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Alternating quaternary algebra structures on irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$

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ABSTRACT

We determine the multiplicity of the irreducible representation $V(n)$ of the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ as a direct summand of its fourth exterior power $\Lambda^4 V(n)$. The multiplicity is 1 (resp. 2) if and only if $n = 4, 6$ (resp. $n = 8, 10$). For these n we determine the multilinear polynomial identities of degree $\leqslant 7$ satisfied by the $\mathfrak{sl}_2(\mathbb{C})$-invariant alternating quaternary algebra structures obtained from the projections $\Lambda^4 V(n) \to V(n)$. We represent the polynomial identities as the nullspace of a large integer matrix and use computational linear algebra to find the canonical basis of the nullspace.

1. Introduction

The theory of n-ary generalizations of Lie algebras has been studied since the late 1960s, beginning with fundamental work by Russian mathematicians; see the survey articles by Kurosh [24] and Baranovich and Burgin [3]. This theory also appeared naturally and independently in various domains of theoretical physics. Indeed, the discovery of Nambu mechanics [27] in 1973, as well as the work of Fillippov [14] in 1985, gave impulse to a significant development of this theory. From a physical point of view, we mention the work of Takhtajan [30], Michor and Vinogradov [26], de Azcárraga and Pérez-Bueno [13], Gautheron [17], Vaisman [31], Curtright and Zachos [11], Curtright et al. [10], and Ataguema et al. [1]. From an algebraic point of view, we mention the work of Kasymov [23], Ling [25], Hanlon and Wachs [20], Gnedbaye [18], Bremner [5], Filippov [15], Bremner and Hentzel [6], and Pozhidaev [28]. The latest development in mathematical physics related to n-ary algebras is the influential work

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of Bagger and Lambert [2] and Gustavsson [19], which aims at a world-volume theory of multiple M2-branes. For a very recent comprehensive survey of this area, referring to both the physical and mathematical literature, see de Azcárraga and Izquierdo [12].

The main difficulty in this theory is to find a useful generalization of the Jacobi identity. There are two principal candidates: (1) the derivation identity, which states that the multiplication operators in the algebra are derivations of the n-ary structure, and (2) the alternating sum identity, which states that the alternating sum over all possible nested pairs of operations is identically zero. For $n = 2$ both of these identities reduce to the familiar Jacobi identity for Lie algebras.

Alternating n-ary algebras which satisfy the derivation identity were called n-Lie algebras by Filippov; a disadvantage of this theory is that for $n \geqslant 3$ it was shown by Ling [25] that there is only one simple finite-dimensional object over an algebraically closed field of characteristic 0. That is, n-Lie algebras generalize the three-dimensional simple Lie algebra to the n-ary case for $n \geqslant 3$, but not any of the other simple Lie algebras. It seems that the definition of n-Lie algebra is too restrictive: the derivation identity is too strong. From this point of view, the alternating sum identity is a good candidate for a weaker identity which can still be regarded as a natural generalization of the Jacobi identity. It is an open problem to classify the simple alternating n-ary algebras which satisfy the alternating sum identity, but it is clear from the results in Bremner and Hentzel [8] and the present paper that there is more than one simple object, at least for $n = 3$ and $n = 4$.

Our approach in this paper is to use the representation theory of Lie algebras to construct new alternating algebra structures and to discover natural n-ary generalizations of Lie algebras. In particular, we study the exterior powers of an irreducible representation of a simple Lie algebra; these are important objects in invariant theory, algebraic geometry, and the theory of Lie groups (see for example Fulton and Harris [16]). If $\Lambda^n V$, the nth exterior power of an irreducible representation V of a simple Lie algebra L, contains V itself as a direct summand, then the projection $\Lambda^n V \to V$ defines an alternating n-ary algebra structure on V which is L-invariant in the sense that the derivation algebra of this n-ary structure contains a subalgebra isomorphic to L. This approach was used by Bremner and Hentzel [8] in the case of the third exterior power of an irreducible representation of the three-dimensional simple Lie algebra. In this paper, we extend this work to the fourth exterior power.

We recall some basic information about representation theory of Lie algebras. The simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has basis $\{H, E, F\}$ and structure constants

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

All other brackets follow from bilinearity and anticommutativity. For $n \in \mathbb{Z}$, $n \geqslant 0$, the irreducible representation $V(n)$ of $\mathfrak{sl}_2(\mathbb{C})$ with highest weight n has dimension $n + 1$; the action of $\mathfrak{sl}_2(\mathbb{C})$ with respect to the basis $\{v_{n-2i} | i = 0, \ldots, n\}$ is

$$H \cdot v_{n-2i} = (n - 2i)v_{n-2i}, \tag{1}$$

$$E \cdot v_n = 0, \quad E \cdot v_{n-2i} = (n - i + 1)v_{n-2i+2} \quad (i = 1, \ldots, n), \tag{2}$$

$$F \cdot v_{n-2i} = (i + 1)v_{n-2i-2} \quad (i = 0, \ldots, n - 1), \quad F \cdot v_{-n} = 0. \tag{3}$$

Any finite dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to $V(n)$ for some n. Any finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to a direct sum of irreducible representations. The multiplicity of $V(n)$ in its kth exterior power $\Lambda^k V(n)$ is the dimension of the vector space

$$\mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\Lambda^k V(n), V(n))$$

of $\mathfrak{sl}_2(\mathbb{C})$-invariant linear maps P: $\Lambda^k V(n) \to V(n)$. If this multiplicity is positive then P defines an alternating k-ary algebra structure on $V(n)$,

$$[x_1, \ldots, x_k] = P(x_1 \wedge \cdots \wedge x_k),$$

which is $\mathfrak{sl}_2(\mathbb{C})$-invariant in the sense that the action of any $L \in \mathfrak{sl}_2(\mathbb{C})$ is a derivation of the k-ary multiplication: for any $x_1, \ldots, x_k \in V(n)$ we have

$$L \cdot [x_1, \ldots, x_i, \ldots, x_k] = \sum_{i=1}^{k} [x_1, \ldots, L \cdot x_i, \ldots, x_k].$$

For the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ we refer to Humphreys [22].

2. Multiplicity formulas

Bremner and Hentzel [7] studied the case $k = 2$ corresponding to alternating binary algebra structures on $V(n)$. We have

$$\dim \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\Lambda^2 V(n), V(n)) = \begin{cases} 1 & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

In this case $n = 2$ gives the three-dimensional adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$, $n = 6$ gives the seven-dimensional simple non-Lie Malcev algebra, and $n = 10$ gives a new 11-dimensional anticommutative algebra satisfying a polynomial identity of degree 7. Bremner and Hentzel [8] studied the case $k = 3$ corresponding to alternating ternary algebra structures on $V(n)$. For $n = 6q + r$ $(0 \leqslant r \leqslant 5)$ we have

$$\dim \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\Lambda^3 V(n), V(n)) = \begin{cases} q & \text{if } r = 0, 1, 2, 4, \\ q+1 & \text{if } r = 3, 5. \end{cases}$$

The multiplicity is 1 for $n = 3, 5, 6, 7, 8, 10$; the corresponding $V(n)$ provide new examples of alternating ternary algebras.

In this paper, we consider the case $k = 4$: we study alternating quaternary algebra structures on $V(n)$ obtained from $\mathfrak{sl}_2(\mathbb{C})$-invariant linear maps $\Lambda^4 V(n) \to V(n)$. We first obtain a closed formula for the multiplicity $\dim \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\Lambda^4 V(n), V(n))$ using a general approach which applies to arbitrary exterior powers.

Theorem 1. *If n is odd then* $\dim \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\Lambda^4 V(n), V(n)) = 0$. *If n is even then $n = 24q + r$ with $0 \leqslant r < 24$ (r even) and we have*

$$\dim \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}\left(\Lambda^4 V(n), V(n)\right) = \frac{1}{1152} \left\{ \begin{array}{ll|ll} 30n^2 + 96n & \text{if } r = 0, 16 & 30n^2 - 120n + 120 & \text{if } r = 2, \\ 30n^2 + 96n + 288 & \text{if } r = 4, 12 & 30n^2 - 120n + 792 & \text{if } r = 6, 22, \\ 30n^2 + 96n - 384 & \text{if } r = 8 & 30n^2 - 120n + 504 & \text{if } r = 10, 18, \\ 30n^2 - 120n + 408 & \text{if } r = 14 & 30n^2 + 96n - 96 & \text{if } r = 20. \end{array} \right.$$

Proof. A proof using Pólya enumeration is given in Section 9. $\square$

Corollary 2. *The representation $V(n)$ of $\mathfrak{sl}_2(\mathbb{C})$ occurs in $\Lambda^4 V(n)$ with multiplicity 1 (resp. 2) if and only if $n = 4$ or $n = 6$ (resp. $n = 8$ or $n = 10$).*

Proof. The vertices of the parabolas in Theorem 1 occur at either $n = -8/5$ or $n = 2$, so for each r the multiplicity is an increasing function of q. $\square$

For $n = 4, 6, 8, 10$, we use computational linear algebra to find all the multilinear polynomial identities of degree $\leqslant 7$ satisfied by the resulting quaternary algebras.

3. Quaternary algebra structures

In this section, we explain how to compute explicitly the decomposition of $\Lambda^4 V(n)$ as a direct sum of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$, together with an explicit multiplication table for the alternating quaternary algebra structure on $V(n)$ obtained from a projection $\Lambda^4 V(n) \to V(n)$. Recall that $V(n)$ has the vector space basis $\{v_{n-2i} \mid i = 0, 1, \ldots, n\}$ and that the subscript on v_{n-2i} is its weight: its eigenvalue for the action of $H \in \mathfrak{sl}_2(\mathbb{C})$.

