Motion around $L_{4,5}$ in the relativistic R3BP with smaller triaxial primary

Nakone Bello$^a,*$, Jagdish Singh$^b$

$^a$Department of Mathematics, Faculty of Science, Usmanu Danfodiyo University, Sokoto, Nigeria
$^b$Department of Mathematics, Faculty of Science, Ahmadu Bello University, Zaria, Nigeria

Received 23 April 2015; received in revised form 24 June 2015; accepted 1 July 2015
Available online 28 July 2015

Abstract

This paper deals with the triangular points $L_{4,5}$ of the relativistic restricted three-body problem (R3BP) when the smaller primary is assumed triaxial. It is noticed that the locations and stability of the triangular points are affected by both relativistic and triaxiality perturbations. It can be easily seen that the range of stability region of these points is reduced by the effects of relativistic and triaxiality factors and more especially decreases with the increase of triaxiality factor.

© 2015 The Authors. Production and Hosting by Elsevier B.V. on behalf of Nigerian Mathematical Society. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

Keywords: Celestial mechanics; Triaxiality; Relativity; R3BP

1. Introduction

The restricted three-body problem concerns the study of the motion of one infinitesimal celestial body in the gravitation field of two other bodies (conventionally called the primaries) moving along circular keplerian orbits around their center of mass. The third body has small mass with respect to others, and is treated like a test particle whose motion results determined by the two bodies, yet without affecting their motion in turn. It is an approximation of the three-body problem, which regards the study of the dynamics of three masses interacting by means of the gravitational force. This problem possesses five points called Lagrangian points, three of them are called collinear points $L_1$, $L_2$ and $L_3$ and are unstable, they lie on the line joining the primaries, the other two are called the triangular points $L_4$ and $L_5$ and are stable for the mass ratio $\mu \leq 0.038520 \ldots$, Szebehely [1].

It is known that celestial bodies are irregular bodies and cannot be always considered as spherical in the restricted three-body problem, because the shape of the body affects the locations as well as the motion around equilibrium points. In most cases the planets and their natural satellites are extended bodies which are triaxial or oblate spheroids; this problem has wide applications in many astrophysical problems, Trojan asteroids, around the triangular points of

* Corresponding author.
E-mail addresses: bnakone@yahoo.com (N. Bello), jgds2004@yahoo.com (J. Singh).
the Sun-Jupiter system are examples of this. The lack of asphericity, triaxiality or oblateness of the celestial bodies causes large perturbations in a two-body orbit.

Some studies, which are related to the Lagrangian points by considering one or both primaries are oblate spheroids or triaxial, are discussed by SubbaRao and Sharma [2]; Sharma et al. [3]; Singh [4]; AbdulRaheem and Singh [5]; Sharma et al. [6], and Abouelmagd [7].

The theory of the general relativity is currently the most successful gravitational theory describing the nature of space and time, and well confirmed by observations [8]. For the application in celestial mechanics, the most important problem of general relativity is the problem of motion of material bodies.

For a test particle, the equations of motion are determined by the geodesic principle.

Brumberg [9,10] studied the problem in more details and collected most of the important results on relativistic celestial mechanics. He did not obtain only the equation of motion for the general problem of three bodies, but also deduced the equations of motion for the restricted problem of three bodies.

Bhatnagar and Hallan [11] studied the existence and linear stability of the triangular points $L_{4,5}$ in the relativistic R3BP, and found that $L_{4,5}$ are always unstable in the whole region $0 \leq \mu < \frac{1}{2}$ in contrast to the classical R3BP in which they are stable for $\mu < \mu_0$, where $\mu$ is the mass ratio and $\mu_0 = 0.038520 \ldots$ is the Routh’s value.

Douskos and Perdios [12] investigated the stability of the triangular points in the relativistic R3BP and contrary to the result of Bhatnagar and Hallan [11], they obtained a region of linear stability in the parameter space as $0 \leq \mu < \mu_0 - \frac{17\sqrt{3}}{496\sqrt{2}}$ where $\mu_0 = 0.03852 \ldots$ is Routh’s value.

