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Some observations and remarks on differential operators generated by first-order boundary value problems

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Abstract

This paper deals with the study of the set of all self-adjoint differential operators which are generated from first-order, linear, ordinary boundary value problems. These operators are defined on a weighted Hilbert function space and are examined as an application of the result obtained by Everitt and Markus in their paper in 1997. An investigation is given so that first-order self-adjoint boundary value problems are transformed to a study of the nature of the spectrum of associated self-adjoint operators. However, the analysis of this paper is restricted to consideration of conditions under which the spectral properties of these operators yield a discrete spectrum, and consequently to the determination of conditions under which the construction of Kramer analytic kernels, from the above boundary value problems, can be accomplished.

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1. Introduction

In this paper we are concerned with self-adjoint operators in a weighted Hilbert function space which are produced by first-order, linear, ordinary, boundary value problems and we apply the classical Hilbert space theory for such operators in the light of the result in [5]. The theory we consider here is the main tool that leads to the generation of Kramer analytic kernels from these

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first-order boundary value problems. However, we point out that the development of our operator theory, as a source for the construction of the above kernels, is not given in this paper; see [8] for details of these Kramer kernels.

1.1. Notations

Let $\mathbb{R}$ and $\mathbb{C}$ represent the real and complex number fields. Let $I=(a,b)$ be an arbitrary open interval on $\mathbb{R}$; the use of 'loc' restricts a property to compact subintervals of $\mathbb{R}$. Let AC denote absolute continuity and L Lebesgue integration. All functions $f:(a,b)\to\mathbb{C}$ are taken to be Lebesgue measurable on (a,b). All integrals are in the sense of Lebesgue.

Throughout this paper $w:I\to\mathbb{R}$ is a weight function on I, i.e. w is Lebesgue measurable on I, and $w(x)>0$ for almost all $x\in I$. Thus $L^2(I;w)$ represents the class of all complex-valued, Lebesgue measurable functions $f:I\to\mathbb{C}$ such that

$$\int_a^b w|f|^2 \equiv \int_a^b w(x)|f(x)|^2\,\mathrm{d}x < +\infty. \tag{1.1}$$

Then, with due regard to equivalence classes, the norm and inner-product are given by

$$\|f\|_w^2 := \int_I w|f|^2 \quad \text{and} \quad (f,g)_w := \int_a^b w(x)f(x)\bar{g}(x)\,\mathrm{d}x. \tag{1.2}$$

1.2. Brief survey of literature

First-order linear, ordinary, boundary value problems represented by self-adjoint operators are best considered in the context of the general theory of quasi-derivatives; see [3,11]. All the classical differential expressions appear as special cases of quasi-differential expressions. This statement requires the support of detailed analysis; to confirm, see [6, Section I, Appendices A and B, 7, 9]. Earlier work in the general area of both real and complex-valued quasi-differential expressions has been given in [13,15–17]. The results of Shin for arbitrary quasi-differential expressions were re-discovered independently by Zettl in 1965 but not published until the paper Zettl [20]. For this reason Everitt and Markus in [4] proposed to call the collection of such expressions by the name of Shin–Zettl. The work in [20] was continued in [10] and then in [11]. It is conjectured that the most general ordinary linear differential expressions so far defined, for order $n\in\mathbb{N}$, and $n\geqslant 2$, are the Shin–Zettl quasi-differential expressions; for details see [3,9,11]. Note that in [13,14, Chapter V] the order of the quasi-differential expressions is even and the coefficients are all real-valued; such expressions are special cases of the general Shin–Zettl theory.

1.3. Quasi-differential expressions and operators

Let I be an arbitrary open interval of the real line $\mathbb{R}$. Let M_n be a linear ordinary differential expression (or a formal differential operator); in the classical case M_n is of finite order $n\geqslant 1$ on I; in the quasi-differential case $M_n := M_A$ is determined by a $n\times n$ Shin–Zettl complex-valued matrix

$A=[a_{rs}]\in Z_n(I)$, with $n\geqslant 2$, and

$$a_{rs}\in L^1_{\text{loc}}(I)\quad (r,s=1,2,\ldots,n),$$

$$a_{r,r+1}(x)\neq 0\quad (\text{almost all } x\in I \text{ and } r=1,2,\ldots,n-1),$$

$$a_{rs}(x)=0\quad (\text{almost all } x\in I;\ s=r+2,\ldots,n \text{ and } r=1,2,\ldots,n-2).$$