Lemma 3 [8, Lemma 5.1]**.** *Let M be an $\mathfrak{sl}_2(\mathbb{C})$-module with $\dim M < \infty$. For $n \in \mathbb{Z}$ let $M_n = \{v \in M \mid H \cdot v = nv\}$ be the subspace of all vectors of weight n together with 0. For $n \geqslant 0$ the multiplicity of $V(n)$ in the decomposition of M as a direct sum of simple $\mathfrak{sl}_2(\mathbb{C})$-modules is $\dim M_n - \dim M_{n+2}$.*

Definition 4. The **tensor basis** of $\Lambda^4 V(n)$ consists of $\binom{n+1}{4}$ **quadruples**:

$$v_p \wedge v_q \wedge v_r \wedge v_s = \sum_{\sigma \in S_4} \epsilon(\sigma)(v_{\sigma(p)} \otimes v_{\sigma(q)} \otimes v_{\sigma(r)} \otimes v_{\sigma(s)}),$$

where $n \geqslant p > q > r > s \geqslant -n$ with $p, q, r, s \equiv n \pmod 2$ and $\epsilon : S_4 \to \{\pm 1\}$ is the sign homomorphism. We usually abbreviate $v_p \wedge v_q \wedge v_r \wedge v_s$ by $[p, q, r, s]$. The action of $L \in \mathfrak{sl}_2(\mathbb{C})$ satisfies the derivation property,

$$\begin{aligned} L \cdot (v_p \wedge v_q \wedge v_r \wedge v_s) = {} & L \cdot v_p \wedge v_q \wedge v_r \wedge v_s + v_p \wedge L \cdot v_q \wedge v_r \wedge v_s \\ & + v_p \wedge v_q \wedge L \cdot v_r \wedge v_s + v_p \wedge v_q \wedge v_r \wedge L \cdot v_s, \end{aligned} \tag{4}$$

and hence the **weight** of the quadruple $T = [p, q, r, s]$ is $w(T) = p + q + r + s$. The **standard order** of the quadruples is given by decreasing weight, and within each weight by reverse lex order: $T = [p, q, r, s]$ precedes $T' = [p', q', r', s']$ if and only if either $w(T) > w(T')$ or $w(T) = w(T')$ and $t > t'$ where t, t' are the components of T, T' in the leftmost position where the components are not equal.

Remark 5. If we apply Lemma 3 to the tensor basis of $V(n)$ for $n = 4, 6, 8, 10$ then we obtain the decomposition of $\Lambda^4 V(n)$ as a direct sum of irreducible representations:

$$\begin{aligned} & \Lambda^4 V(4) \cong V(4), \quad \Lambda^4 V(6) \cong V(12) \oplus V(8) \oplus V(6) \oplus V(4) \oplus V(0), \\ & \Lambda^4 V(8) \cong V(20) \oplus V(16) \oplus V(14) \oplus 2V(12) \oplus V(10) \oplus 2V(8) \oplus V(6) \oplus 2V(4) \oplus V(0), \\ & \Lambda^4 V(10) \cong V(28) \oplus V(24) \oplus V(22) \oplus 2V(20) \oplus V(18) \oplus 3V(16) \oplus 2V(14) \\ & \qquad \oplus 3V(12) \oplus 2V(10) \oplus 3V(8) \oplus V(6) \oplus 3V(4) \oplus V(0). \end{aligned}$$

The next step is to determine the highest weight vectors for the irreducible summands of $\Lambda^4 V(n)$ as linear combinations of the quadruples in the tensor basis.

Lemma 6. *The quadruple $[n, n-2, n-4, n-6]$ is the quadruple with highest weight in $\Lambda^4 V(n)$ and is a highest weight vector for the summand $V(4n-12)$.*

Proof. This follows directly from Eqs. (2) and (4). $\square$

Example 7. For $n = 4$ we have $\Lambda^4 V(4) \cong V(4)$, and so the quadruple $[4, 2, 0, -2]$ is the only highest weight vector in $\Lambda^4 V(4)$. If we identify $[4, 2, 0, -2]$ with the highest weight vector v_4 of $V(4)$, and repeatedly apply F using Eqs. (3) and (4), then we obtain the weight vectors of $\Lambda^4 V(4)$ corresponding to the basis vectors v_2, v_0, v_{-2}, v_{-4} of $V(4)$:

$$\begin{aligned} & v_4 = [4, 2, 0, -2], \quad v_2 = F \cdot v_4 = 4[4, 2, 0, -4], \quad v_0 = \frac{1}{2!} F^2 \cdot v_4 = 6[4, 2, -2, -4], \\ & v_{-2} = \frac{1}{3!} F^3 \cdot v_4 = 4[4, 0, -2, -4], \quad v_{-4} = \frac{1}{4!} F^4 \cdot v_4 = [2, 0, -2, -4]. \end{aligned}$$

The matrix expressing the weight vectors in $V(4)$ in terms of the quadruples in $\Lambda^4 V(4)$ is $C = \mathrm{diag}(1, 4, 6, 4, 1)$. The matrix expressing the quadruples in terms of the weight vectors is $C^{-1} = \mathrm{diag}\left(1, \frac{1}{4}, \frac{1}{6}, \frac{1}{4}, 1\right)$. We now have the structure constants for the $\mathfrak{sl}_2(\mathbb{C})$-invariant alternating quaternary algebra structure on $V(4)$, which we denote by $[v_p, v_q, v_r, v_s]$:

$$[v_4, v_2, v_0, v_{-2}] = v_4, \quad [v_4, v_2, v_0, v_{-4}] = \tfrac{1}{4}v_4, \quad [v_4, v_2, v_{-2}, v_{-4}] = \tfrac{1}{6}v_4,$$
$$[v_4, v_0, v_{-2}, v_{-4}] = \tfrac{1}{4}v_4, \quad [v_2, v_0, v_{-2}, v_{-4}] = v_4.$$

The LCM of the denominators of the coefficients is 12. Taking $a = \sqrt[3]{12}$ and setting $v'_t = v_t/a$, we obtain integral structure constants:

$$[v'_4, v'_2, v'_0, v'_{-2}] = 12v'_4, \quad [v'_4, v'_2, v'_0, v'_{-4}] = 3v'_4, \quad [v'_4, v'_2, v'_{-2}, v'_{-4}] = 2v'_4,$$
$$[v'_4, v'_0, v'_{-2}, v'_{-4}] = 3v'_4, \quad [v'_2, v'_0, v'_{-2}, v'_{-4}] = 12v'_4.$$

In general, for all other weights $w < 4n - 12$ we need to find a basis for the subspace of highest weight vectors of weight w in $\Lambda^4 V(n)$. The dimension of this subspace is the multiplicity of $V(w)$ as a summand of $\Lambda^4 V(n)$.

Definition 8. Suppose that $4n - 14 \geqslant w \geqslant 0$ (w even). Let $d(w)$ be the dimension of the weight space of weight w in $\Lambda^4 V(n)$: the number of quadruples of weight w. We define the matrix $E_w^{(n)}$ of size $d(w+2) \times d(w)$ by setting the (i,j) entry equal to the coefficient of the ith quadruple of weight $w+2$ in the expression for the action of $E \in \mathfrak{sl}_2(\mathbb{C})$ on the jth quadruple of weight w. We call this the **E-action matrix** for weight w of $\Lambda^4 V(n)$; the nonzero vectors in its nullspace are the highest weight vectors of weight w in $\Lambda^4 V(n)$. We compute the row canonical form (RCF) and extract the canonical integral basis (CIB) by setting the free variables equal to the standard unit vectors, clearing denominators, and canceling common factors.

Example 9. For $\Lambda^4 V(6)$ we use the weight space basis of $V(6)$ and obtain

$$E_8^{(6)} = \begin{bmatrix} 2 & 4 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & 2 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} -2 & 1 \end{bmatrix},$$
$$E_6^{(6)} = \begin{bmatrix} 1 & 4 & \cdot \\ \cdot & 2 & 5 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & \cdot & -10 \\ \cdot & 1 & 5/2 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} 20 & -5 & 2 \end{bmatrix},$$
$$E_4^{(6)} = \begin{bmatrix} 4 & \cdot & \cdot & \cdot \\ 1 & 3 & 5 & \cdot \\ \cdot & \cdot & 2 & 6 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & -5 \\ \cdot & \cdot & 1 & 3 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} \cdot & 5 & -3 & 1 \end{bmatrix},$$
$$E_0^{(6)} = \begin{bmatrix} 2 & 5 & \cdot & \cdot & \cdot \\ \cdot & 3 & \cdot & 6 & \cdot \\ \cdot & 1 & 4 & \cdot & 6 \\ \cdot & \cdot & \cdot & 1 & 3 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & 15 \\ \cdot & 1 & \cdot & \cdot & -6 \\ \cdot & \cdot & 1 & \cdot & 3 \\ \cdot & \cdot & \cdot & 1 & 3 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} -15 & 6 & -3 & -3 & 1 \end{bmatrix}.$$