Katour et al. [13] obtained new locations of the triangular points in the framework of relativistic R3BP with oblateness and photo-gravitational corrections to triangular equilibrium points.

In the present work, we study the existence of the triangular points and their linear stability by considering the less massive primary as a triaxial rigid body.

This paper is organized as follows: In Section 2, the equations governing the motion are presented; Section 3 describes the positions of triangular points, while their linear stability is analyzed in Section 4; the obtained results are discussed in Section 5, finally Section 6 conveys the main findings of this paper.

2. Equations of motion

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP in a barycentric synodic coordinate system $(\xi, \eta)$ and dimensionless variables with origin at the center of mass of the primaries can be written as Brumberg [9] and Bhatnagar and Hallan [11]:

\[
\begin{align*}
\ddot{\xi} - 2n\dot{\eta} &= \frac{\partial W}{\partial \xi} - d \frac{d}{dt} \left( \frac{\partial W}{\partial \dot{\xi}} \right) \\
\ddot{\eta} + 2n\dot{\xi} &= \frac{\partial W}{\partial \eta} - d \frac{d}{dt} \left( \frac{\partial W}{\partial \dot{\eta}} \right)
\end{align*}
\]

with

\[
W = \frac{1}{2}(\xi^2 + \eta^2) + \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{1}{c^2} \left[ -\frac{3}{2} \left( 1 - \frac{1}{3}\mu(1 - \mu) \right)(\xi^2 + \eta^2) \\
+ \frac{1}{8} \left\{ \frac{3}{2}(\xi^2 + \eta^2 + 2(\xi\dot{\eta} - \eta\dot{\xi}) + (\xi^2 + \eta^2) \right\}^2 \\
+ \frac{3}{2} \left( \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \right)(\xi^2 + \eta^2 + 2(\xi\dot{\eta} - \eta\dot{\xi}) + (\xi^2 + \eta^2)) - \frac{1}{2} \left( \frac{(1 - \mu)^2}{\rho_1^2} + \frac{\mu^2}{\rho_2^2} \right) \\
+ \mu(1 - \mu) \left\{ (4\dot{\eta} + \frac{7}{2}\xi)(\frac{1}{\rho_1} - \frac{1}{\rho_2}) - \frac{\eta^2}{2} \left( \frac{\mu}{\rho_1^3} + \frac{1 - \mu}{\rho_2^3} \right) + \left( \frac{-1}{\rho_1\rho_2} + \frac{3\mu - 2}{2\rho_1} + \frac{1 - 3\mu}{2\rho_2} \right) \right\} \right]
\]

\[
n = 1 - \frac{3}{2c^2} \left( 1 - \frac{1}{3}\mu(1 - \mu) \right)
\]

\[
\rho_1^2 = (\xi + \mu)^2 + \eta^2 \\
\rho_2^2 = (\xi + \mu - 1)^2 + \eta^2
\]
where \( 0 < \mu \leq \frac{1}{2} \) is the ratio of the mass of the smaller primary to the total mass of the primaries; \( \rho_1 \) and \( \rho_2 \) are distances of the infinitesimal mass from the bigger and smaller primary, respectively; \( n \) is the mean motion of the primaries; \( c \) is the velocity of light.

By introducing the triaxiality factors of the smaller primary through the parameters \( \sigma_i \) \( (i = 1, 2) \) where

\[
\sigma_1 = \frac{a^2-h^2}{5R^2}, \quad \sigma_2 = \frac{b^2-h^2}{5R^2} \quad [14]
\]

with \( a, b, h \) as lengths of its semi-axes and \( R \) is the distance between the primaries.