In the classical case M_n has complex coefficients and is of the form

$$M_n[f]=p_nf^{(n)}+p_{n-1}f^{(n-1)}+\cdots+p_1f'+p_0f, \tag{1.3}$$

where $p_j:I\to\mathbb{C}$ with $p_j\in L^1_{\text{loc}}(I)$, $j=0,1,\ldots,n-1,n$ and further $p_n\in AC_{\text{loc}}(I)$ with $p_n(x)\neq 0$ for almost all $x\in I$. For the special case $n=1$ see details in [5].

In the more general quasi-differential case the expression M_n is defined as in [5,7, Section I, 11]. For $n\geqslant 2$ the expression $M_n:=M_A$ is determined by a complex Shin–Zettl matrix A of size n with the domain $D(M_n)$ of M_A defined by

$$D(M_A):=\{f:I\to\mathbb{C}: f_A^{[r-1]}\in \text{AC}_{\text{loc}}(I) \text{ for } r=1,2,\ldots,n\}$$

and

$$M_A[f]:=i^nf_A^{[n]}\quad \text{for all } f\in D(M_A), \tag{1.4}$$

where the quasi-derivatives $f_A^{[j]}$ for $j=1,2,\ldots,n$ are taken relative to the matrix $A\in Z_n(I)$. If the matrix A has certain smoothness properties, then the quasi-derivatives of f can be written in terms of the classical derivatives $f^{(j)}$ for $j=1,2,\ldots,n$.

With every matrix $A\in Z_n(I)$, we associate the Lagrange adjoint matrix A^+ defined by

$$A^+:=-L_n^{-1}A^*L_n,$$

with A^* the normal adjoint matrix of A, and $L_n=[l_{rs}]$ the $n\times n$ matrix defined by

$$l_{r,n+1-r}=\begin{cases}(-1)^{r-1}, & r=1,2,\ldots,n,\\ 0, & r,s \text{ otherwise.}\end{cases}$$

It may be shown that $A^+\in Z_n(I)$ and $(A^+)^+=A$. Also, we assume in this paper that M_A is Lagrange symmetric in the notation in [3,9] (in the older notation this extends the idea of formal self-adjointness of classical differential expressions); that is $A^+=A$ and $M_A^+=M_A$. For these results and additional properties see the notes [3].

Quasi-differential expressions and operators of order 1 need to be defined separately; there are no Shin–Zettl matrices of order 1. The general Lagrange symmetric (formally self-adjoint) first-order differential expression on the interval $I=(a,b)$ is of the form

$$M_1[f]:=-\frac{1}{i}\rho f'+\left(-\frac{\rho'}{2i}+q\right)f\quad \text{for all } f\in D(M), \tag{1.5}$$

where

$$\begin{aligned} &\text{(i)}\quad \rho,q:(a,b)\to\mathbb{R},\\ &\text{(ii)}\quad \rho\in \text{AC}_{\text{loc}}(a,b)\quad\text{and}\quad \rho(x)>0 \text{ for all } x\in(a,b),\\ &\text{(iii)}\quad q\in L^1_{\text{loc}}(a,b)\end{aligned} \tag{1.6}$$

and

$$D(M_1) := \{f : (a,b) \to \mathbb{C} : f \in \mathrm{AC}_{\mathrm{loc}}(a,b)\}. \tag{1.7}$$

Every classical ordinary linear differential expression M_n, as in (1.3) above, can be written as a quasi-differential expression M_A, as in (1.4), with the same order $n \geqslant 2$. The first-order differential expressions are essentially classical in form. Therefore, we can assume that when $n \geqslant 2$, M_n is a quasi-differential expression specified by an appropriate Shin–Zettl matrix $A \in Z_n(I)$. When $n = 1$ we take M_1 as a classical expression. In this investigation we deal only with the first-order case.