Example 10. For $\Lambda^4 V(8)$ we use the weight space basis of $V(8)$ and obtain

$$E_{16}^{(8)} = \begin{bmatrix} 4 & 6 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & 3/2 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} -3 & 2 \end{bmatrix},$$
$$E_{14}^{(8)} = \begin{bmatrix} 3 & 6 & \cdot \\ \cdot & 4 & 7 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & \cdot & -7/2 \\ \cdot & 1 & 7/4 \end{bmatrix} \xrightarrow{\text{CIB}} \begin{bmatrix} 14 & -7 & 4 \end{bmatrix},$$
$$E_{12}^{(8)} = \begin{bmatrix} 2 & 6 & \cdot & \cdot & \cdot \\ \cdot & 3 & 5 & 7 & \cdot \\ \cdot & \cdot & \cdot & 4 & 8 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & \cdot & -5 & \cdot & 14 \\ \cdot & 1 & 5/3 & \cdot & -14/3 \\ \cdot & \cdot & \cdot & 1 & 2 \end{bmatrix}$$
$$\xrightarrow{\text{CIB}} \begin{bmatrix} 15 & -5 & 3 & \cdot & \cdot \\ -42 & 14 & \cdot & -6 & 3 \end{bmatrix},$$
$$E_{10}^{(8)} = \begin{bmatrix} 1 & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & 5 & 7 & \cdot & \cdot \\ \cdot & \cdot & 3 & \cdot & 7 & \cdot \\ \cdot & \cdot & \cdot & 3 & 5 & 8 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 4 \end{bmatrix} \xrightarrow{\text{RCF}} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & 70 & \cdot \\ \cdot & 1 & \cdot & \cdot & -35/3 & \cdot \\ \cdot & \cdot & 1 & \cdot & 7/3 & \cdot \\ \cdot & \cdot & \cdot & 1 & 5/3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

$$\xrightarrow{\text{CIB}} \begin{bmatrix} -210 & 35 & -7 & -5 & 3 & \cdot \end{bmatrix},$$

$$E_8^{(8)} = \begin{bmatrix} 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 5 & \cdot & 7 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & 4 & \cdot & 7 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 5 & \cdot & 8 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 3 & 6 & \cdot & 8 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 5 \end{bmatrix}$$

$$\xrightarrow{\text{RCF}} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & 7 & \cdot & 56/3 \\ \cdot & \cdot & 1 & \cdot & \cdot & -7 & \cdot & -14 \\ \cdot & \cdot & \cdot & 1 & \cdot & -5 & \cdot & -40/3 \\ \cdot & \cdot & \cdot & \cdot & 1 & 2 & \cdot & 8/3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 5/3 \end{bmatrix}$$

$$\xrightarrow{\text{CIB}} \begin{bmatrix} \cdot & -7 & 7 & 5 & -2 & 1 & \cdot & \cdot \\ \cdot & -56 & 42 & 40 & -8 & \cdot & -5 & 3 \end{bmatrix},$$

$$E_6^{(8)} = \begin{bmatrix} 5 & \cdot & 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 4 & \cdot & 7 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & 7 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 5 & \cdot & \cdot & 8 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 4 & 6 & \cdot & 8 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & 8 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 5 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 6 \end{bmatrix}$$

$$\xrightarrow{\text{RCF}} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -224/3 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 70/3 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 160/3 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -8/3 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -20/3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 8/3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -5 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{CIB}} \begin{bmatrix} 224 & -70 & -160 & 8 & 20 & -8 & 15 & -6 & 3 \end{bmatrix},$$

$$E_4^{(8)} = \begin{bmatrix} 4 & \cdot & 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 3 & \cdot & 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot & 8 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 4 & 6 & \cdot & \cdot & 8 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & 6 & \cdot & \cdot & 8 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & 4 & \cdot & \cdot & \cdot & 8 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 5 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 4 & 6 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & 7 \end{bmatrix}$$

$$\xrightarrow{\text{RCF}} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -28 & \cdot & 98 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 14 & \cdot & -245/3 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 16 & \cdot & -56 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & 21 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -4 & \cdot & 14/3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 2 & \cdot & -7 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -10 & \cdot & 35 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 & \cdot & -7 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 7/3 \end{bmatrix}$$

$$\xrightarrow{\text{CIB}} \begin{bmatrix} 28 & -14 & -16 & 2 & 4 & -2 & 10 & -2 & 1 & \cdot & \cdot \\ -294 & 245 & 168 & -63 & -14 & 21 & -105 & 21 & \cdot & -7 & 3 \end{bmatrix},$$

$$E_0^{(8)} = \begin{bmatrix} 2 & 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 3 & 6 & \cdot & \cdot & 8 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 6 & \cdot & \cdot & 8 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & 8 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 3 & 5 & \cdot & \cdot & \cdot & 8 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & 8 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & 6 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 3 & \cdot & 6 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 4 & \cdot & 7 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 5 & \cdot & 7 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{RCF}} \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 224 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -64 \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 16 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -8 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 12 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -8 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & -3/2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\text{CIB}} \begin{bmatrix} -448 & 128 & -32 & -24 & 16 & -24 & 2 & 16 & 3 & -4 & -4 & 2 \end{bmatrix}.$$

Definition 11. We construct the **weight vector basis** of $\Lambda^4 V(n)$ as follows. We first determine the highest weight vectors for each summand of $\Lambda^4 V(n)$ as described in Definition 8 and Examples 9 and 10. For each highest weight vector X (of weight w, say) we apply $F \in \mathfrak{sl}_2(\mathbb{C})$ repeatedly w times to obtain weight vectors of weights $w-2, w-4, \ldots, -w$ forming a basis of the summand isomorphic to $V(w)$:

$$X, F \cdot X, \frac{1}{2!}F^2 \cdot X, \ldots, \frac{1}{w!}F^w \cdot X.$$

The set of all these weight vectors is the weight vector basis of $\Lambda^4 V(n)$. The **standard order** on this basis is as follows: We order the weight vectors first by decreasing weight of the corresponding highest weight vector and then by increasing power of F within each summand. (When there is more than one highest weight vector with the same weight, we order them as in the canonical integral basis.)

Definition 12. The **weight vector matrix** C is the $\binom{n+1}{4} \times \binom{n+1}{4}$ matrix which expresses the weight vector basis in terms of the tensor basis: the (i,j) entry is the coefficient of the ith quadruple in the jth element of the weight vector basis.

Definition 13. The **alternating quaternary algebra structure** on $V(n)$ is defined in terms of structure constants as follows. The inverse C^{-1} of the weight vector matrix expresses the tensor basis in terms of the weight vector basis. Let $[p,q,r,s]$ be the jth quadruple in the tensor basis. Column j of C^{-1} expresses $[p,q,r,s]$ as a linear combination of the elements of the weight vector basis. Suppose that the highest weight vector for the summand of $\Lambda^4 V(n)$ isomorphic to $V(n)$ is the kth element of the weight vector basis. The entries of C^{-1} in column j and rows $i = k, \ldots, k+n$ are the coefficients of the projection of $[p, q, r, s]$ onto the summand isomorphic to $V(n)$. Let $P: \Lambda^4 V(n) \to V(n)$ be this surjective homomorphism of $\mathfrak{sl}_2(\mathbb{C})$-modules. The quadruple $[p, q, r, s]$ has weight $p+q+r+s$, and the summand isomorphic to $V(n)$ has (at most) one basis vector of this weight. Hence there is at most one nonzero entry in C^{-1} in column j and rows $i = k, \ldots, k+n$. If all these entries are zero then $P(v_p \wedge v_q \wedge v_r \wedge v_s) = 0$. If there is a nonzero entry, say in row ℓ, then $P(v_p \wedge v_q \wedge v_r \wedge v_s) = (C^{-1})_{i\ell} v_{p+q+r+s}$. The resulting alternating quaternary algebra structure on $V(n)$ is denoted by $[v_p, v_q, v_r, v_s]$ and defined by $[v_p, v_q, v_r, v_s] = P(v_p \wedge v_q \wedge v_r \wedge v_s)$.

Example 14. For $n = 6$, from rows 23 to 29 of the matrix inverse we obtain the structure constants for the alternating quaternary algebra structure on $V(6)$. We ignore the equations for which $|p+q+r+s| > 6$ since in these cases the result is obviously zero: there is no vector of the given weight in $V(6)$. The LCM of the denominators of the coefficients is 120, so we can scale the basis vectors of $V(n)$ by setting $v'_t = v_t/\sqrt[3]{120}$ to obtain integral structure constants.

4. Polynomial identities and computational methods

Definition 15. The nonassociative polynomial $I(x_1, \ldots, x_n)$ is a **polynomial identity** for the algebra A if $I(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in A$.

We are concerned with multilinear polynomial identities of degree n for an alternating quaternary algebra. This means that each term consists of a coefficient and a monomial, where the monomial is some permutation of n distinct variables $x_1, x_2, \ldots, x_n$ together with some association type, by which we mean some placement of brackets representing the quaternary operation. For any k-ary operation, the degree of a monomial has the form $n = 1 + \ell(k-1)$ where ℓ is the number of occurrences of the operation in the monomial. Thus for a quaternary operation the degree of a monomial is congruent to 1 modulo 3.

