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Numerical treatment of a mathematical model arising from a model of neuronal variability

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Abstract

In this paper, we describe a numerical approach based on finite difference method to solve a mathematical model arising from a model of neuronal variability. The mathematical modelling of the determination of the expected time for generation of action potentials in nerve cells by random synaptic inputs in dendrites includes a general boundary-value problem for singularly perturbed differential–difference equation with small shifts. In the numerical treatment for such type of boundary-value problems, first we use Taylor approximation to tackle the terms containing small shifts which converts it to a boundary-value problem for singularly perturbed differential equation. A rigorous analysis is carried out to obtain priori estimates on the solution of the problem and its derivatives up to third order. Then a parameter uniform difference scheme is constructed to solve the boundary-value problem so obtained. A parameter uniform error estimate for the numerical scheme so constructed is established. Though the convergence of the difference scheme is almost linear but its beauty is that it converges independently of the singular perturbation parameter, i.e., the numerical scheme converges for each value of the singular perturbation parameter (however small it may be but remains positive). Several test examples are solved to demonstrate the efficiency of the numerical scheme presented in the paper and to show the effect of the small shift on the solution behavior.
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Keywords: Singular perturbation; Action potential; Fitted mesh; Differential–difference equation; Positive shift; Negative shift; Boundary layer

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1. Introduction

A human brain consists of approximately 10^{11} computing elements called neurons. A typical neuron has three major regions: soma, axon and the dendrites. Dendrites form a dendritic tree, which is very fine bush of thin fibers around the neuron's body. Dendrites receive information from neurons through axons long fibers that serve as transmission lines. An axon is a long cylindrical connection that carries impulses from the neuron. The end part of an axon splits into a fine arborization. Each branch of it terminates in a small end-bulb almost touching the dendrites of neighboring neurons. The axon-dendrite contact organ is called a synapse. The synapse is where the neuron introduces its signal to the neighboring neuron. The neurons communicate through a connection network of axons and synapses having a density of approximately 10^4 synapses per neuron. The hypothesis regarding the modeling of the natural nervous system is that neurons communicate with each other by means of electrical impulses. The neurons operate in chemical environment that is even more important in terms of actual brain behavior. The input to the network is provided by sensory receptors. Receptors deliver stimuli both from within the body, as well as from sense organs when the stimuli originate in the external world. The stimuli are in the form of electrical impulses that convey the information into the network of neurons. As a result of information processing in the central nervous systems, the effectors are controlled and give human responses in the form of diverse actions. We thus have a three stage system, consisting of receptors, neural network, and effectors.

On the theoretical side there have been many advanced model of nerve membrane potential in the presence of random synaptic input. Reviews can be found in J.P. Segundo et al. [4], S.E. Fienberg [5], Holden [6]. Due to the analytic difficulties in solving any realistic model, computer simulation has played an important role as a first step. Stein have given a differential–difference equation model incorporating stochastic effects due to neuronal variability and approximate the solution using Monte Carlo techniques [1]. Stein's model contains the following assumptions:

(i) Excitatory impulses arrive according to a Poisson process $\pi(f_e, t)$, each event of which leads to an instantaneous increase in the membrane depolarization $V(t)$ by a_e, whereas inhibitory current impulses arrive at event times in a second Poisson process $\pi(f_i, t)$, which is independent of $\pi(f_e, t)$ and causes $V(t)$ to decreases by a_i.
(ii) If depolarization reaches a threshold of r units, the neuron fires an impulse.
(iii) After each neuronal firing there is a refractory period of duration, t_0, during which the impulses have no effect and the membrane depolarization, $V(t)$, is reset to zero.
(iv) At times $t > t_0$, each impulse produces unit depolarization.
(v) For sub-threshold levels, the depolarization decays exponentially among impulses with time constant μ.

In 1967, Stein generalized this model to deal with a distribution of postsynaptic potential amplitudes [7]. Johannesma [8] and Tuckwell [9] included the reversal potentials into account. Various other models for neuronal activity have been proposed and many are discussed in Holden's book [6].

The depolarization in Stein's model is a continuous time, continuous state space Markov process whose sample paths have discontinuities of the first kind. The time for generation of action potential in a nerve cell is the time of first passage to level at or above a threshold of r units and determinations of this random variable have proven to be difficult due to the fact that the equations involved are differential–difference equations. For this reason, presumably, diffusion models, in which the discontinuities of $V(t)$ are smoothed out, have been considered as approximations to Stein's model, the initial efforts in this direction being those of Gluss [10], Johannesma [8] and Roy and Smith [11]. In 1980, Tuckwell and Cope [12] studied the Stein's model and compared with diffusion model. Lange and Miura [2] presented mathematical model of the determination of expected time for generation of action potentials in nerve cell by random synaptic inputs in the dendrites. In Stein's model, the distribution representing inputs is taken as a Poisson process with exponential decay. If in addition, there are inputs that can be modeled as a Wiener process with variance parameter σ and drift parameter μ, then the problem for expected first-exit time y, given initial membrane potential $x \in (x_1, x_2)$, can be formulated as a general boundary-value problem for linear second order differential–difference equation (DDE)

$$\frac{\sigma^2}{2} y''(x) + (\mu - x)y'(x) + \lambda_e y(x + a_e) + \lambda_i y(x - a_i) - (\lambda_e + \lambda_i)y(x) = -1, \tag{1.1}$$

where the values $x = x_1$ and $x = x_2$ correspond to the inhibitory reversal potential and to the threshold value of membrane potential for action potential generation, respectively. Here σ and μ are variance and drift parameters, respectively, y is the expected first-exit time and the first order derivative term $-xy'(x)$ corresponds to exponential decay between synaptic inputs. The undifferentiated terms correspond to excitatory and inhibitory synaptic inputs, modeled as Poisson process with mean rates λ_e and λ_i, respectively, and produce jumps in the membrane potential of amounts a_e and a_i, respectively, which are small quantities and could be dependent on voltage. The boundary condition is

$$y(x) \equiv 0, \quad x \notin (x_1, x_2).$$

Pertaining to the above biological phenomena, we present a numerical study of boundary-value problems for singularly perturbed differential–difference equations. Earlier Lange and Miura [2,3,13–16] have given an asymptotic approach to study such types of boundary-value problems.

Now we state a model problem for a general boundary-value problem for a singularly perturbed differential–difference equation containing both type of shifts (negative as well as positive shifts) and let $\Omega = (0, 1)$,

$$\varepsilon y_\varepsilon''(x) + a(x)y_\varepsilon'(x) + \alpha(x)y_\varepsilon(x - \delta) + \omega(x)y_\varepsilon(x) + \beta(x)y_\varepsilon(x + \eta) = f(x), \tag{1.2}$$

$\forall x \in \Omega$ and subject to interval conditions

$$\begin{aligned} &y_\varepsilon(x) = \phi(x) \quad \text{on } -\delta \leqslant x \leqslant 0, \\ &y_\varepsilon(x) = \gamma(x) \quad \text{on } 1 \leqslant x \leqslant 1 + \eta, \end{aligned} \tag{1.3}$$

where ε is small parameter $0 < \varepsilon \ll 1$, δ and η are of $o(\varepsilon)$; $a(x)$, $\alpha(x)$, $\beta(x)$, $\omega(x)$, $f(x)$, $\phi(x)$ and $\gamma(x)$ all are smooth functions. When the shifts are zero (i.e., $\delta = 0$, $\eta = 0$), the solution of the corresponding ordinary differential equation exhibits layer behavior or turning point behavior depending on the coefficient of the convection term, i.e., if $a(x)$ does not change the sign or changes the sign on Ω. Here, we consider the problems whose solution exhibits the layer behavior. The layer will be on the left or the right end of the domain depending on the sign of the coefficient of convection term, i.e., according to $a(x) < 0$ or $a(x) > 0$ on the $\bar{\Omega} = [0, 1]$, respectively. The layer is maintained for $\delta \neq 0$, $\eta \neq 0$ but sufficiently small.

The singularly perturbed boundary-value problems cannot be solved numerically in a satisfactory manner by standard finite difference methods on uniform mesh. This encourages the need for the methods that behave uniformly well, i.e., which converges independent of the singular perturbation parameter. Such methods are referred as ε-uniform or parameter uniform methods, where ε is the singular perturbation parameter. In the construction of an ε-uniform method, there are mainly two approaches. The first are the fitted operator methods which comprise of specially designed finite difference operator which reflects the singularly perturbed nature of the solution. Such fitted operator methods were first suggested by de G. Allen and Southwell [17] for solving the problem of viscous fluid flow past a cylinder. An extensive account of ε-uniform fitted operator methods is discussed in Doolan et al. [18]. The second are the fitted mesh methods which comprise of standard finite difference operators on fitted piecewise-uniform meshes condensing in the boundary layers [19].