Ignoring second and higher power of \( \sigma_i \) and neglecting also their products, we take equations of motion as:

\[
\begin{align*}
\ddot{\xi} - 2n\dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{d}{dt} \left( \frac{\partial W}{\partial \xi} \right) \\
\ddot{\eta} + 2n\dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{d}{dt} \left( \frac{\partial W}{\partial \eta} \right)
\end{align*}
\]

where \( W \) is the potential-like function of the relativistic R3BP. As Katour et al. \[13\], we have not included the parameters \( \sigma_i \) \( (i = 1, 2) \) in the relativistic part of \( W \) because the magnitude of these terms being very small due to \( c^{-2} \).

Hence

\[
W = \frac{1}{2} \left(1 + \frac{3}{2}(2\sigma_1 - \sigma_2)\right) \left(\xi^2 + \eta^2\right) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{\mu}{2\rho_2^2} (2\sigma_1 - \sigma_2) + \frac{3\mu\eta^2}{2\rho_2} (\sigma_2 - \sigma_1)
\]

\[
+ \frac{1}{c^2} \left[ -\frac{3}{2} \left\{1 - \frac{1}{3} \mu(1-\mu)\right\} \left(\xi^2 + \eta^2\right) + \frac{1}{8} \left(\xi^2 + \eta^2 + 2(\xi\dot{\eta} - \eta\dot{\xi}) (\xi^2 + \eta^2)\right)^2 \right.

\]

\[
+ \frac{3}{2} \left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right) \left[\xi^2 + \eta^2 + 2(\xi\dot{\eta} - \eta\dot{\xi}) (\xi^2 + \eta^2)\right] - \frac{1}{2} \left(\frac{(1-\mu)^2}{\rho_1^2} + \frac{\mu^2}{\rho_2^2}\right)
\]

\[
+ \mu(1-\mu) \left\{ \left(8\dot{\eta} + \frac{7}{2}\xi\right) \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) - \eta^2 \left(\frac{2\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3}\right) + \left(\frac{1-\mu}{\rho_1\rho_2} + \frac{3\mu - 2}{2\rho_1} + \frac{1-3\mu}{2\rho_2}\right)\right\},
\]

and \( n \) the perturbed mean motion of the primaries is given by

\[
n = 1 + \frac{3}{4} (2\sigma_1 - \sigma_2) - \frac{3}{2c^2} \left(1 - \frac{1}{3} \mu(1-\mu)\right).
\]

### 3. Location of triangular points

The libration points are obtained from Eq. (5) after putting \( \dot{\xi} = \dot{\eta} = \ddot{\xi} = \ddot{\eta} = 0 \).

These points are the solutions of the equations

\[
\frac{\partial W}{\partial \xi} = 0 = \frac{\partial W}{\partial \eta} \quad \text{with} \quad \dot{\xi} = \dot{\eta} = 0.
\]

That is

\[
\begin{align*}
\xi - \frac{(1-\mu)(\xi + \mu)}{\rho_1^3} - \frac{\mu(\xi - 1 + \mu)}{\rho_2^3} + \left(3\sigma_1 - \frac{3}{2}\sigma_2\right) \xi - \frac{3\mu(\xi - 1 + \mu)(2\sigma_1 - \sigma_2)}{2\rho_2^4} \\
- 15\mu (\xi - 1 + \mu)(\sigma_2 - \sigma_1)\eta^2 + \frac{1}{c^2} \left[ -3\xi \left(1 - \frac{(1-\mu)}{3}\right) + \frac{1}{2} \xi (\xi^2 + \eta^2) - \frac{3}{2} (\xi^2 + \eta^2) \right]
\end{align*}
\]

\[
\times \left\{ \frac{(1-\mu)(\xi + \mu)}{\rho_1^3} + \frac{\mu(\xi - 1 + \mu)}{\rho_2^3} \right\} + 3 \left(\frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}\right) \xi + \frac{(1-\mu)^2(\xi + \mu)}{\rho_1^3} + \frac{\mu^2(\xi - 1 + \mu)}{\rho_2^3}
\]