In degree 4, we have only the single association type $[-,-,-,-]$; the alternating property implies that we have only the single monomial $[x_1, x_2, x_3, x_4]$. In degree 7, the alternating property implies that we have only one association type $[[-,-,-,-],-,-,-]$ and only $\binom{7}{4} = 35$ distinct multilinear monomials,

$$[[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}], x_{\sigma(5)}, x_{\sigma(6)}, x_{\sigma(7)}],$$

where $\sigma \in S_7$ is a (4, 3)-shuffle; that is, $1 \leqslant \sigma(1) < \sigma(2) < \sigma(3) < \sigma(4) \leqslant 7$ and $1 \leqslant \sigma(5) < \sigma(6) < \sigma(7) \leqslant 7$. In degree 10, the alternating property implies that we have two association types,

$$[[[-,-,-,-],-,-,-],-,-,-], \quad [[-,-,-,-],[-,-,-,-],-,-],$$

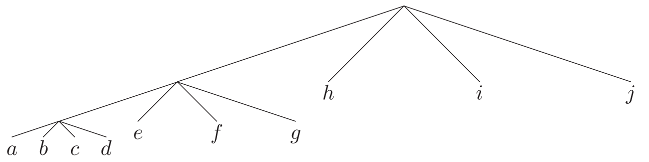
and that the corresponding numbers of distinct multilinear monomials are

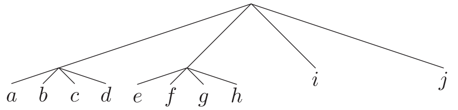
$$\binom{10}{4,3,3} + \frac{1}{2}\binom{10}{4,4,2} = 4200 + 1575 = 5775.$$

The number $T(\ell)$ of association types which involve ℓ occurrences of an alternating quaternary product is equal to the number of rooted trees with ℓ internal vertices in which each internal vertex has four children; see Sloane [29], sequence A036718. The first terms in this sequence are

ℓ	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$T(\ell)$	1	1	1	2	4	9	19	45	106	260	643	1624	4138	10683

The monomials $[[[a, b, c, d], e, f, g], h, i, j]$ and $[[a, b, c, d], [e, f, g, h], i, j]$ correspond, respectively, to the following trees:





4.1. Fill-and-reduce algorithm

Suppose we wish to find all the multilinear polynomial identities of degree n satisfied by an algebra A of dimension d. We assume that we have chosen a basis of A and that we know the structure constants with respect to this basis. We write m for the number of distinct multilinear monomials of degree n, and we assume that these monomials are ordered in some way. We create a matrix X of size $(m + d) \times m$ and initialize it to zero; the columns of M correspond bijectively to the monomials. We choose two parameters p and s: we generate pseudorandom integers in the range 0 to $p - 1$.

We perform the following "fill-and-reduce" algorithm until the rank of the matrix X has remained stable for s iterations:

(1) Generate n pseudorandom elements $a_1, \ldots, a_n$ of A: vectors of length d in which the components are integers in the range 0 to $p - 1$.
(2) For j from 1 to m do:
  (a) Evaluate monomial j by setting the variable x_k equal to the vector a_k for $k = 1, \ldots, n$ and using the structure constants for A, obtaining another vector b of length d.
  (b) Store b as a column vector in column j of X in rows $m + 1$ to $m + d$.
(3) Compute the row canonical form of X; the last d rows are now zero.

After this process has terminated, if the nullspace of X is not zero then it contains candidates for polynomial identities satisfied by A. We usually find that $s = 10$ is a sufficient number of iterations after the rank has stabilized, but we use $s = 100$ to increase our confidence in the results. We now compute the canonical integral basis of the nullspace.

4.2. Module generators algorithm

We assume that we have the canonical integral basis of the nullspace of the matrix X used in the fill-and-reduce algorithm. Let r be the number of these basis vectors; these are the coefficient vectors of polynomial identities satisfied by the algebra A. These identities are linearly independent over $\mathbb{Q}$ but they are not necessarily independent as generators of the S_n-module of identities. We want to find a minimal set of module generators. We start by sorting the basis vectors by increasing Euclidean norm. We create a new matrix M of size $(m + n!) \times m$ and initialize it to zero.

We perform the following algorithm for k from 1 to r:

(1) For i from 1 to $n!$ apply permutation i of the variables $\{x_1, \ldots, x_n\}$ to basis identity k and store the result in row $m + i$ of M. More precisely, for each nonzero coefficient c of the identity, apply permutation i to the corresponding monomial, use the alternating property to straighten the monomial, obtain a standard basis monomial (with index j say) and store $\pm c$ in position $(m + i, j)$ of M (straightening may introduce a sign change).
(2) Compute the row canonical form of M. If the rank has increased from the previous iteration, then we record basis identity k as a new generator.

5. Multiplicity 1: representation $V(4)$

In this section and the next we describe computer searches for polynomial identities satisfied by the two irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ which admit an alternating quaternary structure which is unique up to a scalar multiple; we determine all their identities of degree 7, and show that there are no new identities in degree 10. For all our calculations we use the computer algebra system `Maple`, especially the packages `LinearAlgebra` and `LinearAlgebra[Modular]`.

Theorem 16. *The vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure on $V(4)$ has dimension 21.*

Proof. We use the fill-and-reduce algorithm with $n = 7,\ d = 5,\ m = 35,\ p = 10$ and $s = 100$. We create a matrix X of size 40×35 consisting of an upper block of size 35×35 and a lower block of size 5×35; the columns are labeled by the ordered basis of multilinear monomials in degree 7 for an alternating quaternary operation. We generate seven random elements of $V(4)$ and evaluate the 35 monomials on these seven elements. We put the 35 resulting elements of $V(4)$ as column vectors into the lower block of the matrix. Each of the last five rows of the matrix now contains a linear relation that must be satisfied by the coefficients of any identity for the alternating quaternary structure on $V(4)$. We repeat the fill-and-reduce process until the rank of the matrix stabilizes. The rank reached 14 and did not increase further. Therefore, the nullspace of the matrix has dimension 21. $\square$

Theorem 17. *Every multilinear polynomial identity in degree 7 for the alternating quaternary structure on $V(4)$ is a consequence of the alternating property in degree 4 together with the quaternary derivation identity in degree 7*:

$$\begin{aligned}&[a, b, c, [d, e, f, g]]\\ &\quad = [[a, b, c, d], e, f, g] + [d, [a, b, c, e], f, g] + [d, e, [a, b, c, f], g] + [d, e, f, [a, b, c, g]].\end{aligned}$$

Proof. We use the module generators algorithm, slightly modified to use less memory. We create a matrix of size 59×35 with an upper block of size 35×35 and a lower block of size 24×35. We generate all 5040 permutations of seven letters and divide them into 210 groups of 24 permutations. For each of the 21 basis identities, we perform the following computation. For each group of permutations, we apply the corresponding 24 permutations to the identity, obtain 24 new identities which we store in the lower block of the matrix, and compute the row canonical form of the matrix. After all 210 groups of permutations have been processed, the rank of the matrix is equal to the dimension of the module generated by all the identities up to and including the current identity. After the first identity has been processed, the rank of the matrix is 21, which is the same as the entire nullspace; the rank does not increase as we process the remaining identities. Therefore, every identity is a consequence of the first identity, which has the form

$$\begin{aligned}&[[a, b, c, d], e, f, g] - [[a, b, c, e], d, f, g] + [[a, b, c, f], d, e, g] - [[a, b, c, g], d, e, f]\\ &\quad + [[d, e, f, g], a, b, c].\end{aligned}$$

Applying the alternating property of the quaternary product, we see that this identity can be written in the stated form. $\square$

Remark 18. The alternating property in degree 4 and the quaternary derivation identity together define the case $n = 4$ of the variety of n-Lie algebras introduced by Filippov [14]. Thus the isomorphism $\Lambda^4 V(4) \cong V(4)$ makes $V(4)$ into an alternating quaternary algebra isomorphic to one of the five-dimensional 4-Lie algebras in Filippov's classification of $(n+1)$-dimensional n-Lie algebras.

For $n = 7$, we can use rational arithmetic for these calculations since the matrix X is not large. We can extend these calculations to $n = 10$, but we need to use modular arithmetic to save memory, since the matrix X is very large: it has 5775 columns (the number of alternating quaternary monomials in degree

10). The fill-and-reduce algorithm stabilizes at rank 660, and so the nullspace has dimension 5115. We need to determine which of these identities in degree 10 are consequences of the quaternary derivation identity in degree 7, which we denote by $D(a,b,c,d,e,f,g)$. Since this polynomial alternates in a,b,c we only need to consider six consequences in degree 10, using the variables $\{a,b,c,d,e,f,g,h,i,j\}$:

$$D([a,h,i,j],b,c,d,e,f,g), \quad D(a,b,c,[d,h,i,j],e,f,g), \quad D(a,b,c,d,[e,h,i,j],f,g),$$
$$D(a,b,c,d,e,[f,h,i,j],g), \quad D(a,b,c,d,e,f,[g,h,i,j]), \quad [D(a,b,c,d,e,f,g),h,i,j].$$

We use a modification of the module generators algorithm to determine that these identities generate a module of dimension 5115. Since this equals the dimension of the nullspace from the fill-and-reduce algorithm, it follows that the alternating quaternary structure on $V(4)$ satisfies no new identities in degree 10; that is, every identity in degree 10 is a consequence of the known identities in lower degrees. We used $p=101$ for these calculations; since the group algebra of S_n is semisimple over any field of characteristic $p>n$, and we are studying identities of degree $n=10$, it follows that any prime $p>10$ would give the same dimensions.