The fitted mesh methods have probably received less detailed attention in the literature, than the construction of an appropriate finite difference fitted operator or finite element subspace methods. In 1996 [19], Miller et al. established the great importance of fitted mesh methods for solving singular perturbation problems. There are some problems for which no ε-uniform method can be constructed using a fitted operator approach on a uniform mesh while for such problem an ε-uniform fitted mesh method can be constructed (see [20, Problem 3.6]).

In this paper, an ε-uniform numerical scheme is constructed for a class of boundary-value problems for singularly perturbed differential–difference equations with small shifts. The numerical method comprises a standard upwind finite difference operator on a fitted piecewise-uniform mesh which is condensed in the boundary layers. We first approximate the terms containing small shift by Taylor series and then apply the fitted mesh method, provided shifts are of $o(\varepsilon)$. Finally, we carry out some numerical experiments to demonstrate the accuracy of our scheme and to examine the effect of the small shifts on solution.

Through out this paper, C denotes generic positive constant that is independent of ε and in the case of discrete problems, also independent of the mesh parameter N which may assume different values but remains to be constant. $\|.\|$ denotes the global maximum norm over the appropriate domain of the independent variable, i.e.,

$$\|f\| = \max_{x \in \bar{\Omega}} \left| f(x) \right|.$$

2. Numerical treatment

In this section, we consider the numerical treatment for the model problem (1.2), (1.3). The first step in this direction is the use of Taylor approximations for the terms containing the small shifts in the problem (1.2), (1.3) which converts it to the following boundary-value problem for singularly perturbed differential equation:

$$\varepsilon z_\varepsilon''(x) + \big(a(x) + \beta(x)\eta - \alpha(x)\delta\big)z_\varepsilon'(x) + \big(\alpha(x) + \beta(x) + \omega(x)\big)z_\varepsilon(x) = f(x), \tag{2.1}$$

$$z_\varepsilon(0) = \phi_0, \quad \phi_0 = \phi(0), \tag{2.2a}$$

$$z_\varepsilon(1) = \gamma_1, \quad \gamma_1 = \gamma(1), \tag{2.2b}$$

which differ from the original problem (1.2), (1.3) by terms of $O(\delta^2 z'', \eta^2 z'')$. Here, we assume shifts, i.e., δ and η are sufficiently small, so the solution z_ε of the problem (2.1), (2.2) will provide a good approximation to the solution y_ε of the problem (1.2), (1.3). The differential operator L_ε corresponding to the boundary-value problem (2.1), (2.2) is defined by

$$L_\varepsilon z_\varepsilon(x) = \varepsilon z_\varepsilon''(x) + \big(a(x) + \beta(x)\eta - \alpha(x)\delta\big)z_\varepsilon'(x) + \big(\alpha(x) + \beta(x) + \omega(x)\big)z_\varepsilon(x).$$

2.1. Left side boundary layer

In this section, we consider the case when the solution of the model problem (2.1), (2.2) exhibit layer behavior on the left side on the interval $[0, 1]$, i.e., it is assumed that $(a(x) + \beta(x)\eta - \alpha(x)\delta) \geqslant M > 0$ throughout the interval $[0, 1]$, where M is a positive constant.

2.1.1. Analytical results

Continuous minimum principle. *Let ψ be a smooth function satisfying $\psi(0) \geqslant 0$, $\psi(1) \geqslant 0$ and $L_\varepsilon\psi(x) \leqslant 0$ $\forall x \in \Omega$. Then $\psi(x) \geqslant 0$ $\forall x \in \bar{\Omega}$.*

Proof. Suppose $x^* \in \bar{\Omega}$ be such that $\psi(x^*) = \min_{x\in\bar{\Omega}} \psi(x)$ and assume that $\psi(x^*) < 0$. Since $\psi(0) \geqslant 0$ and $\psi(1) \geqslant 0$, therefore x^* cannot be 0 or 1. Thus $\psi'(x^*) = 0$, $\psi''(x^*) \geqslant 0$ and clearly $L_\varepsilon\psi(x^*) > 0$, which contradicts the hypothesis. Therefore $\psi(x^*) \geqslant 0$ and since $x^* \in \bar{\Omega}$ is chosen arbitrarily, thus $\psi(x) \geqslant 0$ $\forall x \in \bar{\Omega}$. □

Lemma 1. *The solution $z_\varepsilon(x)$ of the boundary-value problem* (2.1), (2.2) *is bounded and satisfies the following estimate*:

$$\|z_\varepsilon\| \leqslant \frac{1}{\theta}\|f\| + \max\big(|\phi_0|, |\gamma_1|\big).$$

Proof. Suppose ψ^+ and ψ^- be the two barrier functions defined by

$$\psi^\pm(x) = \frac{1}{\theta}\|f\| + \max\big(|\phi_0|, |\gamma_1|\big) \pm z_\varepsilon(x), \qquad x \in \bar{\Omega}.$$

Then $\psi^\pm(0) \geqslant 0$, $\psi^\pm(1) \geqslant 0$ and

$$\begin{aligned} L_\varepsilon\psi^\pm(x) = {} & \big(\alpha(x) + \beta(x) + \omega(x)\big)\theta^{-1}\|f\| \\ & + \big(\alpha(x) + \beta(x) + \omega(x)\big)\max\big(|\phi_0|, |\gamma_1|\big) \pm L_\varepsilon z_\varepsilon(x) \end{aligned}$$

using the definition of the continuous operator L_ε and the inequality $(\alpha(x)+\beta(x)+\omega(x))\theta^{-1} \leqslant -1$, we get $L_\varepsilon\psi(x) \leqslant 0$ for all $x \in \Omega$. Therefore the continuous minimum principle implies that

$$\psi^{\pm}(x) = \frac{1}{\theta}\|f\| + \max\bigl(|\phi_0|, |\gamma_1|\bigr) \pm z_\varepsilon(x) \geqslant 0, \quad x \in \bar{\Omega},$$

which on simplification gives the required estimate. □

Theorem 1. *The derivatives of the solution z_ε of the boundary-value problem* (2.1), (2.2) *satisfy the following estimates for $k = 1, 2, 3$,*

$$\bigl\|z_\varepsilon^{(k)}\bigr\| \leqslant C\varepsilon^{-k}.$$

Proof. For $x \in \Omega$ and construct a neighborhood $N_x = (c, c+\varepsilon)$, where c is a positive constant chosen so that $x \in N_x$ and $N_x \subset \Omega$. Then by the mean value theorem, for some $u \in N_x$, we have

$$z'_\varepsilon(u) = \frac{z_\varepsilon(c+\varepsilon) - z_\varepsilon(c)}{\varepsilon}$$

and so

$$\bigl|\varepsilon z'_\varepsilon(u)\bigr| \leqslant 2\|z_\varepsilon\|. \tag{2.3}$$

We have

$$\int_z^x z''_\varepsilon(t)\,dt = z'_\varepsilon(x) - z'_\varepsilon(u),$$

i.e.,

$$z'_\varepsilon(x) = z'_\varepsilon(u) + \int_z^x z''_\varepsilon(t)\,dt$$

using Eq. (2.1) in the above equation, we obtain

$$\varepsilon z'_\varepsilon(x) = \varepsilon z'_\varepsilon(u) + \int_u^x f(t)\,dt - \int_u^x \bigl(a(t) + \beta(t)\eta - \alpha(t)\delta\bigr) z'_\varepsilon(t)\,dt$$
$$- \int_u^x \bigl(\alpha(t) + \beta(t) + \omega(t)\bigr) z_\varepsilon(t)\,dt.$$

Taking modulus on both the sides and using the fact that the maximum norm of a function is always greater than the value of the function over the domain of consideration, we get

$$\bigl|\varepsilon z'_\varepsilon(x)\bigr| \leqslant \bigl|\varepsilon z'_\varepsilon(u)\bigr| + \|f\||x-u| + \left|\int_u^x \bigl(a + \beta(t)\eta - \alpha(t)\delta\bigr) z'_\varepsilon(t)\,dt\right|$$
$$+ \|\alpha + \beta + \omega\|\|z_\varepsilon\||x-u|. \tag{2.4}$$

By integration by parts, we have

$$\int_u^x \big(a(t)+\beta(t)\eta-\alpha(t)\delta\big)z_\varepsilon'(t)\,dt$$
$$= \big(a(t)+\beta(t)\eta-\alpha(t)\delta\big)z_\varepsilon(t)\big|_u^x - \int_u^x \big(a'(t)+\beta'(t)\eta-\alpha'(t)\delta\big)z_\varepsilon(t)\,dt.$$

Taking modulus on both the sides and using the fact that the maximum norm of a function is always greater than the value of the function over the domain of consideration, we get

$$\left|\int_u^x \big(a(t)+\beta(t)\eta-\alpha(t)\delta\big)z_\varepsilon'(t)\,dt\right|$$
$$\leqslant \big(2\|a+\beta\eta-\alpha\delta\| + \|a'+\beta'\eta-\alpha'\delta\||x-u|\big)\|z_\varepsilon\|. \tag{2.5}$$