\[
+ \mu(1-\mu) \left[ \frac{7}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \right] + \frac{7}{2} \xi \left(\frac{\xi + \mu}{\rho_1^3} + \frac{(\xi - 1 + \mu)}{\rho_2^3}\right)
\]
\[
\begin{align*}
+ \frac{3}{2} \eta^2 \left( \frac{\mu(\xi + \mu)}{\rho_1^3} + \frac{(1 - \mu)(\xi - 1 + \mu)}{\rho_2^3} \right) \\
+ \frac{(\xi + \mu)}{\rho_1^3 \rho_2} + \left[ \frac{\xi - 1 + \mu}{\rho_1^3} + \frac{(3\mu - 2)(\xi + \mu)}{2\rho_1^3} \right] = 0
\end{align*}
\] (8a)

and
\[
\eta F = 0, \tag{8b}
\]

with
\[
F = \left( 1 - \frac{1 - \mu}{\rho_1^3} - \frac{\mu}{\rho_2^3} \right) + \left( 3\sigma_1 - \frac{3}{2}\sigma_2 \right) + \frac{3\mu}{\rho_2^3} \left( \frac{3}{2}\sigma_2 - 2\sigma_1 \right) - \frac{15\mu(\sigma_2 - \sigma_1)}{2\rho_2^3} \eta^2 \\
+ \frac{1}{c^2} \left[ -3 \left( 1 - \frac{\mu(1 - \mu)}{3} \right) + \frac{1}{2} (\xi^2 + \eta^2) + 3 \left( \frac{1 - \mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right) - \frac{3}{2} (\xi^2 + \eta^2) \right] \left( \frac{1 - \mu}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right) \\
+ \left( \frac{(1 - \mu)^2}{\rho_1^4} + \frac{\mu^2}{\rho_4^3} \right) + \mu(1 - \mu) \left\{ \frac{7}{2} \xi \left( -\frac{1}{\rho_1^3} + \frac{1}{\rho_2^3} \right) - \left( \frac{\mu}{\rho_1^3} + \frac{1 - \mu}{\rho_2^3} \right) + \frac{3}{2} \xi \left( \frac{\mu}{\rho_1^3} + \frac{1 - \mu}{\rho_2^3} \right) \right\}.
\]

The triangular points are the solutions of Eqs. (8a) and (8b) with \( \eta \neq 0 \). Since \( \frac{1}{\sqrt{3}} \ll 1 \) and in the case \( \frac{1}{\sqrt{3}} \to 0 \) and in the absence of triaxiality (i.e. \( \sigma_1 = \sigma_2 = 0 \)) one can obtain \( \rho_1 = \rho_2 = 1 \); we assume in the relativistic R3BP that \( \rho_1 = 1 + x \) and \( \rho_2 = 1 + y \) where, \( x, y \ll 1 \). may be depending upon the relativistic and triaxiality factors. Substituting these values in the Eqs. (4), solving them for \( \xi, \eta \) and ignoring terms of second and higher powers of \( x \) and \( y \), we get
\[
\begin{align*}
\xi &= x - y + \frac{1 - 2\mu}{2}, \\
\eta &= \pm \left( \frac{\sqrt{3}}{2} + \frac{x + y}{\sqrt{3}} \right).
\end{align*}
\]

Substituting the values of \( \rho_1, \rho_2, \xi, \eta \) from the above in Eqs. (8a) and (8b) with \( \eta \neq 0 \), and neglecting second and higher terms in \( x, y, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \sigma_1, \sigma_2 \) and their products, we have
\[
\begin{align*}
&\frac{3}{2} (1 - \mu) x - \frac{3\mu}{2} y + \frac{3}{4} (2\sigma_1 - \sigma_2) + \frac{(57\sigma_2 - 69\sigma_1)}{16} \mu + \frac{1}{c^2} \left( -\frac{9\mu}{16} + \frac{27\mu^2}{16} - \frac{9\mu^3}{8} \right) = 0 \\
&3 (1 - \mu) x + 3\mu y + \frac{3}{2} (2\sigma_1 - \sigma_2) - \frac{3}{8} (\sigma_1 + 3\sigma_2) \mu + \frac{21 (\mu - \mu^2)}{8c^2} = 0.
\end{align*}
\] (9)