6. Multiplicity 1: representation $V(6)$

Theorem 19. *The vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure on $V(6)$ has dimension 1.*

Proof. We use the fill-and-reduce algorithm with $n=7,\ d=7,\ m=35,\ p=10$ and $s=100$. The details of the computations are similar to those described in the proof of Theorem 16. The rank reached 34 and did not increase further. Therefore, the nullspace of the matrix has dimension 1. □

Theorem 20. *Every multilinear polynomial identity in degree 7 for the alternating quaternary structure on $V(6)$ is a consequence of the alternating property in degree 4 together with the quaternary alternating sum identity in degree 7*:

$$\sum_{\sigma\in S_7} \epsilon(\sigma)[[a^\sigma,b^\sigma,c^\sigma,d^\sigma],e^\sigma,f^\sigma,g^\sigma].$$

Proof. Since the nullspace has dimension 1, this is an immediate corollary of Theorem 19; we do not need to apply the module generators algorithm. □

Remark 21. The referee provided the following alternative proof. The quaternary alternating sum identity is an alternating multilinear function of 7 variables. Evaluating this function on the seven-dimensional space $V(6)$ gives a map $\alpha\colon \Lambda^7 V(6) \to V(6)$. But $\Lambda^7 V(6)$ is 1-dimensional (it is isomorphic to $V(0)$ as an $\mathfrak{sl}_2(\mathbb{C})$-module), and α is invariant under the action of $\mathfrak{sl}_2(\mathbb{C})$. Hence the image of α is an $\mathfrak{sl}_2(\mathbb{C})$-submodule which has dimension 0 or 1. Since $V(6)$ is irreducible, it has no submodule of dimension 1, and so α must be the zero map.

Remark 22. It is shown in Bremner [4] (Theorems 3 and 4) that the quaternary alternating sum identity in degree 7 is satisfied by the following multilinear operation (the alternating quaternary sum) in every totally associative quadruple system,

$$[x_1,x_2,x_3,x_4] = \sum_{\pi\in S_4} \epsilon(\pi)\, x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}x_{\pi(4)},$$

and that the quaternary alternating sum identity of Theorem 20 is a consequence of the quaternary derivation identity of Theorem 16.

We can extend these calculations to $n=10$ using modular arithmetic. The fill-and-reduce algorithm stabilizes at rank 1903, and so the nullspace has dimension 3872. We need to determine which of these identities in degree 10 are consequences of the quaternary alternating sum identity in degree 7, which we denote by $S(a,b,c,d,e,f,g)$. Since this polynomial alternates in all 7 variables, we only need to consider two consequences in degree 10:

$$S([a,h,i,j],b,c,d,e,f,g), \quad [S(a,b,c,d,e,f,g),h,i,j].$$

We can use an elementary argument to find an upper bound on the dimension of the submodule generated by these two identities. The first identity alternates in $a,\ h,\ i,\ j$ and also in $b,\ c,\ d,\ e,\ f,\ g$; hence any permutation of the first identity is equal, up to a sign, to one of $\binom{10}{4} = 210$ possibilities. Similarly, the second identity alternates in $a,\ b,\ c,\ d,\ e,\ f,\ g$ and in $h,\ i,\ j$; hence any permutation of the second identity is equal, up to a sign, to one of $\binom{10}{7} = 120$ possibilities. Altogether we see that the submodule generated by these two identities has dimension at most 330. (In fact our computations show that this submodule has dimension 329.) Since this is less than the dimension of the nullspace from the fill-and-reduce algorithm, the alternating quaternary structure on $V(6)$ satisfies new identities in degree 10 that are not consequences of the known identities in lower degrees. It is an open problem to determine generators for the S_{10}-module of new identities in degree 10.

7. Multiplicity 2: representation $V(8)$

In this section and the next we describe computer searches for polynomial identities satisfied by the two irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ which admit a two-dimensional space of alternating quaternary structures; we determine all their identities of degree 7.

Any $\mathfrak{sl}_2(\mathbb{C})$-module homomorphism $\Lambda^4 V(8) \to V(8)$ is a linear combination of the structures f and g corresponding to the summands isomorphic to $V(8)$ generated by the highest weight vectors computed in Example 10. Up to a scalar multiple, we need to consider only the single structure g and the one-parameter family of structures $f + xg$ for $x \in \mathbb{C}$. For g our methods are similar to those used for $V(4)$ and $V(6)$. For $f + xg$ we need to use the Smith normal form to determine the values of the parameter x which produce a nonzero nullspace. For this we use the Maple command `linalg[smith]` instead of `LinearAlgebra[SmithForm]` since the former is much more efficient than the latter.

Theorem 23. *The vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure g on $V(8)$ has dimension 1 and is spanned by the quaternary alternating sum identity.*

Proof. Similar to the proofs of Theorems 19 and 20. □

Theorem 24. *For any $x \in \mathbb{C}$, the vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure $f + xg$ on $V(8)$ has dimension 1 and is spanned by the quaternary alternating sum identity.*

Proof. In order to determine how the space of identities depends on the parameter x, we use the Smith normal form of a matrix over the polynomial algebra $\mathbb{C}[x]$. Since the computation of the Smith form performs not only row operations but also column operations, we must fill the matrix using a suitable number of trials, and then compute the Smith form once. In the general case, we create a matrix of size $t(n+1) \times m$ where n is the highest weight (recall that $V(n)$ has dimension $n+1$) and m is the number of multilinear monomials in degree d; the matrix consists of t blocks of size $(n+1) \times m$. We choose t so that $t(n+1) > m$ in order to guarantee that we have enough nonzero rows in the matrix to eliminate false nullspace vectors. We perform the following algorithm:

(1) For b from 1 to t do:
  (a) Generate d pseudorandom vectors of length $n+1$ representing elements of $V(n)$.
  (b) For j from 1 to m do:

(i) Evaluate the jth alternating quaternary monomial on the d pseudorandom vectors to obtain another vector of length $n+1$ with components which are polynomials in the parameter x.
(ii) Put the resulting vector into column j of block t.

(2) Compute the Smith normal form of the matrix.

For $n=8$ and $d=7$ we have $m=35$ and we choose $t=4$. The entries of the resulting 36×35 matrix are quadratic polynomials in the parameter x since each monomial involves two occurrences of the quaternary operation. In the Smith normal form of the matrix, the diagonal entries are 1 (34 times) and 0 (once). It follows that the matrix has a one-dimensional nullspace for every value of x. In Bremner [4] (Proposition 3) it is shown that there a unique one-dimensional S_7-submodule of the 35-dimensional module with basis consisting of the alternating quaternary monomials in degree 7, and this submodule is spanned by the quaternary alternating sum identity. Hence the nullspace basis does not depend on the value of the parameter x, and this completes the proof. We checked this result independently by evaluating the quaternary alternating sum identity on pseudorandom vectors for the product $f+xg$ with indeterminate x and verifying that the result was zero. □

Remark 25. It is an open problem to determine whether the alternating quaternary structures on $V(8)$ are isomorphic for all values of the parameter x.

8. Multiplicity 2: representation $V(10)$

As in the previous section, any $\mathfrak{sl}_2(\mathbb{C})$-module homomorphism $\Lambda^4 V(10) \to V(10)$ is a linear combination of two structures f and g, and we consider separately the single structure g and the one-parameter family of structures $f+xg$ for $x \in \mathbb{C}$.

Theorem 26. *The vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure g on $V(10)$ has dimension 0: every identity is a consequence of the alternating properties in degree 4.*

Proof. Similar to the proofs of Theorem 23 except that the matrix achieves the full rank of 35. □

Theorem 27. *For $x=\frac{5}{4}$, the vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure $f+xg$ on $V(10)$ has dimension 1 and is spanned by the quaternary alternating sum identity. For all other $x \in \mathbb{C}$, the vector space of multilinear polynomial identities in degree 7 for the alternating quaternary structure $f+xg$ on $V(10)$ has dimension 0.*

Proof. Similar to the proof of Theorem 24 except that now $n=10$. As before, the entries of the resulting 44×35 matrix are quadratic polynomials in the parameter x. In the Smith normal form of this matrix, the diagonal entries are 1 (28 times) and $x-\frac{5}{4}$ (7 times). It follows that the matrix has zero nullspace except in the case $x=\frac{5}{4}$. We now specialize to this value of x and consider the structure $f+\frac{5}{4}g$; the rest of the proof is similar to that of Theorems 19 and 20. □

9. Proof of multiplicity formula

In this section, we prove the multiplicity formula of Theorem 1. We reduce the problem to a combinatorial question and apply the theory of Pólya enumeration.

Lemma 28 [8, Lemma 5.2]. *Let $M=\Lambda^k V(n)$ be the kth exterior power of $V(n)$. If $w \in \mathbb{Z}$ with $kn \geqslant w \geqslant -kn$ and $w \equiv kn \pmod 2$ then the dimension of the weight space M_w is the number of sequences $(w_1, w_2, \ldots, w_k) \in \mathbb{Z}^k$ satisfying*

$$n \geqslant w_1 > w_2 > \cdots > w_k \geqslant -n; \quad w_1 + w_2 + \cdots + w_k = w; \quad w_1, \ldots, w_k \equiv n \pmod 2.$$

We now specialize to $k = 4$ since we are interested in the fourth exterior power. To compute the multiplicity of $V(n)$ as a direct summand of $\Lambda^4 V(n)$ using Lemmas 3 and 28, we must determine the number of quadruples (p, q, r, s) satisfying

$$n \geqslant p > q > r > s \geqslant -n; \quad p + q + r + s = w; \quad p, q, r, s \equiv n \pmod 2 \tag{5}$$

for $w = n$ and $w = n + 2$. Let n be a non-negative integer and let w be a weight of $\Lambda^4 V(n)$: thus w is an integer satisfying

$$4n \geqslant w \geqslant -4n, \quad w \equiv 0 \pmod 2.$$

For integers p, q, r, s satisfying (5) we define

$$P' = p + n, \quad Q' = q + n, \quad R' = r + n, \quad S' = s + n.$$

Then (P', Q', R', S') is a quadruple of even integers satisfying

$$2n \geqslant P' > Q' > R' > S' \geqslant 0, \quad P' + Q' + R' + S' = W', \quad W' = w + 4n.$$

We need to count the number of partitions of W' into four distinct non-negative even parts less than or equal to $2n$. We only need $W' = 5n$ and $W' = 5n + 2$ corresponding to $w = n$ and $w = n + 2$. It is clear that if n is odd then there are no solutions in both cases, so $V(n)$ does not occur as a summand of $\Lambda^4 V(n)$: the multiplicity is zero. Therefore, we may assume that n is even and define

$$P = \frac{p+n}{2}, \quad Q = \frac{q+n}{2}, \quad R = \frac{r+n}{2}, \quad S = \frac{s+n}{2}, \quad W = \frac{w+4n}{2}.$$

Then (P, Q, R, S) is a quadruple of integers satisfying

$$n \geqslant P > Q > R > S \geqslant 0, \quad P + Q + R + S = W.$$

Definition 29 *[32, p. 612]***.** If G is a subgroup of the symmetric group S_n then the **cycle index** of G is the following polynomial in the indeterminates $x_1, x_2, \ldots, x_n$:

$$Z_G(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{\sigma \in G} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n};$$

here b_i is the number of cycles of length i in the disjoint cycle factorization of σ.