Using inequalities (2.3) and (2.5) in inequality (2.4), we get

$$\begin{aligned}\big|\varepsilon z_\varepsilon'(x)\big| \leqslant{}& 2\|z_\varepsilon\| + \|f\||x-u| \\ &+ \big(2\|a+\beta\eta-\alpha\delta\| + \|a'+\beta'\eta-\alpha'\delta\||x-u|\big)\|z_\varepsilon\| \\ &+ \|\alpha+\beta+w\|\|z_\varepsilon\||x-u|.\end{aligned} \tag{2.6}$$

Using Lemma 1 for the bound on z_ε and the inequality $0 < |x-u| \leqslant 1$ in the above inequality (2.6), we get

$$\big|z_\varepsilon'(x)\big| \leqslant C\varepsilon^{-1}$$

which gives $\|z_\varepsilon'\| \leqslant C\varepsilon^{-1}$, where

$$\begin{aligned}C = \|f\| + \big(&2 + 2\|a+\beta\eta-\alpha\delta\| + \|a'+\beta'\eta-\alpha'\delta\| \\ &+ \|\alpha+\beta+w\|\big)\big(\theta^{-1}\|f\| + \max\big(|\phi_0|, |\gamma_1|\big)\big).\end{aligned}$$

Similarly the bounds for z_ε'' and z_ε''' can be obtained by using the differential equation and the bounds on z_ε and z_ε'. □

These bounds for the derivatives of z_ε were first obtained by Miller et al. [19], using techniques based on Kellogg et al. [21]. However in order to prove that the numerical method is ε-uniform, one needs more precise information on the behavior of the exact solution of the boundary-value problem (2.1), (2.2). This is obtained by decomposing the solution z_ε into a smooth component v_ε and a singular component w_ε as follows:

$$y = v_\varepsilon + w_\varepsilon,$$

where v_ε can be written in the form $v_\varepsilon(x) = v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x)$ and $v_0(x)$, $v_1(x)$ and $v_2(x)$ are defined to be the solutions of the problems

$$\big(a(x)+\beta(x)\eta-\alpha(x)\delta\big)v_0'(x)+\big(\alpha(x)+\beta(x)+\omega(x)\big)v_0(x)=f(x),\quad x\in\Omega,$$
$$v_0(1)=z_\varepsilon(1), \tag{2.7a}$$
$$\big(a(x)+\beta(x)\eta-\alpha(x)\delta\big)v_1'(x)+\big(\alpha(x)+\beta(x)+\omega(x)\big)v_1(x)=-v_0''(x),\quad x\in\Omega,$$
$$v_1(1)=0, \tag{2.7b}$$
$$L_\varepsilon v_2(x)=-v_1''(x),\quad x\in\Omega,\qquad v_2(0)=0,\quad v_2(1)=0. \tag{2.7c}$$

Thus the smooth component $v_\varepsilon(x)$ is the solution of

$$L_\varepsilon v_\varepsilon(x)=f(x),\quad x\in\Omega,\qquad v_\varepsilon(0)=v_0(0)+\varepsilon v_1(0),\quad v_\varepsilon(1)=z_\varepsilon(1) \tag{2.8}$$

and consequently the singular component $w_\varepsilon(x)$ is the solution of the homogeneous problem

$$L_\varepsilon w_\varepsilon(x)=0,\quad x\in\Omega,\qquad w_\varepsilon(0)=z_\varepsilon(0)-v_\varepsilon(0),\quad w_\varepsilon(1)=0. \tag{2.9}$$

Theorem 2. *Let z_ε be the solution of boundary-value problem* (2.1), (2.2) *and let $z_\varepsilon = v_\varepsilon + w_\varepsilon$. Then for sufficiently small ε; v_ε, w_ε and their derivatives satisfy the following bounds*, $0\leqslant k\leqslant 3$:

$$\big\|v_\varepsilon^{(k)}\big\|_{\bar\Omega}\leqslant C\varepsilon^{2-k}, \tag{2.10}$$
$$\big|w_\varepsilon^{(k)}(x)\big|\leqslant C\varepsilon^{-k}\exp(-Mx/\varepsilon),\quad x\in\bar\Omega. \tag{2.11}$$

Proof. The problem (2.7a) is a first order linear differential equation in v_0; therefore it has a unique solution given by

$$v_0(x)=\frac{1}{B(x)}\Bigg[\gamma A-\exp\bigg(\int p(t)\,dt\bigg)\times\int_x^1\frac{f(t)}{(a(t)+\beta(t)\eta-\alpha(t)\delta)}\exp\bigg(\int p(t)\,dt\bigg)\,dt\Bigg],$$

where $A=\exp(\int p(t)\,dt)|_{t=1}$, $B(x)=\exp(\int p(t)\,dt)|_{t=x}$ and $p(t)=(\alpha(t)+\beta(t)+\omega(t))/(a(t)+\beta(t)\eta-\alpha(t)\delta)<0$. Now since $\alpha(x)$, $\beta(x)$, $\omega(x)$ and $f(x)$ are bounded for all $x\in[0,1]$, therefore v_0 is bounded. Again from Eq. (2.7a), we have

$$v_0'(x)=f(x)/\big(a(x)+\beta(x)\eta-\alpha(x)\delta\big)-p(x)v_0(x)$$

and the boundedness of v_0 implies that v_0' is bounded. Using the boundedness of v_0 and v_0' and differentiating the differential equation (2.7a) successively, we obtain the bounds on v_0'' and v_0'''. Thus for $0\leqslant k\leqslant 3$, we have

$$\big|v_0^{(k)}(x)\big|\leqslant C\quad\text{for all } x\in\bar\Omega. \tag{2.12}$$

The problem (2.7b) is also a first order linear differential equation in v_1; therefore it has a unique solution

$$v_1(x)=-\int_x^1\frac{v_0''(t)}{(a(t)+\beta(t)\eta-\alpha(t)\delta)}\exp\bigg(\int p(t)\,dt\bigg)\,dt/B(x). \tag{2.13}$$

Since $v_0''(x)$, $(a(x)+\beta(x)\eta-\alpha(x)\delta)$, $(\alpha(x)+\beta(x)+\omega(x))$ and $p(x)<0$ are bounded for all $x\in[0,1]$. Thus by similar arguments as we have used to obtain the bounds on v_1 and its derivatives, we obtain the bounds on v_1 and its derivatives up to the third order. v_1'' is bounded by a constant independent of ε, so the right side of Eq. (2.7c) is bounded independently of ε. Thus v_2 is the solution of a boundary-value problem similar to (2.1), (2.2). Hence by Theorem 1, we have for $0\leqslant k\leqslant 3$,

$$\|v_2^{(k)}\|\leqslant C\varepsilon^{-k}. \tag{2.14}$$

Since v_0 and v_1 are independent of ε, therefore inequality (2.14) gives the required estimate for the regular component v_ε. Now to obtain the required bounds on the singular component w_ε and its derivatives, we use the two barrier functions Ψ^+ and Ψ^- defined by

$$\Psi^{\pm}(x)=\big(\big|y(0)\big|+|v_0|\big)\exp(-xM/\varepsilon)\pm w_\varepsilon(x).$$

Then we have $\Psi^{\pm}(0)\geqslant 0$, $\Psi^{\pm}(1)\geqslant 0$ and $L_\varepsilon\Psi^{\pm}(x)\leqslant 0$ $\forall x\in(0,1)$, therefore by the minimum principle, we get $\Psi^{\pm}(x)\geqslant 0$, $x\in\bar{\Omega}$, which gives

$$\big|w_\varepsilon(x)\big|\leqslant C\exp(-xM/\varepsilon),\quad x\in\bar{\Omega}, \tag{2.15}$$

where $C=(|y(0)|+|v_0(0)|)$. To find the required bounds on the derivatives of the singular part w_ε of the solution z_ε, we go along the same lines as we did in the proof of Theorem 1. For $x\in\Omega$, construct a neighborhood $N_x=(x,x+\varepsilon)$. Therefore by the mean value theorem, there exists a point $u\in N_x$ such that

$$w_\varepsilon'(u)=\big(w_\varepsilon(x+\varepsilon)-w_\varepsilon(x)\big)/\varepsilon,$$

which implies that

$$\big|\varepsilon w_\varepsilon'(u)\big|\leqslant 2\|w_\varepsilon\|_{N_x}. \tag{2.16}$$

Now we have

$$w_\varepsilon'(x)=w_\varepsilon'(u)+\int_u^x w_\varepsilon''(t)\,dt.$$

Using Eq. (2.9) for $w_\varepsilon''(t)$ in the above equation and adopting the similar steps as we did in establishing the bound for z_ε', we get

$$\big|w_\varepsilon'(x)\big|\leqslant C\varepsilon^{-1}\|w_\varepsilon\|_{N_x}. \tag{2.17}$$