Solving these equations for \( x \) and \( y \), we get
\[
\begin{align*}
x &= -\frac{\mu (2 + 3\mu)}{8c^2} - \left( \frac{1}{\sqrt{3}} - \frac{\mu}{2(1 - \mu)} \right) \sigma_1 + \frac{1}{2} \left( -\frac{\mu}{2(1 - \mu)} \right) \sigma_2 \\
y &= -\frac{1}{\sqrt{3}} \sigma_1 + \frac{11}{8} \sigma_2.
\end{align*}
\] (10)

Thus, the coordinates of the triangular points \( (\xi, \pm \eta) \) denoted by \( L_4 \) and \( L_5 \) respectively are,
\[
\begin{align*}
\xi &= \frac{1 - 2\mu}{2} \left( 1 + \frac{5}{4c^2} \right) + \left( \frac{3}{8} + \frac{\mu}{2(1 - \mu)} \right) \sigma_1 - \frac{7}{8} + \frac{\mu}{2(1 - \mu)} \right) \sigma_2 \\
\eta &= \frac{\sqrt{3}}{2} \left[ 1 + \frac{1}{12c^2} (-5 + 6\mu - 6\mu^2) + \frac{2}{3} \left( -\frac{19}{8} + \frac{\mu}{2(1 - \mu)} \right) \sigma_1 + \left( \frac{15}{8} - \frac{\mu}{2(1 - \mu)} \right) \sigma_2 \right].
\end{align*}
\] (11)
4. Stability of \( L_4 \)

Let \( (a, b) \) be the coordinates of the triangular point \( L_4 \).

We set \( \xi = a + \alpha, \eta = b + \beta, (\alpha, \beta \ll 1) \), in the Eqs. (5) of motion.

First, we compute the terms of their R.H.S, neglecting second and higher order terms, we get

\[
\left( \frac{\partial W}{\partial \dot{\xi}} \right)_{\xi=a+\alpha, \eta=b+\beta} = A\dot{\alpha} + B\dot{\beta} + C\ddot{\alpha} + D\ddot{\beta}
\]

where,

\[
A = \frac{3}{4} \left\{ 1 + \frac{1}{2c^2} \left( 2 - 19\mu + 19\mu^2 \right) \right\} + \frac{3 (15\mu^2 - 49\mu + 26)}{16(1 - \mu)} \sigma_1 - \frac{3 (31\mu^2 - 63\mu + 24)}{16(1 - \mu)} \sigma_2,
\]

\[
B = \frac{3\sqrt{3}}{4} (1 - 2\mu) \left[ 1 - \frac{2}{3c^2} \right] + \frac{\sqrt{3} (89\mu^2 - 131\mu + 50)}{16(1 - \mu)} \sigma_1 - \frac{\sqrt{3} (37\mu^2 - 65\mu + 36)}{16(1 - \mu)} \sigma_2,
\]

\[
C = \frac{\sqrt{3}}{2c^2} (1 - 2\mu),
\]

\[
D = \frac{6 - 5\mu + 5\mu^2}{2c^2}.
\]

Similarly, we obtain

\[
\left( \frac{\partial W}{\partial \dot{\eta}} \right)_{\xi=a+\alpha, \eta=b+\beta} = A_1\dot{\alpha} + B_1\dot{\beta} + C_1\ddot{\alpha} + D_1\ddot{\beta}
\]

where,

\[
A_1 = \frac{3\sqrt{3}}{4} (1 - 2\mu) \left[ 1 - \frac{2}{3c^2} \right] + \frac{\sqrt{3} (89\mu^2 - 131\mu + 50)}{16(1 - \mu)} \sigma_1 - \frac{\sqrt{3} (37\mu^2 - 65\mu + 36)}{16(1 - \mu)} \sigma_2,
\]