Let us now consider the spectral differential equation associated with the pair $\{M_1, w\}$ where M_1 is defined by (1.5) to (1.7) and w is a given weight function which defines the space $L^2((a,b); w)$, as in (1.1) and (1.2). Thus we have

$$M_1[y] = \lambda w y \quad \text{on } (a,b),$$

where the parameter $\lambda = \mu + i\nu \in \mathbb{C}$. The solutions of this linear equation of order 1 are considered within the space $L^2((a,b); w)$ (see Proposition 2.1 in Section 2), and determine the deficiency indices of the symmetric operators considered below.

In this study we describe certain self-adjoint linear operators T on domains $D(T) \subset L^2((a,b); w)$ associated with M_1 and specified by a differential boundary condition connected with the Green's formula. Following the theory developed in [5,14, Chapter V] we define two unbounded operators T_0 and T_1 in $L^2((a,b); w)$ determined by the pair $\{M_1, w\}$. Accordingly to definitions and properties given in Section 3 of this paper, T_0 on $D(T_0)$ and T_1 on $D(T_1)$ are the minimal and maximal operators, respectively, generated by the differential expression M_1. Our goal is to show how the general Stone/von Newman theory of symmetric linear operators in Hilbert space is reformulated and adapted to the determination of all self-adjoint extensions T on domains $D(T) \supset D(T_0)$, by means of the generalised Glazman–Krein–Naimark (GKN) theory for differential operators as given in [5, Section 4, Theorem 1]. In particular, we construct a bijective mapping between the set $\{T\}$ of all such self-adjoint operators and the set $\{\beta\}$ of certain non-null elements of the quotient space $D(T_1)/D(T_0)$. The domain $D(T)$ of any self-adjoint extension T of T_0 can be obtained as a restriction of the domain of the maximal operator T_1 by choosing an element $\beta \in D(T_1)$ such that β arises from a non-null member of the quotient space $D(T_1)/D(T_0)$, and satisfies a symmetric property connected with the boundary condition of the self-adjoint problem. Thus the quotient space $D(T_1)/D(T_0)$ effects a classification of self-adjoint extensions T of T_0 on $D(T)$ containing operators specified by the boundary condition functions β.

Of course, this announced goal becomes more vital within the framework of the idea of showing in this paper, how the solution of a first-order self-adjoint boundary value problem consists of considering the possibility of finding the nature of the spectrum of the operator T. But, the spectral properties of T in the space $L^2((a,b); w)$ are not given in this paper; this is performed in [8]. The scope of our theory is illustrated by an example of the boundary condition functions β.

The operator theory required is to be found in [1,14]; for the classical theory for self-adjoint extensions of symmetric operators as based on Hilbert space constructions, see [1,2,12,14,18,19].

The contents of the paper: the general, first-order, Lagrange symmetric, linear differential equation is discussed in Section 2. Section 3 is devoted to definitions and some properties of self-adjoint differential operators in the weighted Hilbert function space $L^2((a,b); w)$. In Section 4, while investigating the definition of the domain of self-adjoint operators connected with first-order boundary value problems, we relate our analysis to the generalised GKN theory. In Section 5 we give information about the nature of the solution of the boundary value problem, we make a few remarks on the spectrum of our operators and we give some examples. Finally, Section 6 deals with an example of our boundary condition functions.

2. Differential equations

The self-adjoint boundary value problems considered in this paper are generated by the first-order Lagrange symmetric (formally self-adjoint), linear differential equation

$$i\rho(x)y'(x) + \tfrac{1}{2}\, i\rho'(x)y(x) + q(x)y(x) = \lambda w(x)y(x) \quad \text{for all } x \in (a,b), \tag{2.1}$$

where for the open interval (a,b) the endpoints satisfy $-\infty \leqslant a < b \leqslant +\infty$, and $\lambda \in \mathbb{C}$ is the spectral parameter.

