized hypergeometric functions on $Gr(2, n + 1)$. In general, such a Gauss–Manin connection can be constructed in association with the generalized hypergeometric functions on $Gr(k + 1, n + 1)$, where the dimension of B is given by $\binom{n-1}{k}$. The study of the Gauss–Manin connection is beyond the scope of this note. Interested readers are advised to see mathematical literature, *e.g.*, Section 3.8 in [15]. In this context, the KZ connection (5.12) can be interpreted as the Gauss–Manin connection for the $Gr(2, n + 1)$-type generalized hypergeometric functions.

So far, we consider the case of $k = 1$. This is natural because it clarifies the relation between Gauss' hypergeometric function and the 4-point KZ solutions. For general $(n + 1)$-point KZ solutions, however, there are no particular reasons to choose $k = 1$ except it leads to the simplest hypergeometric integrals. In principle, we can consider $k \ge 2$ cases and relate the KZ solutions to generalized hypergeometric functions on $Gr(k + 1, n + 1)$. In fact, there is a remarkable result or a theorem by Schechtman and Varchenko [17,18] that, with $\mathfrak{g}$ being the $SL(2, \mathbf{C})$ algebra, the KZ solutions can be expressed by the following hypergeometric-type integral

$$F_J = \int_\Delta \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{1}{\kappa}\Omega_{ij}} \prod_{j=1}^{n} \prod_{s=1}^{k} (t_s - z_j)^{\frac{1}{\kappa}\Omega_j}$$
$$\times \prod_{1 \le r < s \le k} (t_r - t_s)^{\frac{1}{\kappa}} R_J(t, z) \, dt_1 \wedge \cdots \wedge dt_k \tag{5.57}$$

In this expression a multivalued function relevant to Φ_t in (5.54) is given by

$$\widetilde{\Phi}_t = \prod_{j=1}^{n} \prod_{s=1}^{k} (t_s - z_j)^{\frac{1}{\kappa}\Omega_j} = \prod_{j=1}^{n} \widetilde{l}_j(t)^{\frac{1}{\kappa}\Omega_j} \tag{5.58}$$

where

$$\widetilde{l}_j(t) = (t_1 - z_j)(t_2 - z_j) \cdots (t_k - z_j) \tag{5.59}$$

This multivalued function $\widetilde{\Phi}_t$ is defined on the space

$$\widetilde{X} = \mathbf{C}^k - \bigcup_{j=1}^{n} \widetilde{\mathcal{H}}_j \tag{5.60}$$

where

$$\widetilde{\mathcal{H}}_j = \{t \in \mathbf{C}^k ; \ \widetilde{l}_j(t) = 0\} \tag{5.61}$$

Namely, $\widetilde{X}$ represents a coordinate on $\mathbf{C}^k$, eliminating n distinct points $(t_1, t_2, \cdots, t_k) = (z_j, z_j, \cdots, z_j)$ for $1 \leq j \leq n$. This space is analogous to the one in (2.47) except that we have a different $\widetilde{l}_j(t)$ here. From $\widetilde{\Phi}_t$ we can then define the k-th homology and cohomology groups, $H_k(\widetilde{X}, \widetilde{\mathcal{L}}_{\mathfrak{g}}^{\vee})$ and $H^k(\widetilde{X}, \widetilde{\mathcal{L}}_{\mathfrak{g}}^{\vee})$. In the integral (5.57), Δ denotes an element of $H_k(\widetilde{X}, \widetilde{\mathcal{L}}_{\mathfrak{g}}^{\vee})$ or a twisted k-cycle.

According to Schechtman and Varchenko [17,18], $R_J(t, z)$ in (5.57) is a rational function of t_s and z_j and is expressed as follows (see also recent reviews [30,31]). Let J be a set of n non-negative integers $J = (j_1, j_2, \cdots, j_n)$ under the condition $|J| = j_1 + j_2 + \cdots + j_n = k$. Note that there is no particular maximum limit for k (such as $k < n$) in the original derivation [17,18]. (In an alternative derivation with a free field OPE method [20], k corresponds to the number of insertions in an $(n + 1)$-point correlators.) Thus, for $j_i \in \mathbf{Z}_{\geq 0}$ $(i = 1, 2, \cdots, n)$, we have $\binom{n + k - 1}{n - 1}$ different elements in J. This corresponds to the number of possible n-partitions of integer k, allowing an empty set. To be concrete, for each choice of $J = (j_1, j_2, \cdots, j_n)$, we can define an n-partition of the sequence of k integers $(s_1, s_2, \cdots, s_k)$ such that

$$(s_1, s_2, \cdots, s_k) = \Big(\underbrace{1, \cdots, 1}_{j_1}, \underbrace{2, \cdots, 2}_{j_2}, \cdots, \underbrace{n, \cdots, n}_{j_n} \Big) \tag{5.62}$$

Accordingly, we can define a rational function

$$S_J(z_1, \cdots, z_n, t_1, \cdots, t_k) = \frac{1}{(t_1 - z_{s_1})(t_2 - z_{s_2}) \cdots (t_k - z_{s_k})} \tag{5.63}$$

The rational function $R_J(t, z)$ is then defined as

$$R_J(t, z) = \frac{1}{j_1! j_2! \cdots j_n!} \sum_{\sigma \in \mathcal{S}_k} S_J(z_1, \cdots, z_n, t_{\sigma_1}, \cdots, t_{\sigma_k}) \tag{5.64}$$

where the summation of $\mathcal{S}_k$ is taken over the permutations $\sigma = \begin{pmatrix} 1\ 2\ \cdots\ k \\ \sigma_1 \sigma_2 \cdots \sigma_k \end{pmatrix}$. For simple cases, $R_J(t, z)$ are written down as

$$R_{(1,0,\cdots,0)}(z, t) = \frac{1}{t_1 - z_1}, \quad R_{(2,0,\cdots,0)}(z, t) = \frac{1}{(t_1 - z_1)(t_2 - z_1)},$$

$$R_{(1,1,0,\cdots,0)}(z, t) = \frac{1}{(t_1 - z_1)(t_2 - z_2)} + \frac{1}{(t_2 - z_1)(t_1 - z_2)} \tag{5.65}$$

Schechtman and Varchenko show that the $(n + 1)$-point KZ solutions can be given by

$$\sum_{|J|=k} F_J \tag{5.66}$$

for arbitrary k and F_J is defined in (5.57).

In what follows we show that $R_J(t, z)dt_1 \wedge \cdots \wedge dt_k$ can be interpreted as an element of the k-th cohomology group $H^k(\widetilde{X}, \widetilde{\mathcal{L}}_{\mathfrak{g}})$ for $k < n$ and $j_i \in \{0, 1\}$ $(i = 1, 2, \cdots, n)$. The condition $k < n$ is necessary to relate the solutions to generalized hypergeometric functions on $Gr(k + 1, n + 1)$. The other condition $j_i \in \{0, 1\}$ $(i = 1, 2, \cdots, n)$ arises from the fact that the number of the basis for $H^k(\widetilde{X}, \widetilde{\mathcal{L}}_{\mathfrak{g}})$ is given by $\binom{n-1}{k}$ as mentioned in (2.49) and below. Under these conditions the label J can be replaced by a set of k integers $\{j_1, j_2, \cdots, j_k\}$, satisfying $1 \leq j_1 < j_2 < \cdots < j_k \leq n - 1$. The basis of $H^k(\widetilde{X}, \widetilde{\mathcal{L}}_{\mathfrak{g}})$ is then given by

$$
\begin{aligned}
\varphi_{j_1, j_2, \cdots, j_k} &= d \log \widetilde{l}_{j_1} \wedge d \log \widetilde{l}_{j_2} \wedge \cdots \wedge d \log \widetilde{l}_{j_k} \\
&= \left(\frac{dt_1}{t_1 - z_{j_1}} + \cdots + \frac{dt_k}{t_k - z_{j_1}} \right) \wedge \left(\frac{dt_1}{t_1 - z_{j_2}} + \cdots + \frac{dt_k}{t_k - z_{j_2}} \right) \\
&\quad \wedge \cdots \wedge \left(\frac{dt_1}{t_1 - z_{j_k}} + \cdots + \frac{dt_k}{t_k - z_{j_k}} \right) \\
&= \sum_{\sigma \in \mathcal{S}_k} \frac{dt_1 \wedge dt_2 \wedge \cdots \wedge dt_k}{(t_1 - z_{j_{\sigma_1}})(t_2 - z_{j_{\sigma_2}}) \cdots (t_k - z_{j_{\sigma_k}})}
\end{aligned}
\tag{5.67}
$$

In comparison with (5.64), we find that $\varphi_{j_1, j_2, \cdots, j_k}$ are equivalent to $R_I(t, z)$ where the elements of $I = (i_1, i_2, \cdots, i_n)$ are set to $i_l = 1$ for $l = j_1, j_2, \cdots, j_k$ and $i_l = 0$ otherwise. This illustrates a direct relation of the $(n + 1)$-point KZ solutions to the hypergeometric integrals of a form

$$
F_{j_1 j_2 \cdots j_k} = \int_{\Delta} \widetilde{\Phi}_0 \widetilde{\Phi}_t \, \varphi_{j_1 j_2 \cdots j_k}
\tag{5.68}
$$

$$
\widetilde{\Phi}_0 = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{\frac{1}{\kappa} \Omega_{ij}} \prod_{1 \leq r < s \leq k} (t_r - t_s)^{\frac{1}{\kappa}}
\tag{5.69}
$$

$$
\widetilde{\Phi}_t = \prod_{j=1}^{n} \widetilde{l}_j(t)^{\frac{1}{\kappa} \Omega_i}
\tag{5.70}
$$

where $\widetilde{l}_j(t)$ is defined by (5.59). Notice that the basis $\varphi_{j_1 j_2 \cdots j_k}$ in (5.67) is not the unique choice. We can choose different bases which lead to alternative parametrization of the solutions $F_{j_1 j_2 \cdots j_k}$ For example, using the result in (2.50), we can also express the basis of $H^k(\widetilde{X}, \widetilde{\mathcal{L}}_{\mathfrak{g}})$ as

$$
\begin{aligned}
\varphi_{j_1, j_2, \cdots, j_k} &= d \log \frac{\widetilde{l}_{j_1+1}}{\widetilde{l}_{j_1}} \wedge d \log \frac{\widetilde{l}_{j_2+1}}{\widetilde{l}_{j_2}} \wedge \cdots \wedge d \log \frac{\widetilde{l}_{j_k+1}}{\widetilde{l}_{j_k}} \\
&= (z_{j_1+1} - z_{j_1})(z_{j_2+1} - z_{j_2}) \cdots (z_{j_k+1} - z_{j_k}) \\
&\quad \times \sum_{\sigma \in \mathcal{S}_k} \frac{dt_1 \wedge dt_2 \wedge \cdots \wedge dt_k}{(t_1 - z_{j_{\sigma_1}+1})(t_1 - z_{j_{\sigma_1}})(t_2 - z_{j_{\sigma_2}+1})(t_2 - z_{j_{\sigma_2}}) \cdots (t_k - z_{j_{\sigma_k}+1})(t_k - z_{j_{\sigma_k}})}
\end{aligned}
\tag{5.71}
$$

In general, the basis can be chosen as $\varphi_{j_1, j_2, \cdots, j_k} = d \log \frac{\widetilde{l}_{j_1+a}}{\widetilde{l}_{j_1}} \wedge \cdots \wedge d \log \frac{\widetilde{l}_{j_k+a}}{\widetilde{l}_{j_k}}$ where $a \equiv 1, 2, \cdots, n - 1 \pmod{n}$.