Lemma 30 [21, p. 36]**.** *The cycle index of the alternating group A_n is*

$$Z_{A_n}(x_1, x_2, \ldots, x_n) = Z_{S_n}(x_1, x_2, \ldots, x_n) + Z_{S_n}(x_1, -x_2, \ldots, (-1)^{n-1} x_n).$$

Proof. The definition of cycle index gives

$$Z_{A_n}(x_1, x_2, \ldots, x_n) = \frac{2}{n!} \left[\sum_{\sigma \in S_n} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} - \sum_{\sigma \in S_n \backslash A_n} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \right].$$

Since $\sigma \in S_n \backslash A_n$ if and only if σ has an odd number of even length cycles, we get

$$Z_{A_n}(x_1, x_2, \ldots, x_n) = \frac{2}{n!} \cdot \frac{1}{2} \left[\sum_{\sigma \in S_n} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} + \sum_{\sigma \in S_n} x_1^{b_1} (-x_2)^{b_2} \cdots \left((-1)^{n-1} x_n \right)^{b_n} \right].$$

This completes the proof. □

The next result is the special case $k = 4$ of Theorem 2 in Wu and Chao [32]; but note that we allow $0 \in S$.

Proposition 31. *If S is a set of non-negative integers then the number of partitions of an integer n into four distinct parts in S is the coefficient of x^n in*

$$Z_{A_4}\left(\sum_{i\in S}x^i,\sum_{i\in S}x^{2i},\sum_{i\in S}x^{3i},\sum_{i\in S}x^{4i}\right)-Z_{S_4}\left(\sum_{i\in S}x^i,\sum_{i\in S}x^{2i},\sum_{i\in S}x^{3i},\sum_{i\in S}x^{4i}\right).$$

Corollary 32. *If S is a set of non-negative integers then the number of partitions of a positive integer n into four distinct parts in S is the coefficient of x^n in*

$$Z_{S_4}\left(\sum_{i\in S}x^i,-\sum_{i\in S}x^{2i},\sum_{i\in S}x^{3i},-\sum_{i\in S}x^{4i}\right).$$

Proof. Take $n = 4$ in Lemma 30, set $x_j = \sum_{i\in S} x^{ji}$, and apply Proposition 31. $\square$

Definition 33. For us $S = \{0, 1, \ldots, n\}$ so we define the following polynomials:

$$P_n(x)=Z_{S_4}\left(\sum_{i=0}^{n}x^i,-\sum_{i=0}^{n}x^{2i},\sum_{i=0}^{n}x^{3i},-\sum_{i=0}^{n}x^{4i}\right).$$

Lemma 34. *We have*

$$\left(\sum_{i=0}^{n}x^i\right)^t=\sum_{\ell=0}^{nt}\left[\sum_{k=0}^{\min\left(t,\left\lfloor\frac{\ell}{n+1}\right\rfloor\right)}(-1)^k\binom{t}{k}\binom{\ell-(n+1)k+t-1}{t-1}\right]x^{\ell}.$$

Proof. We use these three familiar identities:

$$\frac{1-x^{n+1}}{1-x}=\sum_{i=0}^{n}x^i,\quad \left(1-x^{n+1}\right)^t=\sum_{k=0}^{t}(-1)^k\binom{t}{k}x^{(n+1)k},\quad \frac{1}{(1-x)^t}=\sum_{j=0}^{\infty}\binom{j+t-1}{t-1}x^j.$$

We obtain

$$\left(\sum_{i=0}^{n}x^i\right)^t=\frac{(1-x^{n+1})^t}{(1-x)^t}=\sum_{k=0}^{t}\sum_{j=0}^{\infty}(-1)^k\binom{t}{k}\binom{j+t-1}{t-1}x^{(n+1)k+j}. \tag{6}$$

We set $\ell = (n+1)k + j$ so that $\ell - j = (n+1)k$ and $\ell - j \equiv 0 \pmod{n+1}$. We also have $k = (\ell - j)/(n+1)$ and so $k \leqslant \lfloor \ell/(n+1) \rfloor$. Substituting $j = \ell - (n+1)k$ in (6), and noting that nt is the largest power of x, we obtain the stated formula. $\square$

Definition 35. We use the following notation:

$$\Delta_m^n=\begin{cases}1 & \text{if } n\equiv 0 \pmod m,\\ 0 & \text{otherwise,}\end{cases}\qquad \Delta_{s,m}^n=\begin{cases}1 & \text{if } n\equiv s \pmod m,\\ 0 & \text{otherwise.}\end{cases}$$

Definition 36. We consider the following integer-valued functions of n:

$$\alpha(n)=\left\lceil\frac{n}{4}\right\rceil,\quad \beta(n)=\left\lceil\frac{3n}{4}\right\rceil,\quad \gamma(n)=\left\lfloor\frac{3n-2}{4}\right\rfloor,\quad \delta(n)=\left\lfloor\frac{5n}{6}\right\rfloor.$$

Proposition 37. *For even $n \in \mathbb{Z}$, the number of solutions $P, Q, R, S \in \mathbb{Z}$ to*

$$n\geqslant P>Q>R>S\geqslant 0,\quad P+Q+R+S=\frac{5n}{2},$$

equals

$$\frac{23}{1152}n^3 - \frac{29}{96}n^2 + \frac{1}{288}\left(-36\alpha(n) + 180\beta(n) + 36\gamma(n) + 27\Delta_4^n - 167\right) n$$
$$+\frac{1}{24}\left(6\alpha(n)^2 - 6\beta(n)^2 - 6\gamma(n)^2 + 12\beta(n) - 12\gamma(n) + 8\delta(n) + 3\Delta_4^n - 6\Delta_8^n - 3\right).$$

Proof. By Corollary 32, we need to find the coefficient of $x^{5n/2}$ in the polynomial $P_n(x)$ of Definition 33. The cycle index of S_4 is

$$Z_{S_4}(x_1, x_2, x_3, x_4) = \frac{1}{24}(x_1^4 + 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 + 6x_4).$$

For the first four terms, we need to evaluate the following products:

$$A = \left(\sum_{i=0}^{n} x^i\right)^4, \quad B = \left(\sum_{i=0}^{n} x^i\right)^2 \sum_{i=0}^{n} x^{2i}, \quad C = \sum_{i=0}^{n} x^i \sum_{i=0}^{n} x^{3i}, \quad D = \left(\sum_{i=0}^{n} x^{2i}\right)^2.$$

Lemma 34 gives

$$A = \sum_{\ell=0}^{4n} \sum_{k=0}^{\min\left(4,\left\lfloor \frac{\ell}{n+1}\right\rfloor\right)} (-1)^k \binom{4}{k}\binom{\ell-(n+1)k+3}{3} x^\ell.$$

Similarly,

$$B = \left(\sum_{\ell=0}^{2n} \sum_{k=0}^{\min\left(2,\left\lfloor \frac{\ell}{n+1}\right\rfloor\right)} (-1)^k \binom{2}{k}\left(\ell-(n+1)k+1\right) x^\ell\right) \sum_{i=0}^{n} x^{2i}.$$

The upper limit of k is 0 for $0 \leqslant \ell \leqslant n$, and 1 for $n+1 \leqslant \ell \leqslant 2n$. Hence

$$\begin{aligned} B &= \left(\sum_{\ell=0}^{n}(\ell+1)x^\ell + \sum_{\ell=n+1}^{2n}[(\ell+1) - 2(\ell-n)]x^\ell\right)\sum_{i=0}^{n} x^{2i} \\ &= \left(\sum_{\ell=0}^{n}(\ell+1)x^\ell + \sum_{\ell=n+1}^{2n}(2n-\ell+1)x^\ell\right)\sum_{i=0}^{n} x^{2i} \\ &= \sum_{\ell=0}^{n}\sum_{m=0}^{n}(\ell+1)x^{\ell+2m} + \sum_{\ell=n+1}^{2n}\sum_{m=0}^{n}(2n-\ell+1)x^{\ell+2m}. \end{aligned}$$