The coefficients ρ, q, w satisfy the following conditions:

$$\begin{aligned} &\text{(i)} && \rho, q, w : (a,b) \to \mathbb{R},\\ &\text{(ii)} && \rho \in \mathrm{AC}_{\mathrm{loc}}(a,b) \text{ and } \quad \rho(x) > 0 \text{ for all } x \in (a,b),\\ &\text{(iii)} && q, w \in L^1_{\mathrm{loc}}(a,b),\\ &\text{(iv)} && w(x) > 0 \quad \text{for almost all } x \in (a,b). \end{aligned} \tag{2.2}$$

The endpoint a is defined as regular if $a \in \mathbb{R}$ with $\rho \in \mathrm{AC}_{\mathrm{loc}}[a,b)$ and $q, w \in L^1_{\mathrm{loc}}[a,b)$; similarly for the endpoint b; otherwise endpoints a, b are defined as singular. The differential equation (2.1) is said to be regular if both endpoints a and b are regular.

Under conditions (2.2) the differential equation (2.1) has the following initial value properties; let $c \in (a,b)$ and $\gamma \in \mathbb{C}$, then there exist a unique mapping $y : (a,b) \times \mathbb{C} \to \mathbb{C}$ such that

$$\begin{aligned} &\text{(i)} && y(\cdot,\lambda) \in \mathrm{AC}_{\mathrm{loc}}(a,b) \quad \text{for all } \lambda \in \mathbb{C},\\ &\text{(ii)} && y(x,\cdot) \in \mathbf{H} \quad \text{for all } x \in (a,b),\\ &\text{(iii)} && y(c,\lambda) = \gamma \quad \text{for all } \lambda \in \mathbb{C},\\ &\text{(iv)} && y(\cdot,\lambda) \quad \text{satisfies (2.1) for almost all } x \in (a,b), \text{ and all } \lambda \in \mathbb{C}. \end{aligned} \tag{2.3}$$

This result can be proved along the lines of the classical existence theorem in [14, Chapter V, Section 16.1, Theorem 1]. However direct formal integration shows that the required solution y is given explicitly by

$$y(x,\lambda) = \gamma \sqrt{\frac{\rho(c)}{\rho(x)}} \exp\left(\int_c^x \frac{(\lambda w(t) - q(t))}{i\rho(t)}\, \mathrm{d}t \right) \quad \text{for all } x \in (a,b) \text{ and for all } \lambda \in \mathbb{C}. \tag{2.4}$$

From this explicit form of the solution y, and the conditions (2.2) imposed on the coefficients ρ, q, w, it follows that all the required properties (2.3) are satisfied. Note that, in general, the endpoints a and b are singular points for the equation and the solution $y(\cdot,\lambda)$ in that these endpoints are either infinite, or if finite then the properties (2.3) do not hold at a and/or b, without additional conditions to

(2.2) on the coefficients ρ, q, w. However, if the endpoint a is regular then the initial value properties (2.3) hold at a^+; similarly for b^-.

From the explicit form (2.4) of the solutions of the Eq. (2.1) we obtain

Proposition 2.1. *With the solution $y(\cdot,\lambda)$ of the differential Eq. (2.1) given by the explicit form (2.4) take $\gamma \neq 0$ and $c \in (a,b)$, and write $\lambda = \mu + iv$; then for all $\lambda \in \mathbb{C}$ the solution $y(\cdot,\lambda) \in L^2((a,b);w)$ if and only if*

$$\int_c^b \frac{w(t)}{\rho(t)}\,\mathrm{d}t < +\infty. \tag{2.5}$$

Proof. For the proof of this Proposition see [8].

3. Some properties of differential operators

We turn now to differential operators defined in the Hilbert function space $L^2((a,b);w)$. Define the differential expression $M : D(M) \subset \mathrm{AC}_{\mathrm{loc}}(a,b) \to L^1_{\mathrm{loc}}(a,b)$ by

$$D(M) := \{f : (a,b) \to \mathbb{C} : f \in \mathrm{AC}_{\mathrm{loc}}(a,b)\}$$

and

$$M[f](x) := i\rho(x)f'(x) + \tfrac{1}{2}\,i\rho'(x)f(x) + q(x)f(x) \quad \text{for all } x \in (a,b). \tag{3.1}$$

With coefficient conditions as in (2.2) M is a Lagrange symmetric quasi-differential expression in the sense referred in [14]; see also [5].