\[
B_1 = \frac{9}{4} \left\{ 1 + \frac{7}{6c^2} \left( -2 + 3\mu - 3\mu^2 \right) \right\} - \frac{3 (15\mu^2 - 22\mu - 22)}{16(1 - \mu)} \sigma_1 + \frac{3 \mu (15\mu - 23)}{16(1 - \mu)} \sigma_2
\]

\[
C_1 = \frac{1}{2c^2} \left( -4 + \mu - \mu^2 \right),
\]

\[
D_1 = \frac{\sqrt{3} (1 - 2\mu)}{2c^2}.
\]

\[
\frac{d}{dt} \left( \frac{\partial W}{\partial \dot{\xi}} \right)_{\xi=a+\alpha, \eta=b+\beta} = A_2\dot{\alpha} + B_2\dot{\beta} + C_2\ddot{\alpha} + D_2\ddot{\beta}
\]

where,

\[
A_2 = \frac{\sqrt{3}}{2c^2} (1 - 2\mu),
\]

\[
B_2 = \frac{1}{2c^2} \left( -4 + \mu - \mu^2 \right),
\]

\[
C_2 = \frac{1}{4c^2} \left( 17 - 2\mu + 2\mu^2 \right),
\]

\[
D_2 = -\frac{\sqrt{3}}{4c^2} (1 - 2\mu).
\]

\[
\frac{d}{dt} \left( \frac{\partial W}{\partial \dot{\eta}} \right)_{\xi=a+\alpha, \eta=b+\beta} = A_3\dot{\alpha} + B_3\dot{\beta} + C_3\ddot{\alpha} + D_3\ddot{\beta}
\]

where,

\[
A_3 = \frac{\sqrt{3}}{2c^2} (1 - 2\mu),
\]

\[
B_3 = \frac{1}{2c^2} \left( -4 + \mu - \mu^2 \right),
\]

\[
C_3 = \frac{1}{4c^2} \left( 17 - 2\mu + 2\mu^2 \right),
\]

\[
D_3 = -\frac{\sqrt{3}}{4c^2} (1 - 2\mu).
\]
where,
\[ A_3 = \frac{1}{2c^2} \left(6 - 5\mu + 5\mu^2\right), \]
\[ B_3 = -\frac{\sqrt{3}}{2c^2} \left(1 - 2\mu\right), \]
\[ C_3 = -\frac{\sqrt{3}}{4c^2} \left(1 - 2\mu\right), \]
\[ D_3 = \frac{3(5 - 2\mu + 2\mu^2)}{4c^2}. \]

Thus, the variational equations of motion corresponding to Eqs. (5), on making use of Eq. (7), can be obtained as
\[ P_1\ddot{\alpha} + P_2\ddot{\beta} + P_3\dot{\alpha} + P_4\dot{\beta} + P_5\alpha + P_6\beta = 0, \]
\[ q_1\ddot{\alpha} + q_2\ddot{\beta} + q_3\dot{\alpha} + q_4\dot{\beta} + q_5\alpha + q_6\beta = 0 \]
(12)

where,
\[ P_1 = 1 + C_2, \quad P_2 = D_2, \quad P_3 = A_2 - C, \]
\[ P_4 = \left\{B_2 - 2 \left(1 + \frac{3}{4} (2\sigma_1 - \sigma_2) - \frac{3}{2c^2} \left(1 - \frac{1}{3} \mu(1 + \mu^2)\right)\right) - D\right\}, \]
\[ P_5 = -A_1, \quad P_6 = -B, \]
\[ q_1 = C_3, \quad q_2 = 1 + D_3, \]
\[ q_3 = 2 \left(1 + \frac{3}{4} (2\sigma_1 - \sigma_2) - \frac{3}{2c^2} \left(1 - \frac{1}{3} \mu(1 + \mu^2)\right)\right) - C_1 + A_3, \quad q_4 = B_3 - D_1, \]
\[ q_5 = -A_1, \quad q_6 = -B_1. \]