We now set $p = \ell + 2m$, so that $\ell = p - 2m$. For $0 \leqslant \ell \leqslant n$ we have $0 \leqslant p - 2m \leqslant n$ and so $\frac{1}{2}(p-n) \leqslant m \leqslant \frac{1}{2}p$, but $m \in \mathbb{Z}$ so $\left\lceil \frac{1}{2}(p-n)\right\rceil \leqslant m \leqslant \left\lfloor \frac{1}{2}p\right\rfloor$; since also $0 \leqslant m \leqslant n$ we get $\max\left(0, \left\lceil \frac{1}{2}(p-n)\right\rceil\right) \leqslant m \leqslant \min\left(n, \left\lfloor \frac{1}{2}p\right\rfloor\right)$. Similarly, for $n+1 \leqslant \ell \leqslant 2n$ we obtain $\max\left(0, \left\lceil \frac{1}{2}(p-2n)\right\rceil\right) \leqslant m \leqslant \min\left(n, \left\lfloor \frac{1}{2}(p-(n+1))\right\rfloor\right)$. Therefore

$$B = \sum_{p=0}^{3n} \sum_{m=\max\left(0,\left\lceil \frac{p-n}{2}\right\rceil\right)}^{\min\left(n,\left\lfloor \frac{p}{2}\right\rfloor\right)} (p-2m+1)\, x^p + \sum_{p=n+1}^{4n} \sum_{m=\max\left(0,\left\lceil \frac{p-2n}{2}\right\rceil\right)}^{\min\left(n,\left\lfloor \frac{p-(n+1)}{2}\right\rfloor\right)} (2n-(p-2m)+1)\, x^p.$$

Using a similar change of index we obtain

$$C = \sum_{i=0}^{n}\sum_{j=0}^{n} x^{i+3j} = \sum_{p=0}^{4n} \sum_{m=\max(0,\lceil \frac{p-n}{3}\rceil)}^{\min\left(n,\left\lfloor \frac{p}{3}\right\rfloor\right)} x^p = \sum_{p=0}^{4n}\left[\min\left(n, \left\lfloor \frac{p}{3}\right\rfloor\right) - \max\left(0, \left\lceil \frac{p-n}{3}\right\rceil\right) + 1\right] x^p.$$

Replacing x by x^2 in Lemma 34 gives

$$D=\sum_{\ell=0}^{n}(\ell+1)x^{2\ell}+\sum_{\ell=n+1}^{2n}(2n-\ell+1)x^{2\ell}.$$

We now write

$$E=\sum_{\ell=0}^{n}x^{4\ell},$$

and obtain

$$\begin{aligned}
&A-6B+8C+3D-6E\\
&=\sum_{p=0}^{4n}\sum_{k=0}^{\min\left(4,\left\lfloor\frac{p}{n+1}\right\rfloor\right)}(-1)^k\binom{4}{k}\binom{p-(n+1)k+3}{3}x^p\\
&\quad-6\left[\sum_{p=0}^{3n}\sum_{m=\max\left(0,\left\lceil\frac{p-n}{2}\right\rceil\right)}^{\min\left(n,\left\lfloor\frac{p}{2}\right\rfloor\right)}(p-2m+1)\,x^p+\sum_{p=n+1}^{4n}\sum_{m=\max\left(0,\left\lceil\frac{p-2n}{2}\right\rceil\right)}^{\min\left(n,\left\lfloor\frac{p-(n+1)}{2}\right\rfloor\right)}(2n-(p-2m)+1)\,x^p\right]\\
&\quad+8\left[\sum_{p=0}^{4n}\left[\min\left(n,\left\lfloor\frac{p}{3}\right\rfloor\right)-\max\left(0,\left\lceil\frac{p-n}{3}\right\rceil\right)+1\right]x^p\right]\\
&\quad+3\left[\sum_{\ell=0}^{n}(\ell+1)x^{2\ell}+\sum_{\ell=n+1}^{2n}(2n-\ell+1)x^{2\ell}\right]-6\sum_{\ell=0}^{n}x^{4\ell}.
\end{aligned}$$

We need the coefficient T of $x^{5n/2}$ in the last equation:

$$\begin{aligned}
T=&\sum_{k=0}^{\left\lfloor\frac{5n}{2(n+1)}\right\rfloor}(-1)^k\binom{4}{k}\binom{\frac{5n}{2}-(n+1)k+3}{3}\\
&-6\left[\sum_{m=\left\lceil\frac{3n}{4}\right\rceil}^{n}\left(\frac{5n}{2}-2m+1\right)+\sum_{m=\left\lceil\frac{n}{4}\right\rceil}^{\left\lfloor\frac{3n-2}{4}\right\rfloor}\left(2m-\frac{n}{2}+1\right)\right]\\
&+8\left(\left\lfloor\frac{5n}{6}\right\rfloor-\frac{n}{2}+1\right)+3\delta_4^n\left(0+\frac{3n}{4}+1\right)-6\delta_8^n.
\end{aligned}$$

For $n=0$ and $n=2$ we get $T=0$; this is expected since the $\mathfrak{sl}_2(\mathbb{C})$-modules $V(0)$ and $V(2)$ have dimensions 1 and 3, respectively, so in both cases $\Lambda^4V(n)$ is $\{0\}$. For $n\geqslant 4$ the upper limit of k is 2, and we use the formula

$$\sum_{m=a}^{b}m=\frac{1}{2}(b-a+1)(b+a).$$

We obtain

$$\begin{aligned}
T=&\binom{\frac{5n}{2}+3}{3}-4\left(\tfrac{3n}{2}+2\;3\right)+6\binom{\frac{n}{2}+1}{3}-6\left(n-\left\lceil\frac{3n}{4}\right\rceil+1\right)\left(\frac{5n}{2}+1\right)\\
&+6\left(n-\left\lceil\frac{3n}{4}\right\rceil+1\right)\left(n+\left\lceil\frac{3n}{4}\right\rceil\right)-6\left(\left\lfloor\frac{3n-2}{4}\right\rfloor-\left\lceil\frac{n}{4}\right\rceil+1\right)\left(-\frac{n}{2}+1\right)
\end{aligned}$$

$$-6\left(\left\lfloor\frac{3n-2}{4}\right\rfloor-\left\lceil\frac{n}{4}\right\rceil+1\right)\left(\left\lfloor\frac{3n-2}{4}\right\rfloor+\left\lceil\frac{n}{4}\right\rceil\right)+8\left\lfloor\frac{5n}{6}\right\rfloor-4n+8$$
$$+3\delta_4^n\left(\frac{3n}{4}+1\right)-6\delta_8^n.$$

Expanding this and collecting terms with the same power of n gives

$$\frac{23}{48}n^3-\frac{29}{4}n^2+\frac{1}{12}(-36\alpha(n)+180\beta(n)+36\gamma(n)+27\Delta_4^n-167)n$$
$$+(6\alpha(n)^2-6\beta(n)^2-6\gamma(n)^2+12\beta(n)+8\delta(n)-12\gamma(n)+3\Delta_4^n-6\Delta_8^n-3).$$

We check that this gives $T=0$ for $n=0$ and $n=2$. Finally, we divide by 24. $\square$

Corollary 38. *For even $n\in\mathbb{Z}$, write $n=24q+r$ with $q,r\in\mathbb{Z}$ and $0\leqslant r<24$. The dimension of the weight space of weight n in the $\mathfrak{sl}_2(\mathbb{C})$-module $\Lambda^4V(n)$ is*

$$\dim[\Lambda^4V(n)]_n$$
$$=\frac{1}{1152}\begin{cases}23n^3-42n^2+48n & (r=0)\\ 23n^3-42n^2-60n+104 & (r=2)\\ 23n^3-42n^2+48n+160 & (r=4)\\ 23n^3-42n^2-60n+360 & (r=6)\\ 23n^3-42n^2+48n-256 & (r=8)\\ 23n^3-42n^2-60n+232 & (r=10)\end{cases}\left|\begin{array}{ll}23n^3-42n^2+48n+288 & (r=12),\\ 23n^3-42n^2-60n+104 & (r=14),\\ 23n^3-42n^2+48n-128 & (r=16),\\ 23n^3-42n^2-60n+360 & (r=18),\\ 23n^3-42n^2+48n+32 & (r=20),\\ 23n^3-42n^2-60n+232 & (r=22).\end{array}\right.$$

Proof. The dimension is given by the formula of Proposition 37. The LCM of the denominators of the functions $\alpha(n)$, $\beta(n)$, $\gamma(n)$, $\delta(n)$ and the periods of the functions Δ_4^n and Δ_8^n equals 24. Hence the dimension is given by a cubic polynomial in n which depends on the remainder of n modulo 24. $\square$

Definition 39. We consider the following integer-valued functions of n:

$$\epsilon(n)=\left\lfloor\frac{3n}{4}\right\rfloor,\quad \zeta(n)=\left\lceil\frac{n+2}{4}\right\rceil,\quad \eta(n)=\left\lceil\frac{3n+2}{4}\right\rceil,\quad \theta(n)=\left\lfloor\frac{5n+2}{6}\right\rfloor.$$

Proposition 40. *For even $n\in\mathbb{Z}$, the number of solutions $P,Q,R,S\in\mathbb{Z}$ to*

$$n\geqslant P>Q>R>S\geqslant 0,\quad P+Q+R+S=\frac{5n+2}{2},$$

equals

$$\frac{23}{1152}n^3-\frac{21}{64}n^2+\frac{1}{288}(36\epsilon(n)-36\zeta(n)+180\eta(n)+27\Delta_{4,2}^n-254)n$$
$$+\frac{1}{48}(-12\epsilon(n)^2+12\zeta(n)^2-12\eta(n)^2-12\epsilon(n)-12\zeta(n)+36\eta(n)+16\theta(n)$$
$$+3\Delta_{4,2}^n-12\Delta_{8,6}^n-24).$$

Proof. Similar to the proof of Proposition 37. $\square$

Corollary 41. *For even $n\in\mathbb{Z}$, write $n=24q+r$ with $q,r\in\mathbb{Z}$ and $0\leqslant r<24$. The dimension of the weight space of weight $n+2$ in the $\mathfrak{sl}_2(\mathbb{C})$-module $\Lambda^4V(n)$ is*

$$
\dim[\Lambda^4 V(n)]_{n+2} = \frac{1}{1152}\begin{cases} 23n^3-72n^2-48n & (r=0) \\ 23n^3-72n^2+60n-16 & (r=2) \\ 23n^3-72n^2-48n-128 & (r=4) \\ 23n^3-72n^2+60n-432 & (r=6) \\ 23n^3-72n^2-48n+128 & (r=8) \\ 23n^3-72n^2+60n-272 & (r=10) \end{cases} \left| \begin{array}{ll} 23n^3-72n^2-48n & (r=12), \\ 23n^3-72n^2+60n-304 & (r=14), \\ 23n^3-72n^2-48n-128 & (r=16), \\ 23n^3-72n^2+60n-144 & (r=18), \\ 23n^3-72n^2-48n+128 & (r=20), \\ 23n^3-72n^2+60n-560 & (r=22). \end{array} \right.
$$

Proof. Similar to the proof of Corollary 38. □

Theorem 1 now follows by applying Lemma 3 to Corollaries 38 and 41.