Lastly, we notice that $\widetilde{l}_j(t)$ in (5.59) is linear in terms of the elements of $t = (t_1, t_2, \cdots, t_k)$. This enable us to determine the rank-1 local systems $\widetilde{\mathcal{L}}_{\mathfrak{g}}, \widetilde{\mathcal{L}}_{\mathfrak{g}}^{\vee}$ in association to $\widetilde{\Phi}_t$ and make it straightforward to construct the hypergeometric integrals (5.68). However, if we consider

$$\prod_{1 \leq r < s \leq k} (t_r - t_s)^{\frac{1}{\kappa}} \widetilde{\Phi}_t \tag{5.72}$$

rather than $\widetilde{\Phi}_t$, as a multivalued function of interests, we can not properly determine rank-1 local systems out of it since it can not be factorized into functions linear in t. In order to circumvent this issue, one may regard the above multivalued function as a function on $\mathbf{C}^{n+k}$. But this leads to mixture of variables in t_s and z_i and a resulting hypergeometric integral may be regarded as that on $Gr(1, n + k + 1)$. Thus it is not appropriate to think of (5.72) as a multivalued function of interest when we interpret $F_{j_1 j_2 \cdots j_k}$ as hypergeometric functions.

One may still wonder why $\widetilde{\Phi}_t$ instead of (5.72) should be extracted as the defining multivalued function. The author do not have a satisfying answer to it; this could be an ambiguity in the construction of $F_{j_1 j_2 \cdots j_k}$. This issue arises from the fact that $F_{j_1 j_2 \cdots j_k}$ is not *exactly* defined as generalized hypergeometric function on $Gr(k + 1, n + 1)$. As discussed in the beginning of this section, the configuration space of the KZ solutions $\Psi(z_0, z_1, z_2, \cdots, z_n)$ is equivalent to that of generalized hypergeometric functions on $Gr(2, n + 1)$, which is represented by $n + 1$ distinct points in $\mathbf{CP}^1$. If we relate the $(n + 1)$-point KZ solutions to generalized hypergeometric functions on $Gr(k + 1, n + 1)$ in a rigorous manner, we need to expand the configuration space such that it is represented by $n + 1$ distinct points in $\mathbf{CP}^k$ but this brings about ambiguities with the actual/physical configuration space mentioned above.

In conclusion, we can express solutions of the KZ equation in terms of hypergeometric-type integrals. The $(n + 1)$-point KZ solutions in general can be represented by generalized hypergeometric functions on $Gr(2, n + 1)$ as shown in (5.52)–(5.54). We can generalize this expression to represent the $(n + 1)$-point KZ solution as hypergeometric-type integrals on $Gr(k + 1, n + 1)$, as shown in (5.68)–(5.70), but there exists a subtle ambiguity in rigorous construction of the integrals for $k \geq 2$.

6. Holonomy operators of KZ connections

In the present section we review the construction of holonomy operators of the Knizhnik–Zamolodchikov (KZ) connection, following [21,25], and consider the holonomy operators in relation to cohomology and homology of the physical configuration space $\mathcal{C}$ in (5.16). We first reconsider the KZ equation. We rewrite the KZ equation (5.1) as a differential form (5.11), $D\Psi = (d - \Omega)\Psi = 0$, where the KZ connection Ω is defined as

$$\Omega = \frac{1}{\kappa} \sum_{1 \leq i < j \leq n} \Omega_{ij} \, \omega_{ij} \tag{6.1}$$

$$\omega_{ij} = d \log(z_i - z_j) = \frac{dz_i - dz_j}{z_i - z_j} \tag{6.2}$$

In doing so, we implicitly use the condition $\Omega_{ij} = \Omega_{ji}$. The KZ equation is originally derived from the application of a Ward identity to the current correlators. Action of the operator Ω_{ij} on the Hilbert space $V^{\otimes n} = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is then defined as

$$\sum_{\mu} 1 \otimes \cdots \otimes 1 \otimes \rho_i(I_\mu) \otimes 1 \otimes \cdots \otimes 1 \otimes \rho_j(I_\mu) \otimes 1 \otimes \cdots \otimes 1 \tag{6.3}$$

where I_μ ($\mu = 1, 2, \cdots, \dim\mathfrak{g}$) are elements of the Lie algebra $\mathfrak{g}$ and ρ denotes its representation. Thus, by definition, Ω_{ij} satisfies $\Omega_{ij} = \Omega_{ji}$.

For example, in the conventional choice of $\mathfrak{g}$ being the $SL(2, \mathbf{C})$ algebra Ω_{ij} can be defined as

$$\Omega_{ij} = a_i^{(+)} \otimes a_j^{(-)} + a_i^{(-)} \otimes a_j^{(+)} + 2a_i^{(0)} \otimes a_j^{(0)} \tag{6.4}$$

where the operators $a_i^{(\pm, 0)}$ act on the i-th Fock space V_i and forms the $SL(2, \mathbf{C})$ algebra:

$$[a_i^{(+)}, a_j^{(-)}] = 2a_i^{(0)} \delta_{ij} \,, \quad [a_i^{(0)}, a_j^{(+)}] = a_i^{(+)} \delta_{ij} \,, \quad [a_i^{(0)}, a_j^{(-)}] = -a_i^{(-)} \delta_{ij} \tag{6.5}$$

where δ_{ij} denotes Kronecker's delta. Note that in the case of $i = j$, Ω_{ii} becomes the quadratic Casimir of $SL(2, \mathbf{C})$ algebra which acts on V_i. This defines the operator Ω_i that we have introduced in (5.32).

The resultant KZ solutions in an integral form show that the solutions can be described in terms of Ω_{ij} where $1 \leq i < j \leq n$. This is a natural consequence of our setting that the physical configuration space of the KZ solution $\mathcal{C}$ can be represented by ordered distinct $n + 1$ points in $\mathbf{CP}^1$. We have already considered such a space $\mathcal{C} = X_n / \mathcal{S}_n$ in (5.16) and see that the monodromy representation of the KZ equation is given by the braid group $\mathcal{B}_n = \Pi_1(\mathcal{C})$. From these perspectives we can discard the operators Ω_{ji} $(i < j)$ and begin with $D\Psi = (d - \Omega)\Psi = 0$ as the defining KZ equation. *The study of a KZ system can then be attributed to the classification of the KZ connections which satisfy the infinitesimal braid relations (5.6) and (5.7).* As shown in (5.15), such a KZ connection becomes a flat connection, $D\Omega = 0$. Thus, a general solution of the KZ equation can be given by a holonomy of Ω. In the language of gauge theory the holonomy is given by a Wilson loop operator of a gauge field in question. According to Kohno [21], the holonomy of Ω provides a general linear representation of the braid group on the Hilbert space $V^{\otimes n}$.

Holonomy operators of the KZ connections: a review

The holonomy of Ω can be defined as [21]:

$$\Theta_\gamma = 1 + \sum_{r \geq 1} \oint_\gamma \underbrace{\Omega \wedge \Omega \wedge \cdots \wedge \Omega}_{r} \tag{6.6}$$

where γ represents a closed path on $\mathcal{C}_r = X_r / \mathcal{S}_r$ where X_r is defined in (5.3). In the following we shall denote the physical configuration space, with the number of dimensions being explicit. Since the integrand in (6.6) is an r-form, the corresponding integral is taken over the r-dimensional complex space $\mathcal{C}_r$. Formally, the above integral can be evaluated as an iterated integral of K.-T. Chen [22]. Let the path γ in $\mathcal{C}_r$ be represented by

$$\gamma(t) = (z_1(t), z_2(t), \cdots, z_r(t)) \qquad 0 \leq t \leq 1 \tag{6.7}$$

Denoting the pull-back $\gamma^* \omega_{ij}$ as

$$\gamma^* \omega_{ij} = \omega_{ij}(t) = \frac{dz_i(t) - dz_j(t)}{z_i(t) - z_j(t)}, \tag{6.8}$$

we can explicitly express Θ_γ as an iterated integral

$$\Theta_\gamma = \sum_{r \geq 0} \frac{1}{\kappa^r} \int_{0 \leq t_1 \leq t_2 \leq \cdots \leq t_r \leq 1} \sum_{(i < j)} \Omega_{i_1 j_1} \Omega_{i_2 j_2} \cdots \Omega_{i_r j_r} \bigwedge_{l=1}^{r} \omega_{i_l j_l}(t_l) \tag{6.9}$$

where $(i < j)$ means that the set of indices $(i_1, j_1, \cdots, i_r, j_r)$ are ordered such that $1 \leq i_l < j_l \leq r$ for $l = 1, 2, \cdots, r$.