Then, the characteristic equation is
\[ (P_1q_2 - P_2q_1)\lambda^4 + (P_1q_6 + P_5q_4 + P_3q_4 - P_6q_1 - P_2q_5 - P_4q_3)\lambda^2 + P_5q_6 - P_6q_5 = 0. \]
(13)

Substituting the values of \( P_i, \ q_i, \ i = 1, 2, \ldots, 6 \) in (13), the characteristic equation (13) after normalizing becomes
\[ \lambda^4 + b\lambda^2 + d = 0 \]
(14)

where,
\[ b = \left(1 - \frac{9}{c^2}\right) + 3\sigma_1 - \left(3\mu + \frac{3}{2}\right)\sigma_2, \]
\[ d = \frac{27\mu(1 - \mu)}{4} + \frac{9\mu(-65 + 77\mu - 24\mu^2 + 12\mu^3)}{8c^2} - \frac{9\mu(89\mu - 79)}{16}\sigma_1 + \frac{9\mu(37\mu - 27)}{16}\sigma_2. \]

When \( \frac{1}{c^2} \to 0 \) and in the absence of triaxiality (i.e. \( \sigma_1 = \sigma_2 = 0 \)), (14) reduces to its well-known classical restricted problem form (see e.g. [1]):
\[ \lambda^4 + \lambda^2 + \frac{27\mu(1 - \mu)}{4} = 0. \]

The discriminant of (14) is
\[ \Delta = \frac{-54}{c^4}\mu^4 + \frac{108}{c^2}\mu^3 + \left(27 + \frac{801}{4}\sigma_1 - \frac{333}{4}\sigma_2 - \frac{693}{2c^2}\right)\mu^2 \]
\[ + \left(-27 - \frac{711}{4}\sigma_1 + \frac{219}{4}\sigma_2 + \frac{585}{2c^2}\right)\mu + 1 - \frac{18}{c^2} + 6\sigma_1 - 3\sigma_2. \]
(15)
Its roots are
\[ \lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \]  
where,
\[ b = \left(1 - \frac{9}{c^2}\right) + 3\sigma_1 - \left(\frac{3}{2} + 3\mu\right)\sigma_2. \]

From (15), we have
\[ \frac{d\Delta}{d\mu} = \frac{-216}{c^2}\mu^3 + \frac{324}{c^2}\mu^2 + 2 \left(27 + \frac{801}{4}\sigma_1 - \frac{333}{4}\sigma_2 - \frac{693}{2c^2}\right)\mu \]
\[ + \left(-27 - \frac{711}{4}\sigma_1 + \frac{219}{4}\sigma_2 + \frac{585}{2c^2}\right) < 0 \quad \forall \mu \in \left(0, \frac{1}{2}\right]. \]  

From (17), it can be easily seen that \( \Delta \) is monotone decreasing in \( \left(0, \frac{1}{2}\right] \).

But
\[ (\Delta)_{\mu=0} = 1 + 6\sigma_1 - 3\sigma_2 - \frac{18}{c^2} > 0 \]
\[ (\Delta)_{\mu=\frac{1}{2}} = -\frac{23}{4} + \frac{57}{16}\sigma_2 - \frac{525}{16}\sigma_1 + \frac{207}{4c^2} < 0. \]  

Since \( (\Delta)_{\mu=0} \) and \( (\Delta)_{\mu=\frac{1}{2}} \) are of opposite signs, and \( \Delta \) is monotone decreasing and continuous, there is one value of \( \mu \), e.g. \( \mu_c \) in the interval \( \left(0, \frac{1}{2}\right] \) for which \( \Delta \) vanishes.