We note that for any solution $y(\cdot,\lambda)$ of the differential equation (2.1) we have $y(\cdot,\lambda) \in D(M)$ and, for all $\lambda \in \mathbb{C}$,

$$M[y(\cdot,\lambda)] = \lambda w y(\cdot,\lambda) \quad \text{on } (a,b). \tag{3.2}$$

The differential expression M is Lagrange symmetric and has the following Green's formula, for all compact intervals $[\alpha,\beta] \subset (a,b)$,

$$\int_\alpha^\beta \{\bar{g}M[f] - \overline{fM[g]}\} = [f,g](x)|_\alpha^\beta \quad \text{for all } f,g \in D(M), \tag{3.3}$$

where the symplectic form $[\cdot,\cdot](\cdot) : (a,b) \times D(M) \times D(M) \to \mathbb{C}$ is defined by

$$[f,g](x) := i\rho(x)f(x)\bar{g}(x) \quad \text{for all } x \in (a,b). \tag{3.4}$$

For details of these results see the Everitt–Markus paper [5, Section 3, (3.4)–(3.7)].

The maximal differential operator $T_1 : D(T_1) \subset L^2((a,b);w) \to L^2((a,b);w)$ generated by the differential expression M in the space $L^2((a,b);w)$, is defined by

$$D(T_1) := \{f \in D(M) : f, w^{-1}M[f] \in L^2((a,b);w)\}$$

and

$$T_1 f := w^{-1}M[f] \quad \text{for all } f \in D(T_1).$$

We note that, from Green's formula (3.3) we have the result that the limits

$$[f,g](a^+) := \lim_{x \to a^+} [f,g](x) \quad [f,g](b^-) := \lim_{x \to b^-} [f,g](x) \tag{3.5}$$

both exist in $\mathbb{C}$ and are finite for all $f, g \in D(T_1)$.

The minimal differential operator $T_0 : D(T_0) \subset L^2((a,b);w) \to L^2((a,b);w)$, generated by the differential expression M in the space $L^2((a,b);w)$, is defined by

$$D(T_0) := \{f \in D(T_1) : [f,g](b^-) - [f,g](a^+) = 0 \quad \text{for all } g \in D(T_1)\}$$

and

$$T_0 f := w^{-1}M[f] \quad \text{for all } f \in D(T_0).$$

The operators T_0 and T_1 have the properties, where T^* denotes the Hilbert space adjoint of the operator T,

(i) $T_0 \subseteq T_1$.   (ii) T_0 is closed and symmetric in $L^2((a,b);w)$.

(iii) T_1 is closed in $L^2((a,b);w)$.   (iv) $T_0^* = T_1$ and $T_1^* = T_0$.

For reference to these results see [5, Section 3].

If there are any self-adjoint operators T in $L^2((a,b);w)$ generated by the expression M then all such operators have to satisfy the inclusion relation

$$T_0 \subseteq T = T^* \subseteq T_1 = T_0^*.$$

From the general theory of unbounded operators in Hilbert space, see [14, Chapter IV], such self-adjoint operators exist if and only if the deficiency indices (d^-, d^+) of T_0 are equal, see [14, Chapter IV, Section 14.8, Theorem 8]. The deficiency indices of T_0 are defined by

$$\begin{aligned} d^{\pm} &= \dim\{f \in D(T_0^*) : T_0^* f = \pm i f\} = \dim\{f \in D(T_1) : T_1 f = \pm i f\} \\ &= \dim\{y \in D(M) : M[y] = \pm i w y \text{ on } (a,b) \text{ and } y(\cdot, \pm i) \in L^2((a,b);w)\}. \end{aligned} \tag{3.6}$$

From this last representation (3.6), since the differential equation (3.2) is of the first order, it follows that $0 \leqslant d^{\pm} \leqslant 1$. Thus for self-adjoint extensions of T_0 to exist there are only two possibilities:

$$\text{(i) } d^- = d^+ = 0, \quad \text{(ii) } d^- = d^+ = 1. \tag{3.7}$$

Proposition 3.1. *Let the deficiency indices* (d^-, d^+) *of the minimal operator* T_0 *be equal; then we have (since* $0 \leqslant d^{\pm} \leqslant 1$*)*

(i) $d^- = d^+ = 0$ if and only if for some $c \in (a,b)$, $w/\rho \notin L^1(a,c]$ and $w/\rho \notin L^1[c,b)$,

(ii) $d^- = d^+ = 1$ if and only if $w/\rho \in L^1(a,b)$.