Let the initial point of $\gamma(t)$ be $z_i(0) = z_i$ for $i = 1, 2, \cdots, r$ and the final point be $(z_1(1), z_j(1)) = (z_1, z_{\sigma_j})$ for $j = 2, 3, \cdots, r$ and $\sigma = \begin{pmatrix} 2 & 3 & \cdots & r \\ \sigma_2 & \sigma_3 & \cdots & \sigma_r \end{pmatrix}$. Since $\mathcal{C}_r = X_r / \mathcal{S}_r$ is permutation invariant, the initial and final points are identical and we can naturally interpret γ as a closed path on $\mathcal{C}_r$. By definition γ represents an element of the braid group:

$$\gamma \in \Pi_1(\mathcal{C}_r) = \mathcal{B}_r \tag{6.10}$$

To make Θ_γ permutation invariant explicitly, we now *redefine* the holonomy operator of Ω as an analog of the Wilson loop operator in gauge theory [25]:

$$\Theta_\gamma = \mathrm{Tr}_\gamma \, \mathrm{P} \exp \left[\sum_{r \geq 2} \oint_\gamma \underbrace{\Omega \wedge \Omega \wedge \cdots \wedge \Omega}_{r} \right] \tag{6.11}$$

The meanings of the symbol P and the trace Tr_γ are clarified below. As in (6.9), the exponent in (6.10) can be expanded as

$$\oint_\gamma \underbrace{\Omega \wedge \cdots \wedge \Omega}_{r} = \frac{1}{\kappa^r} \oint_\gamma \sum_{(i<j)} \Omega_{i_1 j_1} \Omega_{i_2 j_2} \cdots \Omega_{i_r j_r} \, \omega_{i_1 j_1} \wedge \omega_{i_2 j_2} \wedge \cdots \wedge \omega_{i_r j_r} \tag{6.12}$$

Action of the symbol P on the above integral imposes the ordering conditions $1 \leq i_1 < i_2 < \cdots < i_r \leq r$ and $2 \leq j_1 < j_2 < \cdots < j_r \leq r + 1$, with $r + 1 \equiv 1 \pmod r$. Thus we have

$$\mathrm{P} \oint_\gamma \underbrace{\Omega \wedge \cdots \wedge \Omega}_{r} = \frac{1}{\kappa^r} \oint_\gamma \Omega_{12} \Omega_{23} \cdots \Omega_{r1} \, \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{r1} \tag{6.13}$$

The trace Tr_γ in (6.11) is carried out by summing over permutations of the indices:

$$\mathrm{Tr}_\gamma \mathrm{P} \oint_\gamma \underbrace{\Omega \wedge \cdots \wedge \Omega}_{r} = \sum_{\sigma \in \mathcal{S}_{r-1}} \frac{1}{\kappa^r} \oint_\gamma \Omega_{1\sigma_2} \Omega_{\sigma_2 \sigma_3} \cdots \Omega_{\sigma_r 1} \, \omega_{1\sigma_2} \wedge \omega_{\sigma_2 \sigma_3} \wedge \cdots \wedge \omega_{\sigma_r 1} := I_r \tag{6.14}$$

This can be interpreted as a trace over generators of the braid group $\mathcal{B}_r$ and is called a braid trace.

Homology and cohomology interpretations of the integral I_r

The holonomy operator (6.11) is essentially calculated by the above integral I_r. We now consider I_r in terms of cohomology and homology of $\mathcal{C}_r$. The cohomology part is relatively straightforward. Since the KZ connection Ω is a flat connection $D\Omega = 0$, it is a closed form with respect to the covariant derivative $D = d - \Omega$. But, by definition, it is not an exact form, that is, $\Omega \neq Df$ for any function f of $(z_1, \cdots, z_r)$. Thus it is an element of the cohomology group $H^1(\mathcal{C}_r)$ whose coefficients are given by $\mathfrak{g} \otimes \mathfrak{g}$,

$$\Omega \in H^1(\mathcal{C}_r, \mathfrak{g} \otimes \mathfrak{g}) \tag{6.15}$$

To generalize, the integrand of I_r can be considered as

$$\underbrace{\Omega \wedge \cdots \wedge \Omega}_{r} \in H^r(\mathcal{C}_r, \mathfrak{g}^{\otimes r}) \tag{6.16}$$

where $\mathfrak{g}^{\otimes r}$ denotes a set of operators acting on the Hilbert space $V^{\otimes r}$.

Since γ is defined as a closed path in $\mathcal{C}_r$, the algebraic coefficients $\Omega_{i_1 j_1} \cdots \Omega_{i_r j_r}$ can be extracted out of the integrand. In order to make sense of Θ_γ as an integral, we need to regard γ as an element of the r-th homology group $H_r(\mathcal{C}_r, \mathbf{R})$, with the coefficients being real number:

$$\gamma \in H_r(\mathcal{C}_r, \mathbf{R}) \tag{6.17}$$

The element γ can be considered as a path in $\mathcal{C}_r$ connecting r hyperplanes $\mathcal{H}_{ij}$ defined in (5.4). Since the KZ equation has branch points at $\mathcal{H}_{ij}$ ($i < j$), (6.17) is in accord with a general concept of the homology group by use of boundary operators as discussed in (2.20)–(2.22). Note that the homology and cohomology considered here are not the twisted ones as before. Thus we can not directly relate I_r to the bilinear construction of hypergeometric-type integrals in (2.23) and (2.24). In the present note, however, we have been equipped with a certain level of understanding of the hypergeometric integrals in terms of homology and cohomology. Namely, we learned that a (co)homology interpretation provides a systematic treatments of analytic continuation or monodromy representation of the solutions to a differential equation of interests. In what follows, we further consider these aspects of the integral I_r.

Let $L\mathcal{C}_r$ be a loop space in $\mathcal{C}_r$. It is known that the fundamental homotopy group of $\mathcal{C}_r$ is isomorphic to the 0-dimensional homology group of $L\mathcal{C}_r$ [32]:

$$\Pi_1(\mathcal{C}_r) \cong H_0(L\mathcal{C}_r) \tag{6.18}$$

With the result (6.10), we then find that γ can also be an element of $H_0(L\mathcal{C}_r)$, with the coefficients being the real number:

$$\gamma \in H_0(L\mathcal{C}_r, \mathbf{R}) \tag{6.19}$$

This suggests that the integrand of I_r can and should be interpreted as an element of the 0-dimensional cohomology group of the loop space $L\mathcal{C}_r$. We can then naturally *assume*

$$\underbrace{\Omega \wedge \cdots \wedge \Omega}_{r} \in H^0(L\mathcal{C}_r, \mathfrak{g}^{\otimes r}) \tag{6.20}$$

In the following, we briefly argue that this assumption is a favorable one. Remember that the basis of $H^{r-1}(\tilde{X}, \widetilde{\mathcal{L}}_{\mathfrak{g}})$ considered in (5.67), with $n = r$ and $k = r - 1$, can be expressed as

$$\varphi_{2,3,\cdots,r} = \sum_{\sigma \in \mathcal{S}_{r-1}} \frac{dt_2 \wedge dt_3 \wedge \cdots \wedge dt_r}{(t_2 - z_{\sigma_2})(t_3 - z_{\sigma_3}) \cdots (t_r - z_{\sigma_r})} \tag{6.21}$$

Adding the parameters (t_1, z_1), we can *redefine* the above basis as

$$\varphi_{1,2,3,\cdots,r} = \sum_{\sigma \in \mathcal{S}_{r-1}} \frac{dt_1 \wedge dt_2 \wedge dt_3 \wedge \cdots \wedge dt_r}{(z_1 - t_1)(z_{\sigma_2} - t_2)(z_{\sigma_3} - t_3) \cdots (z_{\sigma_r} - t_r)} \tag{6.22}$$

By definition (see *e.g.*, (2.45)), the dimension-0 cohomology means there are no extra parameters except $\{z_1, z_2, \cdots, z_r\} \in L\mathcal{C}_r$. The basis of the dimension-0 cohomology group $H^0(L\mathcal{C}_r)$ can then be obtained by identifying $\{t_1, t_2, \cdots, t_r\}$ with $\{z_1, z_2, \cdots, z_r\}$ in the above. To avoid divergence, we need to require $t_i \neq z_i$ in the denominator; since z_i's are permutation invariant we can choose t_i's arbitrarily such that $\varphi_{1,2,\cdots,r}$ becomes finite. One of the simplest nontrivial results can be obtained by setting $t_i = z_{\sigma_{i+1}}$ in the denominator of (6.22), with $t_r = z_{\sigma_{r+1}} = z_1$. We then have

$$\varphi_{1,2,\cdots,r}\big|_{t=z} = \sum_{\sigma \in \mathcal{S}_{r-1}} \frac{dz_1 \wedge dz_2 \wedge \cdots \wedge dz_r}{(z_1 - z_{\sigma_2})(z_{\sigma_2} - z_{\sigma_3})(z_{\sigma_3} - z_{\sigma_4}) \cdots (z_{\sigma_r} - z_1)} \tag{6.23}$$

Apart from the algebraic part involving Ω_{ij}, the above factor is identical to the integral I_r at the level of integrand. Thus, as far as the integrand is concerned, the fact that the basis of $H^0(LC_r)$ is given by (6.23) leads to a favorable confirmation of the assertion (6.20).

As discussed earlier, we need to determine a multivalued function to make a (co)homology interpretation of the integral I_r. The above analyses show that a suitable choice is given by

$$\Phi_0 = \prod_{1 \le i < j \le r} (z_i - z_j)^{\frac{1}{\kappa}\Omega_{ij}} \tag{6.24}$$

which is defined on C_r. As studied in the previous sections, elements of homology groups satisfy the equivalent relation $d \log \Phi_0 \equiv 0$. In particular, from (5.30) we see that this enable us to replace the ordinary derivative ∂_z by the covariant derivative D_z. In a sense this equivalent relation can be thought of as an origin of the minimal coupling principle in gauge theories.