We have

$$\begin{aligned}\|w_\varepsilon\|_{N_x}&=\sup_{x\in N_x}\big|w_\varepsilon(x)\big|\\&\leqslant C\exp(-Mx/\varepsilon),\quad \text{from inequality (2.15).}\end{aligned}$$

Using this value of w_ε in inequality (2.17), we obtain

$$\big|w_\varepsilon'(x)\big|\leqslant C\varepsilon^{-1}\exp(-Mx/\varepsilon),$$

which gives the required result. The estimate for w_ε'' can be easily obtained from the differential equation and the bounds on w_ε and w_ε'. $\square$

2.1.2. Discretization

We discretize the boundary-value problem (2.1), (2.2) using fitted mesh method composed with a finite difference operator on a piecewise uniform mesh which is condensed in the left side boundary layer region. The fitted piecewise-uniform mesh $\bar{\Omega}_\varepsilon^N$ on the interval $[0, 1]$ is constructed by partitioning the interval into two subintervals $(0, \tau)$ and $(\tau, 1)$, on each of these subintervals, a uniform mesh is placed, i.e., the each subinterval is divided into $N/2$ equal parts. The resulting piecewise uniform mesh depends on one parameter (τ) which is called the transition parameter and chosen such that $\tau \equiv \min[1/2, C\varepsilon \ln N]$ with $C > 1/|\theta|$. Assuming that $N = 2^r$ with $r \geqslant 2$, which guarantees that there is at least one point in the boundary layer. The fitted finite difference method for the boundary-value problem (2.1), (2.2), on the piecewise uniform mesh $\bar{\Omega}_\varepsilon^N$ is defined by

$$L_{\varepsilon,l}^N z_\varepsilon^N(x_i) = f(x_i), \quad x_i \in \Omega_\varepsilon^N, \tag{2.18}$$

$$z_\varepsilon^N(x_0) = \phi_0, \tag{2.19a}$$

$$z_\varepsilon^N(x_N) = \gamma_1, \tag{2.19b}$$

where for any mesh function Ψ_i, the discrete operator $L_{\varepsilon,l}^N$ is defined as

$$L_{\varepsilon,l}^N \Psi_i = \varepsilon D^{\pm}\Psi_i + \big(a(x_i) + \beta(x_i)\eta - \alpha(x_i)\delta\big) D^+\Psi_i + \big(\alpha(x_i) + \beta(x_i) + \omega(x_i)\big)\Psi_i,$$

with

$$D^-\Psi_i = (\Psi_i - \Psi_{i-1})/h_i, \qquad D^+ = (\Psi_{i+1} - \Psi_i)/h_{i+1},$$

$$D^{\pm}\Psi_i = \frac{2(D^+\Psi_i - D^-\Psi_i)}{(h_i + h_{i+1})}.$$

Discrete minimum principle. *Suppose $\Psi_0 \geqslant 0$ and $\Psi_N \geqslant 0$. The $L_{\varepsilon,l}^N \Psi_i \leqslant 0$, $i = 1(1)N-1$ implies that $\Psi_i \geqslant 0$, $i = 1(1)N-1$.*

Proof. Let k be such that $\Psi_k = \min_{0 \leqslant i \leqslant N} \Psi_i$ and assume $\Psi_k < 0$. Then we have $\Psi_k - \Psi_{k-1} \leqslant 0$, $\Psi_{k+1} - \Psi_k \geqslant 0$ upon using these inequalities along with definition of discrete operator $L_{\varepsilon,l}^N$, one can easily obtain $L_{\varepsilon,l}^N \Psi_k > 0$ which is a contradiction, therefore our assumption that $\Psi_k < 0$ is wrong, hence $\Psi_k \geqslant 0$, while k is chosen to be fixed but arbitrary, so $\Psi_i \geqslant 0$ for all i, $0 \leqslant i \leqslant N$. □

Lemma 2. *Let U_i be any mesh function such that $U_0 = U_N = 0$. Then for all i, $0 \leqslant i \leqslant N$,*

$$|U_i| \leqslant \frac{1}{|\theta|} \max_{1 \leqslant j \leqslant N-1} \big|L_{\varepsilon,l}^N U_j\big|.$$

Proof. Put $A = \frac{1}{|\theta|} \max_{1 \leqslant j \leqslant N-1} |L_{\varepsilon,l}^N U_j|$ and introduce two barrier functions Ψ_i^+, Ψ_i^- defined by

$$\Psi_i^{\pm} U_i = A \pm U_i.$$

Then we have $\Psi_0^{\pm} \geqslant 0$, $\Psi_N^{\pm} \geqslant 0$ and upon using the definition of the discrete operator $L_{\varepsilon,l}^N$ and the inequality $(\alpha(x_i) + \beta(x_i) + \omega(x_i))/\theta \leqslant -1$ yields $L_{\varepsilon,l}^N \leqslant 0$ for $1 \leqslant i \leqslant N-1$.

Thus an application of discrete minimum principle gives $\Psi_i^{\pm} \geqslant 0$, $\forall i$, $0 \leqslant i \leqslant N$, which proves the required estimate. □

Theorem 3. *The solution $Z^N = \langle (z_\varepsilon)_i \rangle_{i=0}^N$ of the discrete boundary-value problem* (2.18), (2.19) *and the solution $z_\varepsilon(x)$ of the continuous boundary-value problem* (2.1), (2.2) *satisfies the following ε-uniform error estimate*:

$$\sup_{0<\varepsilon\leqslant 1} \|Z^N - z_\varepsilon\| \leqslant CN^{-1}\ln N,$$

where C is a constant independent of ε.

Proof. As in the case of the continuous problem (2.1), (2.2), the solution Y^N of the discrete boundary-value problem (2.18), (2.19) can be decomposed into regular and singular components. Thus

$$Z^N = V_\varepsilon^N + W_\varepsilon^N,$$

where V_ε^N is the solution of the inhomogeneous problem

$$\begin{aligned} &L_{\varepsilon,l}^N V_\varepsilon^N(x_i) = f(x_i) \quad \text{for all } x_i \in \Omega^N, \\ &V_\varepsilon^N(0) = v_\varepsilon(0), \quad V_\varepsilon^N(1) = v_\varepsilon(1), \end{aligned} \tag{2.20}$$

and W_ε^N is the solution of the homogeneous problem

$$\begin{aligned} &L_{\varepsilon,l}^N W_\varepsilon^N(x_i) = 0 \quad \text{for all } x_i \in \Omega^N, \\ &W_\varepsilon^N(0) = w_\varepsilon(0), \quad W_\varepsilon^N(1) = w_\varepsilon(1). \end{aligned} \tag{2.21}$$

The error can be written in the form

$$Z^N - Z_\varepsilon = \left(V_\varepsilon^N - v_\varepsilon\right) + \left(W_\varepsilon^N - w_\varepsilon\right). \tag{2.22}$$

Thus the errors in the regular and singular components of the solution can be estimated separately. To estimate the error for the regular component, from the differential and difference equations, we have

$$\begin{aligned} L_{\varepsilon,l}^N\left(V_\varepsilon^N - v_\varepsilon\right)(x_i) &= f(x_i) - L_{\varepsilon,l}^N v_\varepsilon(x_i) = \left(L_{\varepsilon,l} - L_{\varepsilon,l}^N\right) v_\varepsilon(x_i) \\ &= \varepsilon\left(\frac{d^2}{dx^2} - D^{\pm}\right) v_\varepsilon(x_i) \\ &\quad + \left(a(x_i) + \beta(x_i)\eta - \alpha(x_i)\delta\right)\left(\frac{d}{dx} - D^{+}\right) v_\varepsilon(x_i). \end{aligned} \tag{2.23}$$

Let $x_i \in \Omega^N$. Then for any $\psi \in C^2(\bar{\Omega})$, we have

$$\left|\left(D^{+} - \frac{d}{dx}\right)\psi(x_i)\right| \leqslant (x_{i+1} - x_i)\|\psi^{(2)}\|/2$$

and for any $\psi \in C^3(\bar{\Omega})$,

$$\left|\left(D^{\pm} - \frac{d^2}{dx^2}\right)\psi(x_i)\right| \leqslant (x_{i+1} - x_{i-1})\|\psi^{(3)}\|/3.$$

For the proof of these results, one can see Lemma 1 [19, p. 21]. Using these results in Eq. (2.23), we obtain

$$\begin{aligned}&\left|L_{\varepsilon,l}\left(V_\varepsilon^N - v_\varepsilon\right)(x_i)\right|\\&\quad\leqslant C(x_{i+1}-x_{i-1})\left(\frac{\varepsilon}{3}\left\|v_\varepsilon^{(3)}\right\| + \frac{(a(x_i)+\beta(x_i)\eta-\alpha(x_i)\delta)}{2}\left\|v_\varepsilon^{(2)}\right\|\right)\end{aligned}$$

using the Theorem 2 for the estimates of $v_\varepsilon^{(2)}$ and $v_\varepsilon^{(3)}$ and the fact that $x_{i+1}-x_{i-1}\leqslant 2N^{-1}$, we get

$$\left|L_{\varepsilon,l}\left(V_\varepsilon^N - v_\varepsilon\right)(x_i)\right| \leqslant CN^{-1}, \quad x_i \in \Omega^N. \tag{2.24}$$