Remark 42. We can use Corollary 32 to obtain another proof of the decompositions in Remark 5. For $n = 4, 6, 8, 10$ we compute the polynomial $P_n(x)$ from Definition 33. In each case, the coefficient of $x^{(w+4n)/2}$ is $\dim[\Lambda^4 V(n)]_w$, and we then apply Lemma 3 to find the multiplicity of $V(w)$ in $\Lambda^4 V(n)$:

$$
P_4(x) = x^{10} + x^9 + x^8 + x^7 + x^6,
$$

$$
\begin{aligned}
P_6(x) = {} & x^{18} + x^{17} + 2x^{16} + 3x^{15} + 4x^{14} + 4x^{13} + 5x^{12} + 4x^{11} + 4x^{10} + 3x^9 + 2x^8 \\
& + x^7 + x^6,
\end{aligned}
$$

$$
\begin{aligned}
P_8(x) = {} & x^{26} + x^{25} + 2x^{24} + 3x^{23} + 5x^{22} + 6x^{21} + 8x^{20} + 9x^{19} + 11x^{18} + 11x^{17} \\
& + 12x^{16} + 11x^{15} + 11x^{14} + 9x^{13} + 8x^{12} + 6x^{11} + 5x^{10} + 3x^9 + 2x^8 + x^7 + x^6,
\end{aligned}
$$

$$
\begin{aligned}
P_{10}(x) = {} & x^{34} + x^{33} + 2x^{32} + 3x^{31} + 5x^{30} + 6x^{29} + 9x^{28} + 11x^{27} + 14x^{26} + 16x^{25} \\
& + 19x^{24} + 20x^{23} + 23x^{22} + 23x^{21} + 24x^{20} + 23x^{19} + 23x^{18} + 20x^{17} \\
& + 19x^{16} + 16x^{15} + 14x^{14} + 11x^{13} + 9x^{12} + 6x^{11} + 5x^{10} + 3x^9 + 2x^8 + x^7 + x^6.
\end{aligned}
$$

10. Conclusion

We recovered a five-dimensional 4-Lie algebra from the isomorphism $\Lambda^4 V(4) \cong V(4)$. This algebra satisfies the quaternary derivation identity D, and hence also the quaternary alternating sum identity S. We found that the identity S is also satisfied by the unique structure on $V(6)$, every structure $f + xg$ on $V(8)$, and the structure $f + \frac{5}{4}g$ on $V(10)$. By Bremner [4] it is known that the quaternary alternating sum operation in a totally associative quadruple system also satisfies S. This raises the question whether the structures which satisfy S can be embedded into totally associative quadruple systems if the original associative operation is replaced by the quaternary alternating sum. An affirmative answer to this question would provide a partial generalization of the Poincaré–Birkhoff–Witt theorem for Lie algebras; see Pozhidaev [28] for related work. Simple associative n-tuple systems were classified by Carlsson [9]. In particular, for $n = 4$, any simple associative quadruple system is isomorphic to a subspace of matrices of the form

$$
\begin{bmatrix} 0 & 0 & Z \\ X & 0 & 0 \\ 0 & Y & 0 \end{bmatrix},
$$

where X, Y, Z have sizes $q \times p$, $r \times q$, $p \times r$ (respectively) and p, q, r are positive integers. It is an open problem to determine whether any of the alternating quaternary algebras presented in this paper are isomorphic to such a subspace of matrices under the quaternary alternating sum.

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References

[1] H. Ataguema, A. Makhlouf, S. Silvestrov, Generalization of n-ary Nambu algebras and beyond, J. Math. Phys. 50 (8) (2009) 083501–083515.
[2] J. Bagger, N. Lambert, Modeling multiple M2-branes, Phys. Rev. D 75 (4) (2007) 045020–045026.
[3] T.M. Baranovich, M.S. Burgin, Linear Ω-algebras, Uspehi Mat. Nauk 30 (4(184)) (1975) 61–106.
[4] M.R. Bremner, Varieties of anticommutative n-ary algebras, J. Algebra 191 (1) (1997) 76–88.
[5] M.R. Bremner, Identities for the ternary commutator, J. Algebra 206 (2) (1998) 615–623.
[6] M.R. Bremner, I.R. Hentzel, Identities for generalized Lie and Jordan products on totally associative triple systems, J. Algebra 231 (1) (2000) 387–405.
[7] M.R. Bremner, I.R. Hentzel, Invariant nonassociative algebra structures on irreducible representations of simple Lie algebras, Experiment. Math. 13 (2) (2004) 231–256.
[8] M.R. Bremner, I.R. Hentzel, Alternating triple systems with simple Lie algebras of derivations, Nonassociative Algebra and its Applications, Lect. Notes Pure Appl. Math., vol. 246, Chapman & Hall/CRC, 2006, pp. 55–82.
[9] R. Carlsson, n-Ary algebras, Nagoya Math. J. 78 (1980) 45–56.
[10] T.L. Curtright, X. Jin, L. Mezincescu, Multi-operator brackets acting thrice, J. Phys. A 42 (46) (2009) 462001–462006.
[11] T.L. Curtright, C.L. Zachos, Classical and quantum Nambu mechanics, Phys. Rev. D 68 (2003) 085001.
[12] J.A. de Azcárraga, J.M. Izquierdo, n-Ary algebras: a review with applications, Invited review for J. Phys. A (Math. Theor.), 2010, 118pp. Available from <math-ph/arXiv:1005.1028>.
[13] J.A. de Azcárraga, J.C. Pérez-Bueno, Higher-order simple Lie algebras, Comm. Math. Phys. 184 (3) (1997) 669–681.
[14] V.T. Filippov, n-Lie algebras, Sibirsk. Mat. Zh. 26 (6) (1985) 126–140.
[15] V.T. Filippov, On the n-Lie algebra of Jacobians, Sibirsk. Mat. Zh. 39 (3) (1998) 660–669.
[16] W. Fulton, J. Harris, Representation Theory. A First Course, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.
[17] P. Gautheron, Simple facts concerning Nambu algebras, Comm. Math. Phys. 195 (2) (1998) 417–434.
[18] A.V. Gnedbaye, Opérades des algèbres $(k+1)$-aires, Operads: Proceedings of Renaissance Conferences, Contemp. Math., vol. 202, Amer. Math. Soc., Providence, 1997, pp. 83–113.
[19] A. Gustavsson, One-loop corrections to Bagger–Lambert theory, Nuclear Phys. B 807 (1–2) (2009) 315–333.
[20] P. Hanlon, M. Wachs, On Lie k-algebras, Adv. Math. 113 (2) (1995) 206–236.
[21] F. Harary, E.M. Palmer, Graphical Enumeration, Academic Press, New York, 1973.
[22] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics, vol. 9, 1972, Springer-Verlag, New York.
[23] S.M. Kasymov, On the theory of n-Lie algebras, Algebra i Logika 26 (3) (1987) 277–297.
[24] A.G. Kurosh, Multiple operator rings and algebras, Uspehi Mat. Nauk 24 (1(145)) (1969) 3–15.
[25] W.X. Ling, On the Structure of n-Lie Algebras, Ph.D. Thesis, University of Siegen, 1993.
[26] P.W. Michor, A.M. Vinogradov, n-Ary Lie and associative algebras. Geometrical Structures for Physical Theories, II (Vietri, 1996), Rend. Sem. Mat. Univ. Politec. Torino 54 (4) (2010) 373–392.
[27] Y. Nambu, Generalized Hamiltonian dynamics, Phys. Rev. D 7 (3) (1973) 2405–2412.
[28] A. Pozhidaev, Enveloping algebras of Filippov algebras, Comm. Algebra 31 (2) (2003) 883–900.
[29] N.J.A. Sloane, On-Line Encyclopedia of Integer Sequences. Available from <www.research.att.com/ njas/sequences>.
[30] L. Takhtajan, On foundation of the generalized Nambu mechanics, Comm. Math. Phys. 160 (2) (1994) 295–315.
[31] I. Vaisman, A survey on Nambu-Poisson brackets, Acta Math. Univ. Comenian. (N.S.) 68 (2) (1999) 213–241.
[32] X. Wu, C. Chao, An application of Pólya's enumeration theorem to partitions of subsets of positive integers, Czechoslovak Math. J. 55(130) (3) (2005) 611–623.