In conclusion, the holonomy operator of the KZ connection is essentially given by the integral I_r. Motivated by the bilinear construction of the generalized hypergeometric functions (which we have reviewed in the previous sections), we analyze I_r in terms of (non-twisted) homology and cohomology group of the relevant physical configuration space $C_r = X_r / S_r$ or its loop space LC_r. Following the notation in (2.23), the construction can be written as

$$H_r(C_r, \mathbf{R}) \times H^r(C_r, \mathfrak{g}^{\otimes r}) \longrightarrow \mathbf{C} \tag{6.25}$$

$$H_0(LC_r, \mathbf{R}) \times H^0(LC_r, \mathfrak{g}^{\otimes r}) \longrightarrow \mathbf{C} \tag{6.26}$$

The two interpretations of the integral I_r suggest that an unambiguous constituent of I_r is given by its integrand $I_r(z)$, *i.e.*,

$$I_r(z) = \sum_{\sigma \in S_{r-1}} \frac{1}{\kappa^r} \frac{\Omega_{1\sigma_2}\Omega_{\sigma_2\sigma_3}\cdots\Omega_{\sigma_r 1}}{(z_1 - z_{\sigma_2})(z_{\sigma_2} - z_{\sigma_3})\cdots(z_{\sigma_{r-1}} - z_{\sigma_r})(z_{\sigma_r} - z_1)} \tag{6.27}$$

$$I_r = \oint_\gamma I_r(z)\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_r \tag{6.28}$$

$$= \sum_{\sigma \in S_{r-1}} \frac{1}{\kappa^r} \oint_\gamma \Omega_{1\sigma_2}\Omega_{\sigma_2\sigma_3}\cdots\Omega_{\sigma_r 1}\, \omega_{1\sigma_2} \wedge \omega_{\sigma_2\sigma_3} \wedge \cdots \wedge \omega_{\sigma_r 1} \tag{6.29}$$

The expressions (6.28) and (6.29) (or (6.14)) correspond to the interpretations (6.26) and (6.25), respectively. Since the equation between (6.28) and (6.29) is mathematically subtle, we need to further consider the meaning of this. Obviously, the equation holds when the r-form $\alpha_\sigma^r := d(z_1 - z_{\sigma_2}) \wedge d(z_{\sigma_2} - z_{\sigma_3}) \wedge \cdots \wedge d(z_{\sigma_r} - z_1)$ is equivalent to $dz_1 \wedge \cdots \wedge dz_r$, regardless choices of σ. Since z_i's are coordinates of C_r they satisfy the permutation invariance and $z_i - z_j \ne 0$ $(i \ne j)$. Thus natural non-vanishing coordinates on C_r can be taken by $(z_1 - z_{\sigma_2}, z_{\sigma_2} - z_{\sigma_3}, \cdots, z_{\sigma_r} - z_1)$. These are the coordinates on C_r, that is, these are supposed to be independent of the permutation σ. If we impose the condition $z_i \ne 0$, we can then identify the above non-vanishing coordinates as $(z_1, \cdots, z_r)$. This means that z_i's are the coordinates on $\mathbf{CP}^{r-1}$ rather than $\mathbf{C}^r$; remember that $C_r = X_r / S_r$ and we have defined $X_r = \mathbf{C}^r - \bigcup_{i<j} \mathcal{H}_{ij}$ where $\mathcal{H}_{ij} = \{(z_1, \cdots, z_r) \in \mathbf{C}^r;\ z_i - z_j = 0\, (i \ne j)\}$. This means that as far as we consider the projected complex spaces we can equate the above r-form α_σ^r to the simplest form $dz_1 \wedge \cdots \wedge dz_r$, and the above relations (6.28) and (6.29) hold.

In terms of $I_r(z)$ *integrand* part of the holonomy operator Θ_γ can be written as

$$\Theta_\gamma(z) = \exp\left(\sum_{r \geq 2} I_r(z)\right) \tag{6.30}$$

Simplification of the algebraic part

Having considered analytic aspects of the holonomy operators, we now consider simplification of the algebraic structure of I_r. As emphasized earlier in this section, the algebra of the KZ connections is determined by the infinitesimal braid relations (5.6) and (5.7). Using the $SL(2, \mathbf{C})$ algebra in (6.5), we introduce a bialgebraic operator [25]:

$$A_{ij} = a_i^{(+)} \otimes a_j^{(0)} + a_i^{(-)} \otimes a_j^{(0)}. \tag{6.31}$$

A_{ij} satisfy the infinitesimal braid relations (5.6), (5.7). Thus, the operator

$$A = \frac{1}{\kappa} \sum_{1 \leq i < j \leq n} A_{ij}\, \omega_{ij} \tag{6.32}$$

obeys the flatness condition $DA = dA - A \wedge A = -A \wedge A = 0$ where D is now a covariant exterior derivative $D = d - A$. This relation guarantees the existence of the holonomy operator for A.

The bialgebraic structures of Ω and A are different but the constituents of these remain the same, *i.e.*, they are given by $a_i^{(0)}$ and $a_i^{(\pm)}$. Thus, we can use the same Hilbert space $V^{\otimes n}$ and physical configuration $\mathcal{C}$ for both Ω and A. The KZ equation of A is then given by

$$D\Psi = (d - A)\Psi = 0 \tag{6.33}$$

where Ψ is a function of a set of complex variables $(z_1, z_2, \cdots z_n)$. In analogy with (6.11) the holonomy operator of A can be defined as

$$\Theta_\gamma^{(A)} = \mathrm{Tr}_\gamma\, \mathrm{P}\exp\left[\sum_{r \geq 2} \oint_\gamma \underbrace{A \wedge A \wedge \cdots \wedge A}_{r}\right] \tag{6.34}$$

The algebraic part of $\Theta_\gamma^{(A)}$ can be simplified as follows. We first note that the commutator $[A_{12}, A_{23}]$ can be calculated as

$$
\begin{aligned}
[A_{12}, A_{23}] = {}& a_1^{(+)} \otimes a_2^{(+)} \otimes a_3^{(0)} - a_1^{(+)} \otimes a_2^{(-)} \otimes a_3^{(0)} \\
& + a_1^{(-)} \otimes a_2^{(+)} \otimes a_3^{(0)} - a_1^{(-)} \otimes a_2^{(-)} \otimes a_3^{(0)}
\end{aligned} \tag{6.35}
$$

An analog of (6.13) is then expressed as

$$
\begin{aligned}
\mathrm{P}\oint_\gamma & \underbrace{A \wedge \cdots \wedge A}_{r} \\
&= \oint_\gamma A_{12} A_{23} \cdots A_{r1}\, \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{r1} \\
&= \frac{1}{2^r} \sum_{(h_1, h_2, \cdots, h_r)} (-1)^{h_1 + h_2 + \cdots + h_r}\, a_1^{(h_1)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)} \oint_\gamma \omega_{12} \wedge \cdots \wedge \omega_{r1}
\end{aligned}
$$
$$\tag{6.37}$$

where h_i denotes $h_i = \pm = \pm 1$ $(i = 1, 2, \cdots, r)$. In the above expression, we define $a_1^{(\pm)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)} \otimes a_1^{(0)}$ as

$$
a_1^{(\pm)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)} \otimes a_1^{(0)} := \frac{1}{2}[a_1^{(0)}, a_1^{(\pm)}] \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)}
$$
$$
= \pm \frac{1}{2} a_1^{(\pm)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)} \tag{6.38}
$$

The braid trace over (6.37) is expressed as

$$
\mathrm{Tr}_\gamma \mathrm{P} \oint_\gamma \underbrace{A \wedge \cdots \wedge A}_{r} = \frac{1}{2^r} \sum_{(h_1, h_2, \cdots, h_r)} (-1)^{h_1 + h_2 + \cdots + h_r} a_1^{(h_1)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)}
$$
$$
\times \sum_{\sigma \in \mathcal{S}_{r-1}} \oint_\gamma \omega_{1\sigma_2} \wedge \omega_{\sigma_2 \sigma_3} \wedge \cdots \wedge \omega_{\sigma_r 1} \tag{6.39}
$$

Applying the result (6.30), we find that the *integrand* part of the holonomy operator $\Theta_\gamma^{(A)}$ is expressed as

$$
\Theta_\gamma^{(A)}(z) = \exp \left(\sum_{r \geq 2} I_r^{(A)}(z) \right) \tag{6.40}
$$

where

$$
I_r^{(A)}(z) = \sum_{(h_1, h_2, \cdots, h_r)} \sum_{\sigma \in \mathcal{S}_{r-1}} \left(\frac{1}{2\kappa} \right)^r \frac{(-1)^{h_1 + h_2 + \cdots + h_r} a_1^{(h_1)} \otimes a_2^{(h_2)} \otimes \cdots \otimes a_r^{(h_r)}}{(z_1 - z_{\sigma_2})(z_{\sigma_2} - z_{\sigma_3}) \cdots (z_{\sigma_{r-1}} - z_{\sigma_r})(z_{\sigma_r} - z_1)}
$$
$$
\tag{6.41}
$$

7. Holonomy formalism for gluon amplitudes

In this section we first present an improved description of the holonomy formalism [25], using $\Theta_\gamma^{(A)}(z)$ obtained in the previous section. Essential ingredients of recent developments in the calculation of gluon amplitudes are the spinor-helicity formalism and the supertwistor space. We begin with a brief review of these topics.

Spinor-helicity formalism

Spinor momenta of massless particles, such as gluons and gravitons, are generally given by two-component complex spinors. In terms of four-momentum p_μ $(\mu = 0, 1, 2, 3)$, which obey the on-shell condition $p^2 = \eta^{\mu\nu} p_\mu p_\nu = p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0$, $\eta^{\mu\nu}$ denoting the Minkowski metric, the spinor momenta can be expressed as

$$
u^A = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 - ip_2 \\ p_0 - p_3 \end{pmatrix}, \quad \bar{u}_{\dot{A}} = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 + ip_2 \\ p_0 - p_3 \end{pmatrix} \tag{7.1}
$$

where we follow a convention to express a spinor as a column vector. The spinor momenta are two-component spinors; A and $\dot{A}$ take values of $(1, 2)$. With these, the four-momentum p_μ can be parametrized as a (2×2)-matrix

$$
p_{\dot{A}}^A = (\sigma^\mu)_{\dot{A}}^A p_\mu = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} = u^A \bar{u}_{\dot{A}} \tag{7.2}
$$

where $\sigma^\mu = (\mathbf{1}, \sigma^i)$ where σ^i $(i = 1, 2, 3)$ denotes the (2×2) Pauli matrices and $\mathbf{1}$ is the (2×2) identity matrix. Requiring that p_μ be real, we can take $\bar{u}_{\dot{A}}$ as a conjugate of u^A, *i.e.*, $\bar{u}_{\dot{A}} = (u^A)^*$.

Lorentz transformations of u^A are given by

$$u^A \rightarrow (gu)^A \tag{7.3}$$

where $g \in SL(2, \mathbf{C})$ is a (2×2)-matrix representation of $SL(2, \mathbf{C})$; the complex conjugate of this relation leads to Lorentz transformations of $\bar{u}_{\dot{A}}$. Four-dimensional Lorentz transformations are realized by a combination of these, that is, the four-dimensional Lorentz symmetry is given by $SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$. Scalar products of u^A's or $\bar{u}_{\dot{A}}$'s, which are invariant under the corresponding $SL(2, \mathbf{C})$, are expressed as

$$u_i \cdot u_j := (u_i u_j) = \epsilon_{AB} u_i^A u_j^B, \quad \bar{u}_i \cdot \bar{u}_j := [\bar{u}_i \bar{u}_j] = \epsilon^{\dot{A}\dot{B}} \bar{u}_{i\,\dot{A}} \bar{u}_{j\,\dot{B}} \tag{7.4}$$

where ϵ_{AB} is the rank-2 Levi–Civita tensor. This can be used to raise or lower the indices, *e.g.*, $u_B = \epsilon_{AB} u^A$. Notice that these products are zero when i and j are identical.