Now an application of Lemma 2 to the mesh function $(V_\varepsilon^N - v_\varepsilon)(x_i)$ yields

$$\left|\left(V_\varepsilon^N - v_\varepsilon\right)(x_i)\right| \leqslant \theta^{-1}\max_{1\leqslant j\leqslant N-1}\left|L_{\varepsilon,l}\left(V_\varepsilon^N - v_\varepsilon\right)(x_j)\right|. \tag{2.25}$$

Using inequality (2.24) in inequality (2.25), we get

$$\left|\left(V_\varepsilon^N - v_\varepsilon\right)(x_i)\right| \leqslant CN^{-1}, \quad x_i \in \bar{\Omega}^N. \tag{2.26}$$

Arguments for the estimation of the singular component of the error depends on the value of the transition parameter τ, whether $\tau = 1/2$ or $\tau = C\varepsilon\ln N$, where $C = 1/\theta$. In the first case the mesh is uniform and $C\varepsilon\ln N \geqslant 1/2$. In this case, by using the same arguments as we did in the case of estimation of the regular part of the error, we get for each x_i,

$$\begin{aligned}&\left|L_{\varepsilon,l}^N\left(W_\varepsilon^N - w_\varepsilon\right)(x_i)\right|\\&\quad\leqslant (x_{i+1}-x_{i-1})\left(\frac{\varepsilon}{3}\left\|w_\varepsilon^{(3)}\right\| + \frac{(a(x_i)+\beta(x_i)\eta-\alpha(x_i)\delta)}{2}\left\|w_\varepsilon^{(2)}\right\|\right).\end{aligned}$$

Since $(x_{i+1}-x_{i-1}) = 2N^{-1}$ for the uniform mesh and using Theorem 2 for the estimates of $w_\varepsilon^{(2)}$ and $w_\varepsilon^{(3)}$, the above inequality reduces to

$$\left|L_{\varepsilon,l}^N\left(W_\varepsilon^N - w_\varepsilon\right)(x_i)\right| \leqslant C\varepsilon^{-1}N^{-1}, \quad x_i \in \Omega^N.$$

But in this case $\varepsilon^{-1} \leqslant 2C\ln N$, so the above inequality reduces to

$$\left|L_{\varepsilon,l}^N\left(W_\varepsilon^N - w_\varepsilon\right)(x_i)\right| \leqslant CN^{-1}(\ln N)^2, \quad x_i \in \Omega^N. \tag{2.27}$$

Now applying Lemma 2 to the mesh function $(W_\varepsilon^N - w_\varepsilon)(x_i)$, we get

$$\left|\left(W_\varepsilon^N - w_\varepsilon\right)(x_i)\right| \leqslant \theta^{-1}\max_{1\leqslant j\leqslant N-1}\left|L_\varepsilon\left(W_\varepsilon^N - w_\varepsilon\right)(x_j)\right|. \tag{2.28}$$

Using inequality (2.27) in inequality (2.28), we get

$$\left|\left(W_\varepsilon^N - w_\varepsilon\right)(x_i)\right| \leqslant CN^{-1}(\ln N)^2, \quad x_i \in \Omega^N. \tag{2.29}$$

In the second case the mesh is piecewise uniform with mesh spacing $2\tau/N$ in the subinterval $[0,\tau]$ and $2(1-\tau)/N$ in the subinterval $[\tau,1]$ and $\tau = C\varepsilon\ln N$. We will estimate the singular component of the error in each subinterval separately. First suppose that $x_i \in [\tau,1]$. From the triangular inequality, we have

$$\left|(W_\varepsilon - w_\varepsilon)(x_i)\right| \leqslant \left|W_\varepsilon(x_i)\right| + \left|w_\varepsilon(x_i)\right|. \tag{2.30}$$

From Theorem 2, we have

$$\left|w_\varepsilon(x_i)\right| \leqslant C\exp(-Mx_i/\varepsilon).$$

Since $\exp(-Mx_i)$ is a decreasing function for $x_i \in [\tau, 1]$ and $\tau \leqslant x_i$, therefore using these facts in the above inequality, we obtain

$$\left|w_\varepsilon(x_i)\right| \leqslant C\exp(-M\tau). \tag{2.31}$$

But in this case we have $\tau = C\varepsilon \ln N$. Using this value of τ in the above inequality (2.31), we get

$$\left|w_\varepsilon(x_i)\right| \leqslant CN^{-1}, \quad N/2 \leqslant i \leqslant N. \tag{2.32}$$

To establish a similar bound on W_ε^N, we construct a mesh function $\hat{W}_\varepsilon^N$ defined as the solution of the following problem:

$$\varepsilon D^{\pm}\hat{W}_\varepsilon^N(x_i) + MD^{+}\hat{W}_\varepsilon^N(x_i) + \big(\alpha(x_i) + \beta(x_i) + w(x_i)\big)\hat{W}_\varepsilon^N(x_i) = 0 \tag{2.33}$$

$\forall x_i \in \Omega^N$ under the same boundary conditions as for the W_ε^N. Then by Lemma 5 [19, p. 53], we get

$$\left|W_\varepsilon^N(x_i)\right| \leqslant \left|\hat{W}_\varepsilon^N(x_i)\right|, \quad 0 \leqslant i \leqslant N. \tag{2.34}$$

Again an application of Lemma 3 [19, p. 51] for $\hat{W}_\varepsilon^N$ yields

$$\left|\hat{W}_\varepsilon^N(x_i)\right| \leqslant CN^{-1}, \quad N/2 \leqslant i \leqslant N.$$

Using this estimate for $\hat{W}_\varepsilon^N$ in Eq. (2.34), we obtain

$$\left|W_\varepsilon^N(x_i)\right| \leqslant CN^{-1}, \quad N/2 \leqslant i \leqslant N. \tag{2.35}$$

Using the inequalities (2.32) and (2.35) in the inequality (2.30), we obtain the required bound for the singular component of the error in the outer region $[\tau, 1]$,

$$\left|W_\varepsilon^N - w_\varepsilon(x_i)\right| \leqslant CN^{-1}, \quad N/2 \leqslant i \leqslant N. \tag{2.36}$$

Now we estimate the singular component in the boundary layer region, i.e., in the subinterval $[0, \tau]$. To do this, we use similar arguments as we used in the estimation of the regular component and obtain

$$\left|L_{\varepsilon,l}^N\big(W_\varepsilon^N - w_\varepsilon\big)(x_i)\right| \leqslant 2\tau N^{-1}\varepsilon^{-1}. \tag{2.37}$$

From Eq. (2.21), we have

$$\left|W_\varepsilon^N(0) - w_\varepsilon(0)\right| = 0.$$

From inequalities (2.32) and (2.35), we have

$$\left|W_\varepsilon^N(x_{N/2}) - w_\varepsilon(x_{N/2})\right| \leqslant \left|W_\varepsilon^N(x_{N/2})\right| + \left|w_\varepsilon(x_{N/2})\right| \leqslant CN^{-1}. \tag{2.38}$$

Now let us consider the two barrier functions ψ_i^+ and ψ_i^- defined as

$$\psi_i^{\pm} = (\tau - x_i)C_1\varepsilon^{-2}\tau N^{-1} + C_2N^{-1} \pm \big(W_\varepsilon^N - w_\varepsilon\big)(x_i),$$

where C_1 and C_2 are arbitrary constants. Then we have

$$\psi_0^{\pm} = C_1 \tau N^{-1} \varepsilon^{-2} + C_2 N^{-1} \geqslant 0,$$
$$\psi_{N/2}^{\pm} = C_2 N^{-1} \pm \big(W_\varepsilon^N - w_\varepsilon\big)(x_{N/2}).$$

Since from inequality (2.38), it is clear that $(W_\varepsilon^N - w_\varepsilon)(x_{N/2})$ is bounded, so we choose C_2 so that the first term dominates the second term on the right side of the above equation which gives

$$\psi_{N/2}^{\pm} \geqslant 0.$$

Now consider

$$L_{\varepsilon,l}^N \psi_i^{\pm} = \tau N^{-1} C_1 \varepsilon^{-2} L_{\varepsilon,l}^N (\tau - x_i) + C_2\big(\alpha(x_i) + \beta(x_i) + w(x_i)\big) N^{-1} \pm L_{\varepsilon,l}^N \big(W_\varepsilon^N - w_\varepsilon\big).$$