Solving the equation \( \Delta = 0 \), using (15), we obtain critical value of the mass parameter as
\[ \mu_c = \frac{1}{2} - \frac{1}{18} \sqrt{69} - \frac{17\sqrt{69}}{486c^2} - \frac{1}{2} \left(\frac{5}{6} - \frac{59}{9\sqrt{69}}\right)\sigma_1 + \frac{1}{2} \left(\frac{19}{18} - \frac{85}{9\sqrt{69}}\right)\sigma_2 \]
\[ \mu_c = \mu_0 - \frac{17\sqrt{69}}{486c^2} \left(-\frac{1}{2} \left(\frac{5}{6} - \frac{59}{9\sqrt{69}}\right)\sigma_1 + \frac{1}{2} \left(\frac{19}{18} - \frac{85}{9\sqrt{69}}\right)\sigma_2 \right) \]  

where \( \mu_0 = 0.038520 \ldots \) is the Routh’s value.

We consider the following three regions of the values of \( \mu \) separately.

i. When \( 0 \leq \mu < \mu_c \), \( \Delta > 0 \), the values of \( \lambda^2 \) given by (16) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.

ii. When \( \mu_c < \mu \leq \frac{1}{2} \), \( \Delta < 0 \), the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.

iii. When \( \mu = \mu_c \), \( \Delta = 0 \), the values of \( \lambda^2 \) given by (16) are the same. This induces instability of the triangular points.

Hence, the stability region is
\[ 0 < \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2} - \frac{1}{2} \left(\frac{5}{6} - \frac{59}{9\sqrt{69}}\right)\sigma_1 + \frac{1}{2} \left(\frac{19}{18} - \frac{85}{9\sqrt{69}}\right)\sigma_2. \]  

5. Discussion

We observe that the assumption that the smaller primary is triaxial in shape makes the Eqs. (11), (14) and (19) corresponding to positions, characteristic equation and critical mass respectively to differ from those obtained by Douskos and Perdios [12]. In the absence of relativistic and triaxiality factors, the results correspond to the classical restricted problem. In the case \( \sigma_1 = \sigma_2 = 0 \), our result fully coincide with those of Douskos and Perdios [12] and
disagree with those of Bhatnagar and Hallan [11]. When the smaller primary is oblate i.e. \((\sigma_1 = \sigma_2 = A_2)\), our results disagree with those of Katour et al. [13] when the smaller primary is oblate only and the primaries are non luminous. In the absence of relativistic effects, the results obtained in this study are in agreement with those of Sharma et al. [3,6] when the primaries are non luminous and the smaller primary is triaxial only and when only the smaller primary is triaxial respectively; and those of Singh [4] when the primaries are non luminous and the smaller primary is triaxial only. Eq. (19) gives the critical value of the mass parameter \(\mu_c\) of the system, which depends upon relativistic and triaxiality factors. This critical value is used to determine the size of the region of stability of the triangular points and also helps in analyzing the behavior of the parameters involved therein.

It is obvious from (19) that the relativistic and triaxiality factors all reduce the size of the region of stability.

6. Conclusion

Under the assumptions that the smaller primary is triaxial, we have determined the locations of triangular points and have analyzed their linear stability. It is found that their locations and stability are affected by relativistic and triaxiality factors. It is observed that both factors have a destabilizing tendency. A practical application of this model could be study of the motion of a small particle in the gravitation field of the Earth–Moon system. We have noticed that the expressions for \(A, D, A_2, C_2\) in [11] differ from the present study when the smaller primary is not triaxial i.e. \((\sigma_1 = \sigma_2 = 0)\). Consequently the expressions \(P_1, P_3, P_4, P_5\) and characteristic equation are also different. This led them [11] to conclude that triangular points are unstable contrary to Douskos and Perdios [12] and our results. In addition to that when the smaller primary is oblate (i.e. \(\sigma_1 = \sigma_2\)) our result differs from those of Katour et al. [13] when the primaries are non-luminous and the smaller primary only oblate. It seems that there is an error in the expression of the perturbed mean motion \(n\) in their study.

References