For a theory with conformal invariance, such as Maxwell's electromagnetic theory and $\mathcal{N} = 4$ super Yang–Mills theory, we can impose scale invariance on the spinor momentum, *i.e.*,

$$u^A \sim \lambda u^A, \quad \lambda \in \mathbf{C} - \{0\} \tag{7.5}$$

where λ is non-zero complex number. With this identification, we can regard the spinor momentum u^A as a homogeneous coordinate of the complex projective space $\mathbf{CP}^1$. The local coordinate of $\mathbf{CP}^1$ is represented by a complex variable $z \in \mathbf{C} - \{\infty\}$. This can be related to u^A by the following parametrization:

$$u^A = \frac{1}{\sqrt{p_0 - p_3}} \begin{pmatrix} p_1 - ip_2 \\ p_0 - p_3 \end{pmatrix} := \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ z \end{pmatrix}, \quad z = \frac{\beta}{\alpha} \quad (\alpha \neq 0) \tag{7.6}$$

The local complex coordinate of $\mathbf{CP}^1$ can be taken as $z = \beta/\alpha$ except in the vicinity of $\alpha = 0$, where we can instead use $1/z = \alpha/\beta$ as the local coordinate.

A helicity of a massless particle is generally determined by the so-called Pauli–Lubanski spin vector. In the spinor-momenta formalism, we can define an analog of this spin vector. This enables us to define a helicity operator of massless particles as

$$h = 1 - \frac{1}{2} u^A \frac{\partial}{\partial u^A} \tag{7.7}$$

This means that the helicity of the particle is determined by the degree of homogeneity in u.

Twistor and supertwistor spaces

Twistor space is defined by a four-component spinor $W^I = (\pi^A, v_{\dot{A}})$ $(I = 1, 2, 3, 4)$ where π^A and $v_{\dot{A}}$ are two-component complex spinors. From this definition, it is easily understood that twistor space is represented by the complex homogeneous coordinates of $\mathbf{CP}^3$. Thus, W^I correspond to homogeneous coordinates of $\mathbf{CP}^3$ and satisfy the following relation.

$$W^I \sim \lambda W^I, \quad \lambda \in \mathbf{C} - \{0\} \tag{7.8}$$

In twistor space, the relation between π^A and $v_{\dot{A}}$ is defined as

$$v_{\dot{A}} = x_{\dot{A}A} \pi^A \tag{7.9}$$

where $x_{\dot{A}A}$ represent the local coordinates on S^4. This can be understood from the fact that $\mathbf{CP}^3$ is a $\mathbf{CP}^1$-bundle over S^4. We consider that the S^4 describes a four-dimensional compact

spacetime. Notice that in twistor space the spacetime coordinates $x_{\dot{A}A}$ are emergent quantities. Four-dimensional diffeomorphisms, *i.e.*, general coordinate transformations, is therefore realized by

$$u^A \to u'^A \tag{7.10}$$

rather than $x_{\dot{A}A} \to x'_{\dot{A}A}$.

The key idea of the spinor-helicity formalism in twistor space is the identification of the $\mathbf{CP}^1$ fibre of twistor space with the $\mathbf{CP}^1$ on which the spinor momenta are defined. In other words, we identify π^A with the spinor momenta u^A so that we can essentially describe four-dimensional physics in terms of the coordinates of $\mathbf{CP}^1$. In the spinor-momenta formalism, the twistor-space condition $v_{\dot{A}} = x_{\dot{A}A}\pi^A$ is then expressed as

$$v_{\dot{A}} = x_{\dot{A}A}u^A \tag{7.11}$$

In what follows we use the spinor momenta u^A for the role of π^A in twistor space.

Supertwistor space is defined by the homogeneous coordinates of $\mathbf{CP}^{3|4}$. We can denote the coordinates by

$$W^{\hat{I}} = (u^A, v_{\dot{A}}, \xi^\alpha) \tag{7.12}$$

where we introduce Grassmann variables

$$\xi^\alpha = \theta^\alpha_A u^A \qquad (\alpha = 1, 2, 3, 4) \tag{7.13}$$

in addition to the twistor variables $W^I = (u^A, v_{\dot{A}})$ in (7.8). I and $\hat{I}$ are composite indices that can be labeled as $I = 1, 2, 3, 4$ and $\hat{I} = 1, 2, \cdots, 8$, respectively. Coordinates of a compact four-dimensional spacetime $x_{\dot{A}A}$ and their chiral superpartners θ^α_A arise from the supertwistor space with an imposition of the supertwistor conditions:

$$v_{\dot{A}} = x_{A\dot{A}}u^A, \quad \xi^\alpha = \theta^\alpha_A u^A \tag{7.14}$$

These are a supersymmetric analog of the twistor-space condition (7.11).

As in the case of a superspace formalism, the coordinates $x_{\dot{A}A}$ can be extended to $x_{\dot{A}A} \to x_{\dot{A}A} + 2\bar{\theta}_{\alpha\dot{A}}\theta^\alpha_A$. So a supersymmetric extension of the product $x_{\dot{A}A}p^{A\dot{A}}$ is expressed as

$$
\begin{aligned}
x_{\dot{A}A}p^{A\dot{A}} &\to x_{\dot{A}A}u^A\bar{u}^{\dot{A}} + 2\bar{\theta}_{\alpha\dot{A}}\theta^\alpha_A u^A\bar{u}^{\dot{A}}\Big|_{v_{\dot{A}}=x_{\dot{A}A}u^A, \, \xi^\alpha=\theta^\alpha_A u^A} \\
&= v_{\dot{A}}\bar{u}^{\dot{A}} + 2\bar{\eta}_\alpha\xi^\alpha
\end{aligned}
\tag{7.15}
$$

where we use the supertwistor conditions (7.14). We also define antiholomorphic Grassmann variables $\bar{\eta}_\alpha$ ($\alpha = 1, 2, 3, 4$) as

$$\bar{\eta}_\alpha = \bar{u}_{\dot{A}}\bar{\theta}^{\dot{A}}_\alpha \tag{7.16}$$

Holonomy formalism for MHV amplitudes

Having reviewed the spinor-helicity formalism in twistor space, we now define creation operators of gluons. As mentioned in (7.7), the helicity of a particle is determined by the degree of homogeneity in u. In accordance with (7.7), we define the gluon operators $a_i^{(\pm)}$ of helicity $\pm$ and their superpartners as

$$a_i^{(+)}(\xi_i) = a_i^{(+)}$$

$$a_i^{\left(+\frac{1}{2}\right)}(\xi_i) = \xi_i^\alpha a_{i\alpha}^{\left(+\frac{1}{2}\right)}$$

$$a_i^{(0)}(\xi_i) = \frac{1}{2}\xi_i^\alpha \xi_i^\beta a_{i\alpha\beta}^{(0)} \qquad (7.17)$$

$$a_i^{\left(-\frac{1}{2}\right)}(\xi_i) = \frac{1}{3!}\xi_i^\alpha \xi_i^\beta \xi_i^\gamma \epsilon_{\alpha\beta\gamma\delta} a_i^{\delta\left(-\frac{1}{2}\right)}$$

$$a_i^{(-)}(\xi_i) = \xi_i^1 \xi_i^2 \xi_i^3 \xi_i^4 a_i^{(-)}$$

where $i = 1, 2, \cdots, n$ and $\hat{h}_i = (0, \pm\frac{1}{2}, \pm)$ respectively denote the numbering index and the helicity of the particle. The color factor of gluon can be attached to each of the physical operators:

$$a_i^{(\hat{h}_i)} = t^{c_i} a_i^{(\hat{h}_i)c_i} \qquad (7.18)$$

where t^{c_i}'s are given by the generators of the $SU(N)$ gauge group.

In the coordinate-space (or superspace) representation, the physical operators can be expressed as

$$a_i^{(\hat{h}_i)}(x, \theta) = \int d\mu(p_i)\, a_i^{(\hat{h}_i)}(\xi_i)\, e^{ix_\mu p_i^\mu}\bigg|_{\xi_i^\alpha = \theta_A^\alpha u_i^A} \qquad (7.19)$$

where $d\mu(p)$ denotes the Nair measure:

$$d\mu(p_i) = \frac{d^3 p_i}{(2\pi)^3}\frac{1}{2p_{0i}} = \frac{1}{4}\left[\frac{u_i \cdot du_i}{2\pi i}\frac{d^2\bar{u}_i}{(2\pi)^2} - \frac{\bar{u}_i \cdot d\bar{u}_i}{2\pi i}\frac{d^2 u_i}{(2\pi)^2}\right] \qquad (7.20)$$

The maximally helicity violating (MHV) amplitudes are the scattering amplitudes of $(n-2)$ positive-helicity gluons and 2 negative-helicity gluons. In a momentum-space representation, the MHV tree amplitudes are expressed as

$$\mathcal{A}_{\mathrm{MHV}}^{(r_- s_-)}(u, \bar{u}) = i g^{n-2}(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^n p_i\right) \widehat{A}_{\mathrm{MHV}}^{(r_- s_-)}(u) \qquad (7.21)$$

$$\widehat{A}_{\mathrm{MHV}}^{(r_- s_-)}(u) = \sum_{\sigma \in \mathcal{S}_{n-1}} \mathrm{Tr}(t^{c_1} t^{c_{\sigma_2}} t^{c_{\sigma_3}} \cdots t^{c_{\sigma_n}}) \frac{(u_r u_s)^4}{(u_1 u_{\sigma_2})(u_{\sigma_2} u_{\sigma_3}) \cdots (u_{\sigma_n} u_1)} \qquad (7.22)$$

where r and s denote the numbering indices of the two negative-helicity gluons out of the total n gluons. g is the coupling constant of gluon interactions.