After doing some simplification and using inequality (2.37) for the bound on $L_{\varepsilon,l}^N(W_\varepsilon^N - w_\varepsilon)$ in the above inequality, we obtain

$$L_{\varepsilon,l}^N \psi_i^{\pm} \leqslant -N^{-1}\varepsilon^{-1}(C_1 M \ln N \pm C) + \big(\alpha(x_i) + \beta(x_i) + w(x_i)\big) C_2 N^{-1} + \big(\alpha(x_i) + \beta(x_i) + w(x_i)\big)(\tau - x_i) C_1 \varepsilon^{-2} \tau N^{-1}.$$

We choose C_1 such that $C_1 M \ln N \geqslant C$, where $C = 1/\theta$. Thus all the terms on the right side in the above inequality are negative. Therefore we have

$$L_{\varepsilon,l}^N \psi_i^{\pm} \leqslant 0, \quad 1 \leqslant i \leqslant N/2 - 1.$$

Then by the discrete minimum principle, we have

$$\psi_i^{\pm} = (\tau - x_i) C_1 \varepsilon^{-2} \tau N^{-1} + C_2 N^{-1} \pm \big(W_\varepsilon^N - w_\varepsilon\big)(x_i) \geqslant 0, \quad 0 \leqslant i \leqslant N/2,$$

which gives

$$\big|\big(W_\varepsilon^N - w_\varepsilon\big)(x_i)\big| \leqslant C_1 \varepsilon^{-2} \tau^2 N^{-1} + C_2 N^{-1}.$$

Since $\tau = C\varepsilon \ln N$, where $C = 1/\theta$, we get

$$\big|\big(W_\varepsilon^N - w_\varepsilon\big)(x_i)\big| \leqslant C N^{-1} (\ln N)^2, \quad 0 \leqslant i \leqslant N/2. \tag{2.39}$$

Now combining the separate estimates for the singular component of the error in the two regions, i.e., boundary layer region as well as the outer region, we obtain

$$\big|\big(W_\varepsilon^N - w_\varepsilon\big)(x_i)\big| \leqslant C N^{-1} (\ln N)^2, \quad 0 \leqslant i \leqslant N. \tag{2.40}$$

Finally by combining the two inequalities (2.26) to bound the regular error component and (2.40) to bound the singular error component, we obtain the required error estimate. □

2.2. Layer on the right side

Now we assume that $(a(x) - \beta(x)\eta - \alpha(x)\delta) < -M < 0$ throughout the interval [0, 1], where M is a positive constant. This assumption implies that the boundary layer will be in the neighborhood of 1, i.e., on the right side of the interval [0, 1].

2.2.1. Analytical results

As we have established the estimates for the solution of the continuous problem and its derivatives in the case when the solution of the problem exhibits boundary layer behavior on the left side of the interval $[0, 1]$, one can easily obtain similar estimates in this case following the same lines as we did earlier. The key difference is that in this case, we approximate the first derivative by the backward finite difference operator in place of the forward finite difference operator as we did in the case of the left side boundary layer.

2.2.2. Discretization

In this case, we discretize the boundary-value problem (2.1), (2.2) using the fitted mesh finite difference method composed of a standard backward upwind finite difference operator on a fitted piecewise uniform mesh, condensing at the boundary $x = 1$. The fitted piecewise-uniform mesh $\bar{\Omega}_r^N$ on the interval $[0, 1]$ is constructed by partitioning the interval into two subintervals $[0, (1 - \tau)]$ and $[(1 - \tau), 1]$, where the transition parameter is chosen such that $\tau \equiv \min[0.5, C\varepsilon \ln N]$ with $C = 1/\theta$ and it is assumed that $N = 2^m$, $m \geqslant 2$, is an integer, which guarantees that there is at least one point in the boundary layer. On each of these subintervals, a uniform mesh is placed. A fitted finite difference method for the problem (2.1), (2.2) on the piecewise uniform mesh $\bar{\Omega}_r^N$ is defined by

$$L_{\varepsilon,r}^N z_\varepsilon^N(x_i) = f(x_i), \quad x_i \in \Omega_r^N, \tag{2.41}$$

$$z_\varepsilon^N(x_0) = \phi_0, \tag{2.42a}$$

$$z_\varepsilon^N(x_N) = \gamma_1, \tag{2.42b}$$

where for any mesh function Ψ_i, the discrete operator $L_{\varepsilon,r}^N$ is defined by

$$L_{\varepsilon,r}^N \Psi_i = \varepsilon D^{\pm}\Psi_i + \big(a(x_i) - \alpha(x_i)\delta + \beta(x_i)\eta\big)D^{-}\Psi_i + \big(\alpha(x_i) + \beta(x_i) + \omega(x_i)\big)\Psi_i.$$

Also it can be easily show that the solution of the discretized problem converges uniformly in ε to the solution of the continuous problem. One can obtain the error estimate in this case on the same lines as we have done in Section 2.1 for the case of left side boundary layer.

3. Computational results

Some numerical examples are considered and solved using the methods presented here. The exact solution of the boundary-value problem given by Eq. (2.1), (2.2) for constant coefficients, forcing term and interval conditions, i.e., $\alpha(x) = \alpha$, $\beta(x) = \beta$, $a(x) = a$, $\omega(x) = \omega$, $f(x) = f$, $\phi(x) = \phi$ and $\gamma(x) = \gamma$ are constants, then the solution z_ε is given by

$$z_\varepsilon(x) = c_1 \exp(m_1 x) + c_2 \exp(m_2 x) + f/c,$$

where

$$c_1 = \big[-f + \gamma c + \exp(m2)(f - \phi c)\big]/\big[\big(\exp(m1) - \exp(m2)\big)c\big],$$

$$c_2 = \big[f - \gamma c + \exp(m1)(-f + \phi c)\big]/\big[\big(\exp(m1) - \exp(m2)\big)c\big],$$

$$m_1 = \left[-(a-\alpha\delta+\beta\eta)+\sqrt{(a-\alpha\delta+\beta\eta)^2-4\varepsilon c}\,\right]/2\varepsilon,$$
$$m_2 = \left[-(a-\alpha\delta+\beta\eta)-\sqrt{(a-\alpha\delta+\beta\eta)^2-4\varepsilon c}\,\right]/2\varepsilon,$$
$$c = (\alpha+\beta+\omega).$$

Example 1. $a(x)=1$, $\alpha(x)=2$, $\beta(x)=0$, $\omega(x)=-3$, $f(x)=0$, $\phi(x)=1$, $\gamma(x)=1$ in boundary-value problem (1.2), (1.3).

Example 2. $a(x)=1$, $\alpha(x)=0$, $\beta(x)=2$, $\omega(x)=-3$, $\phi(x)=1$, $f(x)=0$, $\gamma(x)=1$ in boundary-value problem (1.2), (1.3).

Example 3. $a(x)=1$, $\alpha(x)=-2$, $\beta(x)=1$, $\omega(x)=-5$, $\phi(x)=1$, $f(x)=0$, $\gamma(x)=1$ in boundary-value problem (1.2), (1.3).

Table 1
Maximum error ($\varepsilon = 0.1$)

$N \rightarrow$	8	32	128	512
$\delta \downarrow$	Example 1			
0.00	0.09907804	0.03700736	0.00954678	0.00214501
0.05	0.09659609	0.03640566	0.00924661	0.00202998
0.09	0.09277401	0.03556652	0.00895172	0.00192488
$\eta \downarrow$	Example 2			
0.00	0.09907804	0.03700736	0.00954678	0.00214501
0.05	0.09977501	0.03727087	0.00979659	0.00224472
0.09	0.10031348	0.03723863	0.00996284	0.00458698

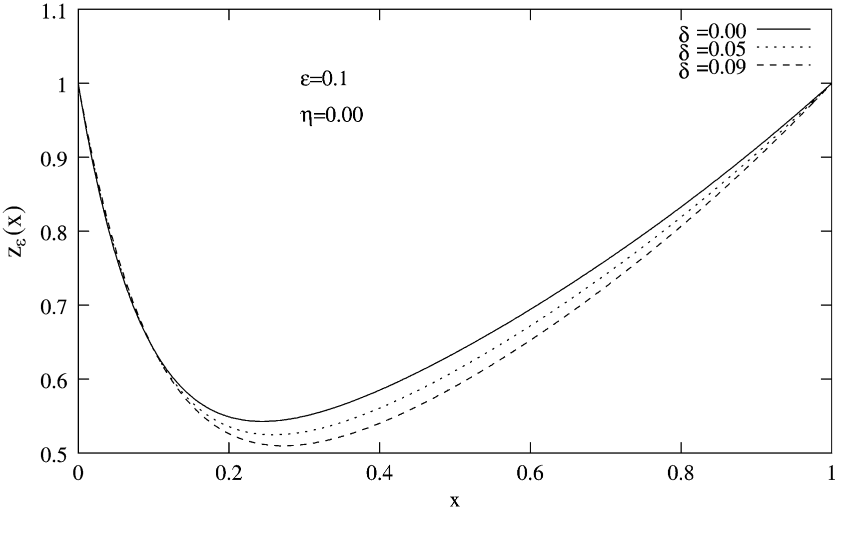

Fig. 1. Numerical solution of Example 1.

Table 2
Maximum error for Example 2 ($\eta = 0.5\varepsilon$)

$\varepsilon \downarrow$	$N \rightarrow 8$	16	32	64	128	256
10^{-1}	0.10233615	0.06103660	0.03823132	0.02299386	0.01295871	0.00664316
10^{-2}	0.16053996	0.09171283	0.05062424	0.02640865	0.01344656	0.00676030
10^{-3}	0.17511397	0.10213037	0.05896661	0.03133175	0.01623376	0.00825735
10^{-4}	0.17669288	0.10327230	0.05991398	0.03189761	0.01656671	0.00843635
10^{-5}	0.17685213	0.10338763	0.06001002	0.03195506	0.01660057	0.00845456
10^{-6}	0.17686807	0.10339917	0.06001964	0.03196081	0.01660396	0.00845639

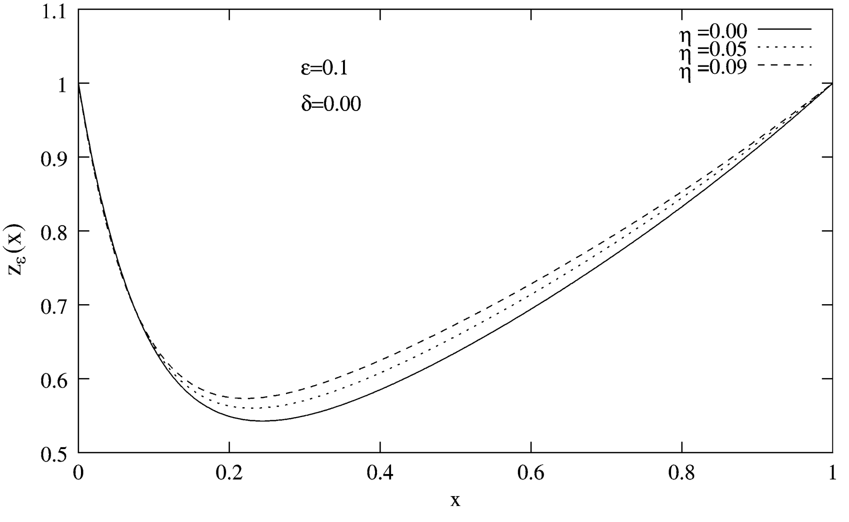

Fig. 2. Numerical solution of Example 2.

Table 3
Maximum error for Example 3 ($\delta = \eta = 0.5\varepsilon$)

$\varepsilon \downarrow$	$N \rightarrow 8$	16	32	64	128	256
10^{-1}	0.12011566	0.07181396	0.04482982	0.02694612	0.01516093	0.00775036
10^{-2}	0.18727108	0.10697821	0.05904116	0.03079689	0.01567964	0.00799076
10^{-3}	0.20429729	0.11915028	0.06879232	0.03655236	0.01893849	0.00963304
10^{-4}	0.20614146	0.12048418	0.06989944	0.03721375	0.01932774	0.00984236
10^{-5}	0.20632746	0.12061888	0.07001167	0.03728089	0.01936732	0.00986365
10^{-6}	0.20634608	0.12063236	0.07002291	0.03728761	0.01937129	0.00986578

Example 4. $a(x) = -1$, $\alpha(x) = -2$, $\beta(x) = 0$, $\omega(x) = 1$, $f(x) = 0$, $\phi(x) = 1$, $\gamma(x) = -1$ in boundary-value problem (1.2), (1.3).

Example 5. $a(x) = -1$, $\alpha(x) = 0$, $\beta(x) = -2$, $\omega(x) = 1$, $f(x) = 0$, $\phi(x) = 1$, $\gamma(x) = -1$ in boundary-value problem (1.2), (1.3).

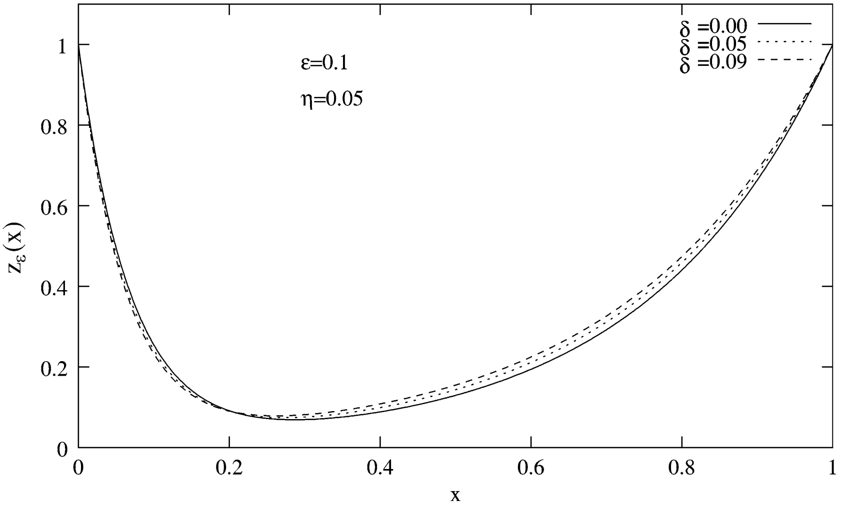

Fig. 3. Numerical solution of Example 3 for different values of δ.

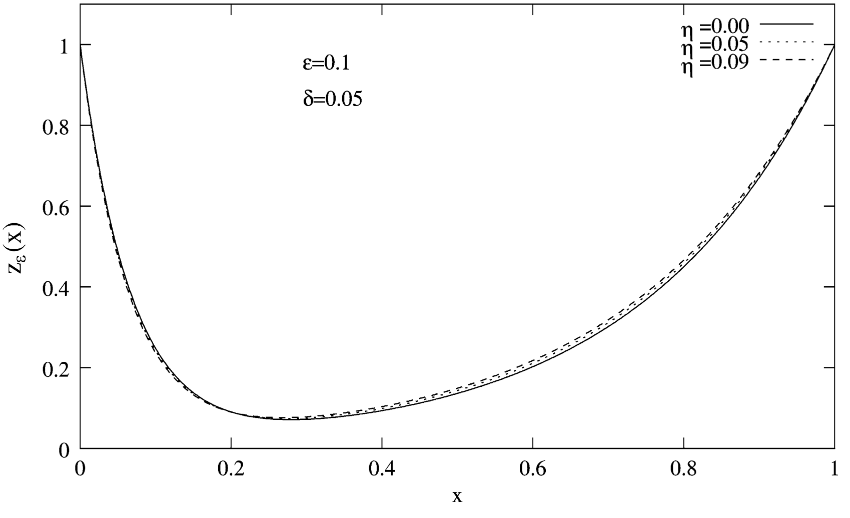

Fig. 4. Numerical solution of Example 3 for different values of η.

Example 6. $a(x) = -(1 + \exp(x^2))$, $\alpha(x) = -x$, $\beta(x) = -(1 - \exp(-x))$, $\omega(x) = x^2$, $f(x) = 1$, $\phi(x) = 1$, $\gamma(x) = -1$ in boundary-value problem (1.2), (1.3).

Example 7. $a(x) = -1$, $\alpha(x) = -2$, $\beta(x) = -2$, $\omega(x) = 1$, $f(x) = 0$, $\phi(x) = 1$, $\gamma(x) = -1$ in boundary-value problem (1.2), (1.3).

Example 8. $a(x) = 1$, $\alpha(x) = -2$, $\beta(x) = -1$, $\omega(x) = 1$, $f(x) = -1$, $\phi(x) = 1$, $\gamma(x) = 1$ in boundary-value problem (1.2), (1.3).