Now it is straightforward to construct an S-matrix functional for the MHV amplitudes by use of the integrand part of the holonomy operator in (6.41). We first replace the operators $a_i^{(h_i)}$ in (6.41) by $a_i^{(\hat{h}_i)}(x, \theta)$ in (7.19). We further use the spinor momenta u_i's for the complex variables z_i's on $\mathbf{CP}^1$. This leads to a supersymmetric versions of $I_r^{(A)}(z)$, i.e.,

$$I_r^{(A)}(u; x, \theta) = \sum_{(\hat{h}_1, \hat{h}_2, \cdots, \hat{h}_r)} \sum_{\sigma \in \mathcal{S}_{r-1}} g^r \frac{(-1)^{\hat{h}_1 + \hat{h}_2 + \cdots + \hat{h}_r}\, a_1^{(\hat{h}_1)}(x, \theta) \otimes \cdots \otimes a_r^{(\hat{h}_r)}(x, \theta)}{(u_1 u_{\sigma_2})(u_{\sigma_2} u_{\sigma_3}) \cdots (u_{\sigma_{r-1}} u_{\sigma_r})(u_{\sigma_r} u_1)}$$

$$(7.23)$$

where we define the coupling constant g by

$$g = \frac{1}{2\kappa} \tag{7.24}$$

In terms of the supersymmetric version of $\Theta_\gamma^{(A)}(z)$,

$$\Theta_\gamma^{(A)}(u; x, \theta) = \exp\left(\sum_{r \geq 2} I_r^{(A)}(u; x, \theta)\right), \tag{7.25}$$

the S-matrix functional for the MHV tree amplitudes can be constructed as

$$\mathcal{F}_{\text{MHV}}\left[a^{(\hat{h})c}\right] = \exp\left[\frac{i}{g^2} \int d^4x \, d^8\theta \, \Theta_\gamma^{(A)}(u; x, \theta)\right] \tag{7.26}$$

Indeed we can check that $\mathcal{F}_{\text{MHV}}$ generates the MHV amplitudes:

$$\frac{\delta}{\delta a_1^{(+)c_1}} \otimes \cdots \otimes \frac{\delta}{\delta a_r^{(-)c_r}} \otimes \cdots \otimes \frac{\delta}{\delta a_s^{(-)c_s}} \otimes \cdots \otimes \frac{\delta}{\delta a_n^{(+)c_n}} \, \mathcal{F}_{\text{MHV}}\left[a^{(h)c}\right]\bigg|_{a^{(h)c}=0}$$

$$= \prod_{i=1}^{n} \int d\mu(p_i) \mathcal{A}_{\text{MHV}}^{(r_- s_-)}(u, \bar{u}) \tag{7.27}$$

where $a^{(h)c}$'s denote the gluon creation operators in the momentum-space representation, which are treated as source functions here. In the above derivation we use the fact that the Grassmann integral over θ's vanishes unless we have the following integrand:

$$\int d^8\theta \, \xi_r^1 \xi_r^2 \xi_r^3 \xi_r^4 \, \xi_s^1 \xi_s^2 \xi_s^3 \xi_s^4 \bigg|_{\xi_i^\alpha = \theta_A^\alpha u_i^A} = (u_r u_s)^4 \tag{7.28}$$

This relation guarantees that only the MHV amplitudes are picked up upon the execution of the Grassmann integral.

Notice that $\mathcal{A}_{\text{MHV}}^{(r_- s_-)}(u, \bar{u})$ contains the delta function

$$(2\pi)^4 \delta^{(4)}(p_1 + \cdots + p_n) = \int d^4x \, e^{i(p_1 + \cdots + p_n)x} \tag{7.29}$$

Thus the resultant expression in (7.27) can be proportional to the products of $\delta^{(4)}(x)$'s. Physically, this is obvious because n gluons are supposed to be scattering at a single point. But, mathematically, we need to be careful about the treatment of the products of the delta functions since there are no rigorous definitions for such products. The problem may be solved if we consider in the momentum representation where the holomorphic MHV amplitudes $\widehat{A}_{\text{MHV}}^{(r_- s_-)}(u)$ in (7.22) can be generated as

$$\frac{\delta}{\delta a_1^{(+)c_1}(x_1)} \otimes \cdots \otimes \frac{\delta}{\delta a_r^{(-)c_r}(x_r)} \otimes \cdots$$

$$\cdots \otimes \frac{\delta}{\delta a_s^{(-)c_s}(x_s)} \otimes \cdots \otimes \frac{\delta}{\delta a_n^{(+)c_n}(x_n)} \, \mathcal{F}_{\text{MHV}}\left[a^{(h)c}\right]\bigg|_{a^{(h)c}(x)=0}$$

$$= i g^{n-2} \, \widehat{A}_{\text{MHV}}^{(r_- r_-)}(u) \tag{7.30}$$

where $a^{(h)c}(x)$'s play the same role as $a^{(h)c}$ in (7.27) except that they are now given by x-space representation of the gluon creation operators:

$$a_i^{(h_i)}(x) = \int d\mu(p_i)\, a_i^{(h_i)}\, e^{ix_\mu p_i^\mu} \tag{7.31}$$

As studied in [26], however, the expression (7.27) turns out to be more useful for the computation of one-loop amplitudes.

CSW rules and non-MHV amplitudes in holonomy formalism

General amplitudes, the so-called non-MHV amplitudes, can be expressed in terms of the MHV amplitudes $\widehat{A}_{\mathrm{MHV}}^{(r_-s_-)}(u)$ at tree level. Prescription for these expressions is called the Cachazo–Svrcek–Witten (CSW) rules [33]. For the next-to-MHV (NMHV) amplitudes, which contain three negative-helicity gluons, the CSW rules can be expressed as

$$\widehat{A}_{\mathrm{NMHV}}^{(r_-s_-t_-)}(u) = \sum_{(i,j)} \widehat{A}_{\mathrm{MHV}}^{(i_+\cdots r_-\cdots s_-\cdots j_+k_+)}(u)\, \frac{\delta_{kl}}{q_{ij}^2}\, \widehat{A}_{\mathrm{MHV}}^{(l_-\,(j+1)_+\cdots t_-\cdots (i-1)_+)}(u) \tag{7.32}$$

where the sum is taken over all possible choices for (i,j) that satisfy the ordering $i < r < s < j < t$. The momentum transfer q_{ij} between the two MHV vertices is given by

$$q_{ij} = p_i + p_{i+1} + \cdots + p_r + \cdots + p_s + \cdots + p_j \tag{7.33}$$

where p's denote four-momenta of gluons as before. General non-MHV amplitudes are then obtained by an iterative use of the relation (7.32).

In terms of the expression (7.27) the CSW rules can be implemented by a contraction operator

$$
\begin{aligned}
\widehat{W}^{(A)}(x) &= \exp\left[-\int d\mu(q) \left(\frac{\delta}{\delta a_p^{(+)}} \otimes \frac{\delta}{\delta a_{-p}^{(-)}} \right) e^{-iq(x-y)} \right]_{y \to x} \\
&= \exp\left[-\int \frac{d^4q}{(2\pi)^4}\, \frac{i}{q^2} \left(\frac{\delta}{\delta a_p^{(+)}} \otimes \frac{\delta}{\delta a_{-p}^{(-)}} \right) e^{-iq(x-y)} \right]_{y \to x}
\end{aligned}
\tag{7.34}
$$

where we take the limit $y \to x$ with $x^0 - y^0 \to 0_+$ (keeping the time ordering $x^0 > y^0$). q denotes a momentum transfer of a virtual gluon. As explicitly parametrized in (7.33), this is off-shell quantity $q^2 \neq 0$. Its on-shell partner can be defined as

$$q_\mu = p_\mu + w\eta_\mu \tag{7.35}$$

where η_μ is a reference null vector ($\eta^2 = 0$) and w is a real parameter. In (4.29) $a_p^{(\pm)}$ denote the creation operators of a pair of virtual gluons at ends of a propagator or at MHV vertices. In the calculation of (4.29) we also use the well-known identity

$$\int d\mu(q) \left[\theta(x^0 - y^0) e^{-iq(x-y)} + \theta(y^0 - x^0) e^{iq(x-y)} \right] = \int \frac{d^4q}{(2\pi)^4}\, \frac{i}{q^2 + i\epsilon}\, e^{-iq(x-y)} \tag{7.36}$$

where ϵ is a positive infinitesimal.

Using the above contraction operator, we can define an S-matrix functional for general non-MHV amplitudes as

$$\mathcal{F}\left[a^{(h)c}\right] = W^{(A)}(x)\, \mathcal{F}_{\mathrm{MHV}}\left[a^{(h)c}\right] \tag{7.37}$$

Generalization of the expression (7.27) can be written as

$$
\frac{\delta}{\delta a_1^{(h_1)c_1}} \otimes \frac{\delta}{\delta a_2^{(h_2)c_2}} \otimes \cdots \otimes \frac{\delta}{\delta a_n^{(h_n)c_n}} \mathcal{F}\left[a^{(h)c}\right]\bigg|_{a^{(h)c}=0}
$$

$$
= \prod_{i=1}^{n} \int d\mu(p_i)\, \mathcal{A}_{\mathrm{N^kMHV}}^{(1_{h_1} 2_{h_2} \cdots n_{h_n})}(u,\bar{u}) \tag{7.38}
$$

where $\mathcal{A}_{\mathrm{N^kMHV}}^{(1_{h_1} 2_{h_2} \cdots n_{h_n})}(u,\bar{u})$ denotes a non-MHV version of the gluon amplitudes in the form of (7.21). This is called $\mathrm{N}^{k-2}\mathrm{MHV}$ amplitudes where $h_i = \pm$ denotes the helicity of the i-th gluon, with the total number of negative helicities being k ($k = 2, 3, \cdots, n-2$). Note that the expression (7.38) is not necessarily limited to tree-level amplitudes; for details on applications of the CSW rules to one-loop amplitudes in the holonomy formalism, see [26]. The above formulation illustrates that the holonomy operator of the KZ connection plays an essential role in the construction of the S-matrix functional $\mathcal{F}\left[a^{(h)c}\right]$ for gluon amplitudes at least at tree level. One of the main purposes for the present note is to study mathematical foundations of the holonomy operator (which we have carried out in the previous section) so as to give an improved description of the holonomy formalism.

8. Grassmannian formulations of gluon amplitudes

In the following we briefly review more *powerful* formulations of gluon amplitudes, known as Grassmannian formulations [1–6]. These are powerful, in particular, in computation of higher loop amplitudes but we here consider basic tree-level formulations.