Table 4
Maximum error for Example 3 ($\varepsilon = 0.1$)

	$N \to 8$	32	128	512
$\delta \downarrow$	$\eta = 0.05$			
0.00	0.09190267	0.03453494	0.01164358	0.00300463
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.11018870	0.04110846	0.01400144	0.00362925
$\eta \downarrow$	$\delta = 0.05$			
0.00	0.09720079	0.03640446	0.01229476	0.00317786
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.10632014	0.03965833	0.01348348	0.00349050

Table 5
Maximum error ($\varepsilon = 0.1$)

	$N \to 8$	32	128	512
$\delta \downarrow$	Example 4			
0.00	0.07847490	0.04678972	0.01727912	0.00443086
0.05	0.09222560	0.03828329	0.01487799	0.00380679
0.09	0.10509460	0.03149275	0.01299340	0.00331935
$\eta \downarrow$	Example 5			
0.00	0.07847490	0.04678972	0.01727912	0.00443086
0.05	0.06834579	0.05516436	0.01972508	0.00506769
0.09	0.08328237	0.06168267	0.02169662	0.00558451

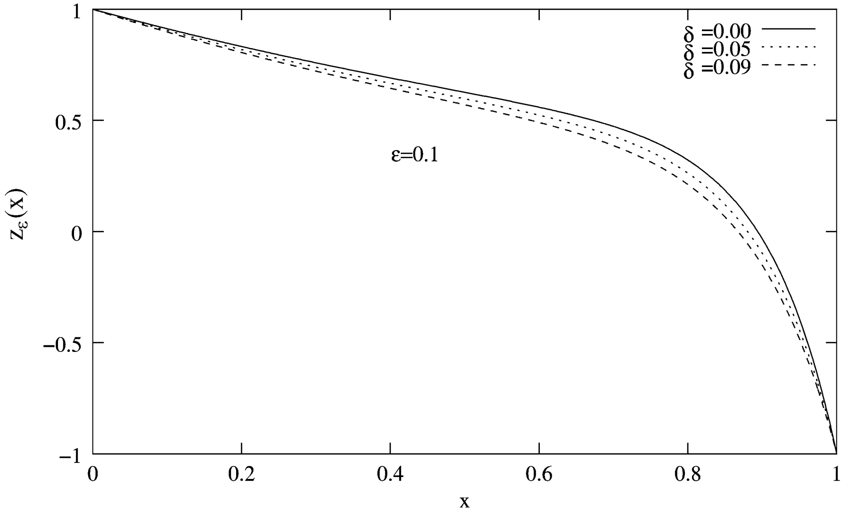

Fig. 5. Numerical solution of Example 5.

4. Conclusion

A finite difference approach has been taken into account to approximate the solution of a more general class of singularly perturbed differential–difference equations which arises

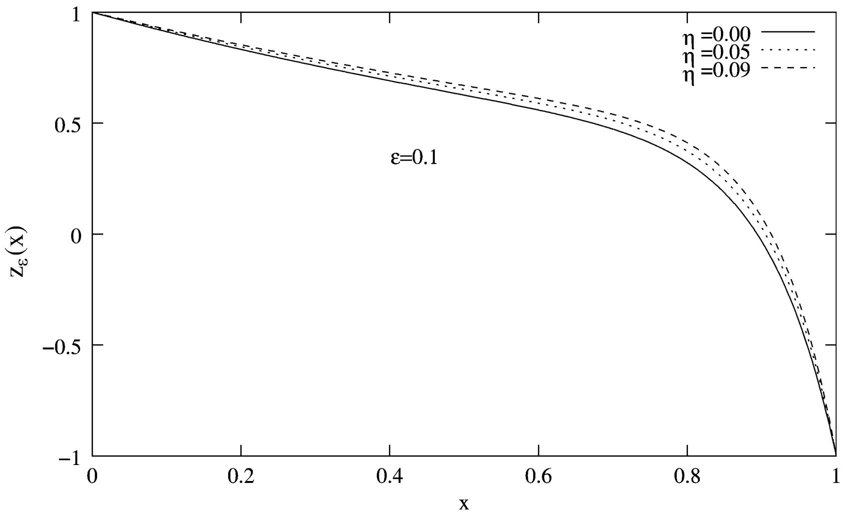

Fig. 6. Numerical solution of Example 6.

Table 6
Maximum error for Example 7 ($\varepsilon = 0.1$)

	$N \to 8$	32	128	512
$\delta \downarrow$	$\eta = 0.05$			
0.00	0.09930002	0.03685072	0.01331683	0.00342882
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.10044578	0.02850398	0.01038902	0.00266379
$\eta \downarrow$	$\delta = 0.05$			
0.00	0.10055269	0.02759534	0.01007834	0.00258299
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.09944067	0.03591410	0.01297367	0.00334044

Table 7
Maximum error for Example 8 ($\delta = \eta = 0.5\varepsilon$)

$\varepsilon \downarrow$	$N \to 8$	16	32	64	128	256
10^{-1}	0.08579690	0.05129568	0.03202130	0.01924723	0.01098354	0.00553597
10^{-2}	0.13376506	0.07641301	0.04217226	0.02199778	0.01119974	0.00570769
10^{-3}	0.14592663	0.08510734	0.04913737	0.02610883	0.01352749	0.00688074
10^{-4}	0.14724390	0.08606013	0.04992817	0.02658125	0.01380553	0.00703026
10^{-5}	0.14737676	0.08615634	0.05000834	0.02662921	0.01383380	0.00704546
10^{-6}	0.14739006	0.08616597	0.05001637	0.02663401	0.01383663	0.00704699

in the mathematical modeling of a model of neuronal variability. A numerical scheme is constructed to solve such type of boundary-value problems. A parameter uniform error estimate is obtained for the presented difference scheme.

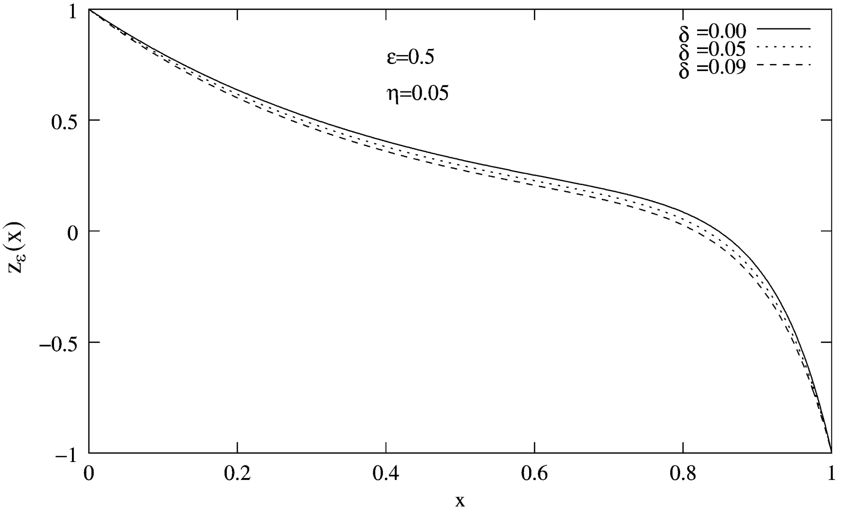

Fig. 7. Numerical solution of Example 7 for different values of δ.

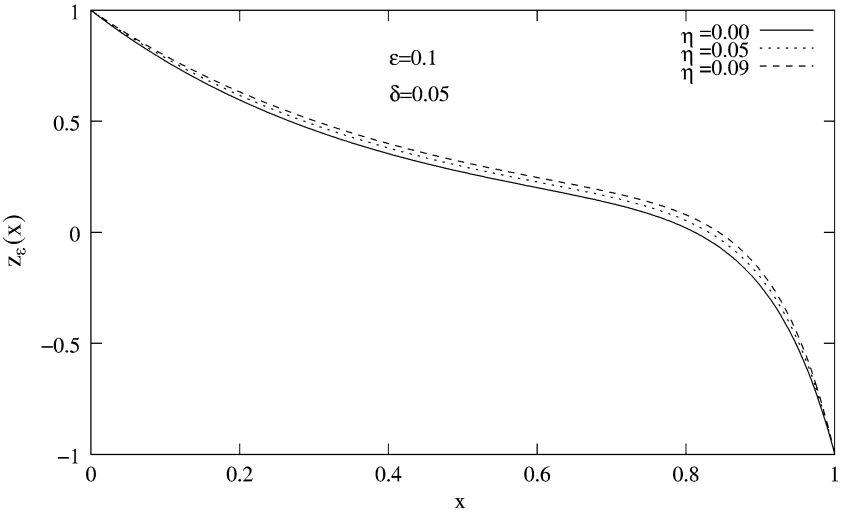

Fig. 8. Numerical solution of Example 7 for different values of η.

A number of numerical experiments are carried out in support of the predicted theory via tabulating the maximum absolute errors in Tables 1–7 for the examples considered and to show the effect of the small shifts on the solution of the problem via plotting the graphs of the solution for different values of negative shift and positive shift for the examples considered, which are reported in the form of Figs. 1–8. We observe from the error Tables 1–5 that the difference scheme converges super-linearly and independently of the singular perturbation parameter which supports the predicted theory.

Figures 1–4 illustrate that in the case when the solution of the boundary-value problem exhibits layer behavior on the left side, the effect of delay or advance on the solution in the boundary layer region is negligible while in the outer region it is considerable and the change in the advance affects the solution in similar fashion as the change in delay affects, but reversely. Figures 5–8 illustrate that in the case when the boundary-value problem exhibits layer behavior on the right side, the changes in delay or advance affect the solution in boundary layer region as well as outer region. The thickness of the layer increases as the size of the delay increases while it decreases as the size of the advance increases.

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