The supertwistor conditions as the momentum conservation laws

We first review various representations of gluon amplitudes, following [25]. The twistor space condition $v_{\dot{A}} = x_{\dot{A}A} u^A$ in (7.11) naturally leads to the relations

$$
v_{\dot{A}}\bar{u}^{\dot{A}} = x_{\dot{A}A} p^{A\dot{A}} = 2(x_0 p_0 - x_1 p_1 - x_2 p_2 - x_3 p_3) = 2 x_\mu p^\mu \tag{8.1}
$$

where we use the rules of scalar products (7.4) between the spinor momenta. We also parametrize $x_{\dot{A}A}$ in terms of the Minkowski coordinates as

$$
x_{\dot{A}A} = x_\mu (\sigma^\mu)_{A\dot{A}} = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix} \tag{8.2}
$$

The product (8.1) suggests that the physical phase space can be spanned by $(v_{\dot{A}}, \bar{u}^{\dot{A}})$, rather than (x_μ, p^μ), in twistor space. We can then relate a function of $(u, \bar{u})$ to a function of (u, v) by Fourier transform integrals

$$
f(u,v) = \frac{1}{4} \int \frac{d^2\bar{u}}{(2\pi)^2} f(u,\bar{u})\, e^{\frac{i}{2} v_{\dot{A}}\bar{u}^{\dot{A}}}, \qquad f(u,\bar{u}) = \int d^2 v\, f(u,v)\, e^{-\frac{i}{2} v_{\dot{A}}\bar{u}^{\dot{A}}} \tag{8.3}
$$

Similarly, by taking conjugates of these, we have

$$
f(\bar{v},\bar{u}) = \frac{1}{4} \int \frac{d^2 u}{(2\pi)^2} f(u,\bar{u})\, e^{\frac{i}{2} \bar{v}_A u^A}, \qquad f(u,\bar{u}) = \int d^2\bar{v}\, f(\bar{v},\bar{u})\, e^{-\frac{i}{2} \bar{v}_A u^A} \tag{8.4}
$$

These integrals are referred to as Fourier transforms in twistor space.

What is remarkable about the use of supertwistor space in gluon amplitudes is that the supertwistor conditions (7.14) automatically arise from the momentum conservation. We shall review this point in the following. We start from n-point N^{k-2}MHV amplitudes of the form

$$\mathcal{A}_{n,k}(u, \bar{u}) = i g^{n-2} (2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^{n} p_i \right) \widehat{A}_{n,k}(u, \bar{u}) \tag{8.5}$$

This is a generalized version of $\mathcal{A}_{\text{MHV}}^{(r-s_-)}(u, \bar{u})$ in (7.21). Notice that $\widehat{A}_{n,k}(u, \bar{u})$ is no more holomorphic to u^A's for $k \geq 1$. The momentum conservation is realized by the delta function:

$$(2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^{n} p_i \right) = \int d^4 x \, e^{-i \, x_\mu \sum_i p_i^\mu} = \int d^4 x \, e^{-\frac{i}{2} x_{\dot{A}A} \sum_i u_i^A \bar{u}_i^{\dot{A}}} \tag{8.6}$$

We now introduce a fermionic partner of the momentum conservation:

$$\delta^{(8)} \left(\sum_{i=1}^{n} p_{i\,\dot{A}}^A \bar{\theta}_\alpha^{\dot{A}} \right) = \delta^{(8)} \left(\sum_{i=1}^{n} u_i^A \bar{\eta}_{i\alpha} \right) = \int d^8 \theta \, e^{-i \, \theta_A^\alpha \sum_i u_i^A \bar{\eta}_{i\alpha}} \tag{8.7}$$

where $\bar{\eta}_\alpha = \bar{u}_{\dot{A}} \bar{\theta}_\alpha^{\dot{A}}$ as defined in (7.16). Adding this fermionic delta function, the amplitudes can be represented by $(u, \bar{u}, \bar{\eta})$:

$$\mathcal{A}_{n,k}(u, \bar{u}, \bar{\eta}) = \delta^{(8)} \left(\sum_i u_i^A \bar{\eta}_{i\alpha} \right) \mathcal{A}_{n,k}(u, \bar{u}) \tag{8.8}$$

The amplitudes can of course be expressed in terms of the supertwistor variables $W^{\hat{I}} = (u^A, v_{\dot{A}}, \xi^\alpha)$ in (7.12). Using Fourier transforms in supertwistor space, we can obtain such representations as

$$\mathcal{A}_{n,k}(u, v, \xi) = \left[\prod_{i=1}^{n} \frac{1}{4} \int \frac{d^2 \bar{u}_i}{(2\pi)^2} d^4 \bar{\eta}_i \right] \mathcal{A}_{n,k}(u, \bar{u}, \bar{\eta}) \exp \left(\frac{i}{2} \sum_i v_{i\dot{A}} \bar{u}_i^{\dot{A}} + i \sum_i \bar{\eta}_{i\alpha} \xi_i^\alpha \right)$$

$$= i g^{n-2} \int d^4 x d^8 \theta \prod_{i=1}^{n} \delta^{(2)} (v_{i\dot{A}} - x_{\dot{A}A} u_i^A) \delta^{(4)} (\xi_i^\alpha - \theta_A^\alpha u_i^A) \widehat{A}_{n,k}(u, \bar{u}) \tag{8.9}$$

where we use (8.6) and (8.7). The inverse transformation is given by

$$\mathcal{A}_{n,k}(u, \bar{u}, \bar{\eta}) = \left[\prod_{i=1}^{n} \int d^2 v_i d^4 \xi_i \, e^{-\frac{i}{2} v_{i\dot{A}} \bar{u}_i^{\dot{A}}} e^{-i \bar{\eta}_{i\alpha} \xi_i^\alpha} \right] \mathcal{A}_{n,k}(u, v, \xi) \tag{8.10}$$

Notice that the supertwistor conditions (7.14) are indeed embedded in $\mathcal{A}_{n,k}(u, v, \xi)$. This illustrates intimate connections between supertwistor space and massless gauge bosons. In the holonomy formalism we consider the physical operators in the coordinate-space representation, see (7.19) or (7.31), and put the supertwistor conditions by hand. This formulation is suitable as far as we rely on the CSW rules where the amplitudes factorize into the MHV vertices $\widehat{A}_{m,1}(u, \bar{u})$ ($m \leq n$) which are holomorphic to u^A's, apart from the contributions from the momentum conservation. To describe the non-MHV amplitudes in a more democratic manner, we need to handle the non-holomorphic part of the amplitudes properly. This can not be done in the holonomy formalism. The most promising formulation is given by the Grassmannian formulations.

Grassmannian formulations of gluon amplitudes

In the Grassmannian formulations the gluon amplitudes are originally considered in terms of the so-called *dual* supertwistor variables

$$\mathcal{W}^{\hat{I}} = (\bar{v}_A, \bar{u}^{\dot{A}}, \bar{\eta}_\alpha) \tag{8.11}$$

The relevant amplitudes $\mathcal{A}_{n,k}(\bar{v}, \bar{u}, \bar{\eta})$ can be Fourier transformed into $\mathcal{A}_{n,k}(u, \bar{u}, \bar{\eta})$ in (8.10) as

$$\mathcal{A}_{n,k}(u, \bar{u}, \bar{\eta}) = \left[\prod_{j=1}^{n} \int d^2 \bar{v}_j \, e^{-\frac{i}{2} \bar{v}_{jA} u_j^A} \right] \mathcal{A}_{n,k}(\bar{v}, \bar{u}, \bar{\eta}) \tag{8.12}$$

Stripping the fermionic part, we can rewrite (8.12) as

$$\mathcal{A}_{n,k}(u, \bar{u}, \bar{\eta}) = i \, g^{n-2} \, (2\pi)^4 \delta^{(4)} \left(\sum_{j=1}^{n} u_j^A \bar{u}^{j\dot{A}} \right) \delta^{(8)} \left(\sum_{j=1}^{n} u_j^A \bar{\eta}_{j\alpha} \right) \widehat{A}_{n,k}(u, \bar{u}) \tag{8.13}$$

$$\widehat{A}_{n,k}(u, \bar{u}) = \left[\prod_{j=1}^{n} \int d^2 \bar{v}_j \, e^{-\frac{i}{2} \bar{v}_{jA} u_j^A} \right] \widehat{A}_{n,k}(\bar{v}, \bar{u}) \tag{8.14}$$

One of the key ideas of the Grassmannian formulation is that the $\bar{v}$-dependence of $\widehat{A}_{n,k}(\bar{v}, \bar{u})$ is factored out by a delta function as follows:

$$\widehat{A}_{n,k}(\bar{v}, \bar{u}) = \delta \left(\sum_{j=1}^{n} \sum_{i=1}^{k} z_{ij} \, \rho_i^A \, \bar{v}_{jA} \right) \widehat{A}_{n,k}(\bar{u}) \tag{8.15}$$

where z_{ij} $(i = 1, 2, \cdots, k; \ j = 1, 2, \cdots, n; \ k = 2, 3, \cdots, n-2)$ denotes an element of $(k \times n)$ complex matrix Z. The delta function is analogous to the momentum-conservation delta functions but it is qualitatively different from them as it involves mixing of the numbering indices. In (8.15) ρ_i^A's are considered as another set of the holomorphic spinors (which are not necessarily spinor momenta). Performing the $\bar{v}$-integral, $\widehat{A}_{n,k}(u, \bar{u})$ are then expressed as

$$\widehat{A}_{n,k}(u, \bar{u}) = \prod_{j=1}^{n} \delta^{(2)} \left(\sum_{i=1}^{k} z_{ij} \, \rho_i^A - u_j^A \right) \widehat{A}_{n,k}(\bar{u}) \tag{8.16}$$

Together with (8.13), this means that on top of the supertwistor conditions we further impose the relation

$$u_j^A = z_{1j} \, \rho_1^A + z_{2j} \, \rho_2^A + \cdots + z_{kj} \, \rho_k^A \quad (j = 1, 2, \cdots, n) \tag{8.17}$$

As mentioned in (7.10), the four-dimensional spacetime is an emergent concept and the spinor momenta are more fundamental quantities in twistor space. In a multigluon system there is a possibility to extend our perspective on the spinor momenta; we may express one entry of the spinor momenta in terms of another set of spinor momenta which are given by two-component holomorphic spinors. The extra condition (8.17) exactly realizes this possibility. One of the remarkable features in the Grassmannian formalism is that the number of linearly independent ρ_i^A's is related to the number of negative helicity gluons in the amplitudes. This makes it possible to describe non-MHV amplitudes in a universal fashion.

Integrating over ρ_i^A's, we see that the above expressions lead to the Grassmannian formulations of the gluon amplitudes. Apart from the color factor, the n-point N^{k-2}MHV gluon amplitudes in the Grassmannian formulation are conjectured in a form of [1]:

$$\mathcal{L}_{n,k}(\mathcal{W}) = \int \frac{d^{k \times n} Z}{\text{vol}[GL(k)]} \frac{\prod_{i=1}^{k} \delta^{4|4}\left(\sum_{j=1}^{n} z_{ij} \mathcal{W}_j^{\hat{I}}\right)}{(12\cdots k)(23\cdots k+1)\cdots(n\,n+1\cdots n+k-1)} \tag{8.18}$$

where $(j_1 j_2 \cdots j_k)$'s denote k-dimensional minor determinants of Z which consist of the j_1-th to j_k-th columns. As explicitly shown in (8.18), j's are ordered such that $(j_1 j_2 \cdots j_k)$ are cyclic to $(12\cdots k)$ in mod n. Thus the above minor determinant is an *ordered* k-dimensional minor determinant of the $k \times n$ matrix Z, which can be computed as

$$(j_1 j_2 \cdots j_k) := \sum_{\sigma \in \mathcal{S}_k} \begin{pmatrix} 1 & 2 & \cdots & k \\ \sigma_1 & \sigma_2 \cdots & \sigma_k \end{pmatrix} z_{\sigma_1 j_1} z_{\sigma_2 j_2} \cdots z_{\sigma_k j_k} \tag{8.19}$$

This is nothing but a Plücker coordinate of the Grassmannian space $Gr(k,n)$. The delta function in (8.18) is defined as

$$\delta^{4|4}\left(\sum_{j=1}^{n} z_{ij} \mathcal{W}_j^{\hat{I}}\right) = \delta^{(2)}\left(\sum_{j=1}^{n} z_{ij} \bar{v}_{jA}\right) \delta^{(2)}\left(\sum_{j=1}^{n} z_{ij} \bar{u}_j^A\right) \delta^{(4)}\left(\sum_{j=1}^{n} z_{ij} \bar{\eta}_{j\alpha}\right) \tag{8.20}$$

In the Grassmannian formulation (8.18) the physical configuration is given by $Z/GL(k, \mathbf{C})$. As discussed in (2.40) (see also (2.25)), this is equivalent to the Grassmannian space $Gr(k,n)$. Following the lines of arguments on $Gr(k,n)$ in Section 2, it is natural to consider $Gr(k,n)$ as a configuration space of n hyperplanes in $\mathbf{CP}^{k-1}$. The modulo of $GL(k, \mathbf{C})$ is essentially required by avoiding the redundancy in the configuration of n hyperplanes. This redundancy is sometimes called "gauge" symmetry in the Grassmannian formulation. With the redundancy eliminated, we also find that the Grassmannian space $Gr(k,n)$ is represented by n distinct points in $\mathbf{CP}^{k-1}$. The integral measure $d^{k \times n} Z/\text{vol}[GL(k)]$ is relevant to this degrees of freedom.

Homology and cohomology interpretations of $\mathcal{L}_{n,k}(\mathcal{W})$

The physical configuration space of $\mathcal{L}_{n,k}(\mathcal{W})$ is analogous to that of the KZ solutions in (5.3) and (5.4) but in the present case the number of variables extends from n to $k \times n$ and the hyperplanes are defined by $(j_1 j_2 \cdots j_k) = 0$. The physical configuration space in the Grassmannian formulation is then expressed as

$$X_{k \times n} = \mathbf{C}^{k \times n} - \bigcup_{j=1}^{n} \mathcal{H}_{(j\, j+1\cdots j+k-1)} \tag{8.21}$$

where, using the notation (8.19), $\mathcal{H}_{(j\, j+1\cdots j+n-1)}$ is defined as

$$\mathcal{H}_{(j\, j+1\cdots j+k-1)} = \{Z \in Mat_{k,n}(\mathbf{C}) \,|\, (j\, j+1 \cdots j+k-1) = 0\} \tag{8.22}$$

Notice that there are no extra parameters except z_{ij}'s. Following the arguments in Section 6 (see (6.18) and below), the cohomology group of interests in the present case is the 0-dimensional cohomology group of a loop space $L\mathcal{C}_{k \times n}$ in $\mathcal{C}_{k \times n}$. By definition an element of the 0-dimensional cohomology group of $L\mathcal{C}_{k \times n}$ can be expressed as

$$\varphi_{1,2,\cdots,n} = d\log(12\cdots k) \wedge d\log(23\cdots k+1) \wedge \cdots \wedge d\log(n1\cdots k-1)$$

$$= \frac{1}{(12\cdots k)(23\cdots k+1)\cdots(n1\cdots k-1)} \frac{d^{k\times n}Z}{\mathrm{vol}[GL(k)]} \tag{8.23}$$

where we use the fact that $(12\cdots k)$'s are the Plücker coordinates of $Gr(k,n)$. Thus, apart from the delta functions (8.20), the integrand of $\mathcal{L}_{n,k}(\mathcal{W})$ is given by the element of $H^0(L\mathcal{C}_{k\times n}, \mathbf{R})$. As discussed in Section 6, it is more natural to consider (8.23) as an element of $H^{k\times n}(\mathcal{C}_{k\times n}, \mathbf{R})$, the $(k \times n)$-dimensional cohomology group of $\mathcal{C}_{k\times n}$. In either case, the dual supertwistor variables enter in a form of delta functions. Thus, to be more precise, the integrand of $\mathcal{L}_{n,k}(\mathcal{W})$ is given by the element of either $H^0(L\mathcal{C}_{k\times n}, \mathcal{W})$ or $H^{k\times n}(\mathcal{C}_{k\times n}, \mathcal{W})$.

Correspondingly, we can naturally interpret the contour of $\mathcal{L}_{n,k}(\mathcal{W})$ as an element of $H_0(L\mathcal{C}_{k\times n}, \mathbf{R})$ or $H_{k\times n}(\mathcal{C}_{k\times n}, \mathbf{R})$. As considered in (6.17), an element of $H_{k\times n}(\mathcal{C}_{k\times n}, \mathbf{R})$ can be considered as a path in $\mathcal{C}_{k\times n}$ connecting n hyperplanes $H_{(j\,j+1\cdots\,j+k-1)}$. This is equivalent to say that the element is given by that of the braid group $\mathcal{B}_n$:

$$\gamma \in H_{k\times n}(\mathcal{C}_{k\times n}, \mathbf{R}) \cong \mathcal{B}_n \tag{8.24}$$

This is also in accord with the relation $\Pi_1(\mathcal{C}_{k\times n}) \cong H_0(L\mathcal{C}_{k\times n})$ in (6.18). Namely, we also have

$$\gamma \in H_0(L\mathcal{C}_{k\times n}, \mathbf{R}) \cong \Pi_1(\mathcal{C}_{k\times n}) \cong \mathcal{B}_n \tag{8.25}$$

In summary, following the notation in (2.23), we can also construct the integral $\mathcal{L}_{n,k}(\mathcal{W})$ as bilinear forms

$$H_{k\times n}(\mathcal{C}_{k\times n}, \mathbf{R}) \times H^{k\times n}(\mathcal{C}_{k\times n}, \mathcal{W}) \longrightarrow \mathbf{C} \tag{8.26}$$

$$H_0(L\mathcal{C}_{k\times n}, \mathbf{R}) \times H^0(L\mathcal{C}_{k\times n}, \mathcal{W}) \longrightarrow \mathbf{C} \tag{8.27}$$

9. Conclusion

Recent developments in the computation of gluon amplitudes provide an intriguing platform for interrelations between modern physics and mathematics. For example, the Grassmannian formulations of gluon amplitudes shed new light on the notion of spacetime and suggest the importance of Grassmannian spaces for our understanding of gluons or particles themselves. Recently, along the lines of these developments, interests in generalized hypergeometric functions on the Grassmannian spaces have been revived. Naively, one would think that a suitable description of a multi-particle system may be given by analytic functions of several complex variables. The generalized hypergeometric functions provide a useful tool to deal with such functions in a form of integrals, which can be constructed by use of the concepts of twisted homology and cohomology. One of the main purposes of the present note is to familiarize ourselves to these concepts and apply them to physical formulations of gluon amplitudes.

Since the generalized hypergeometric functions are not well recognized among physicists, we make this note sort of pedagogic. We first review the definition of Aomoto's generalized hypergeometric functions on $Gr(k+1, n+1)$, interpreting their integral representations in terms of twisted homology and cohomology. We then consider reduction of the general $Gr(k+1, n+1)$ case to particular $Gr(2, n+1)$ cases. The case of $Gr(2, 4)$ leads to Gauss' hypergeometric functions. We carry out a thorough study of this case in section 4. Much of the present note, by nature, deals with reviews of existed literature. But the results in (4.73)–(4.80) are new as far as the author notices.

The case of $Gr(2, 4)$ corresponds to a four-point solution of the Knizhnik–Zamolodchikov (KZ) equation. The $Gr(2, n + 1)$ cases in general lead to $(n + 1)$-point solutions of the KZ equation. In section 5 we have reviewed these solutions. We further find some ambiguities to relate the cases of $Gr(k + 1, n + 1)$ to the previously known Schechtman–Varchenko integral representations of the KZ equation. A system defined by the KZ equations provides a useful description for a multi-particle systems. Since the monodromy representation of the KZ equation is given by the braid group, the KZ system is advantageous especially to analyze the multi-particle system by use of braid groups. The monodromy representation is also given by the holonomy of the KZ connection, which can be expressed in terms of the iterated integral. In section 6 we review the definition of the holonomy operator of the KZ connection. We find that the integral representation of the holonomy operator can be constructed by a set of homology and cohomology groups in two different ways. This interpretation is schematically summarized in (6.25) and (6.26).

Equipped with this mathematical construction of the holonomy operator, in section 7, we present an improved review of the holonomy formalism for gluon amplitudes. We also carry out a similar analysis on the Grassmannian formulations of gluon amplitudes in section 8. After a detailed introduction to the dual supertwistor variables $\mathcal{W}^{\hat{I}}$, we review the definition of the integral representation of the amplitudes $\mathcal{L}_{n,k}(\mathcal{W})$ in (8.18). We find that the integral can also be constructed in bilinear forms in terms of homology and cohomology group of the relevant physical configuration space; see (8.26) and (8.27). From these analyses we observe that the integral contours of $\mathcal{L}_{n,k}(\mathcal{W})$ are given by elements of $\mathcal{B}_n$, the braid group on n strands.

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