Proof. These results follow from Proposition 2.1.

4. The generalized GKN theory

In this section we investigate the two cases of Proposition 3.1; in particular the second case gives the possibility to deal with the problem of defining self-adjoint extensions of closed symmetric operators in Hilbert space. Accordingly, we have

Remark 4.1. 1. In case (i) of (3.7), if we define the operator T by $T := T_0^* = T_0$, then T satisfies $T^* = T$ and T is the (unique) self-adjoint operator in $L^2((a,b);w)$ generated by the differential expression M of (3.1). The self-adjoint boundary value problem, in this case, consists only of the differential equation (2.1); no boundary conditions at the endpoints a and b are required to determine the domain $D(T)$.

2. In case (ii) of (3.7), the general Stone/von Neumann theory of self-adjoint extensions of closed symmetric operators in Hilbert space, see [14, Chapter V, Sections 14.7 and 14.8], proves that there is a continuum of self-adjoint extensions $\{T\}$ of the minimal operator T_0, with $T_0 \subset T \subset T_1$. These extensions can be determined by use of the generalised GKN theory for differential operators as given in [5, Section 4, Theorem 1]. The domain of any self-adjoint extension T of T_0 can be obtained as a restriction of the domain of the maximal operator T_1, see [5, Section 4, (4.2) and (4.3)]. These restrictions are obtained by choosing an element $\beta \in D(T_1)$ such that β arises from a non-null member of the quotient space $D(T_1)/D(T_0)$ with the symmetric property, recalling (3.5),

$$[\beta,\beta](b^-) - [\beta,\beta](a^+) = 0. \tag{4.1}$$

With this boundary condition function $\beta \in D(T_1)$ the domain $D(T)$ is now determined by

$$D(T) := \{f \in D(T_1) : [f,\beta](b^-) - [f,\beta](a^+) = 0\} \tag{4.2}$$

and the self-adjoint operator defined by

$$Tf := w^{-1}M[f] \quad \text{for all } f \in D(T). \tag{4.3}$$

All such self-adjoint extensions T are defined using this method; indeed there is a one-to one mapping between the set $\{T\}$ and the set $\{\beta\}$ of all non-null elements of the quotient space $D(T_1)/D(T_0)$ satisfying the symmetric condition (4.1).

5. Remarks on the spectrum of the operators

The following two remarks concern the spectral properties of self-adjoint extensions T of the minimal operator T_0.

Remark 5.1. *Case* (i) of *Proposition* 3.1. From Proposition 2.1, this case is satisfied if and only if $w/\rho \notin L^1(a,c]$ and $w/\rho \notin L^1[c,b)$ and this condition on the coefficients implies that the differential equation (2.1) has the property that for all nontrivial solutions $y(\cdot,\lambda) \notin L^2((a,b);w)$, for all $\lambda \in \mathbb{C}$.

Let the spectrum of the self-adjoint operator T be denoted by $\sigma(T)$; then it follows that $\sigma(T)$ contains no eigenvalues since, for real $\lambda \in \mathbb{R}$, the differential equation has no solution in $L^2((a,b);w)$; thus the boundary value problem in this case has no eigenvalues; in fact it follows from results in [14, Chapters IV and V] that the spectrum of T is purely continuous and occupies the whole real line, i.e., $\sigma(T) = C\sigma(T) = \mathbb{R}$. We note that this case can give no examples of interest for sampling and interpolation theories.

An example of this case considered in the space $L^2(-\infty, +\infty)$ is

$$iy'(x) = \lambda y(x) \quad \text{for all } x \in (-\infty, +\infty).$$

Remark 5.2. *Case* (ii) of *Proposition* 3.1. This case covers all regular cases of the differential equation (2.1), and all singular cases of the equation when the deficiency indices satisfy (ii) of (3.7), i.e., when the condition (2.5) is satisfied. The self-adjoint boundary value problem consists of considering the possibility of finding nontrivial solutions $y(\cdot, \lambda)$ of the differential equation

$$M[y(\cdot, \lambda)] = \lambda w y(\cdot, \lambda) \quad \text{on } (a, b) \tag{5.1}$$

with the property $y(\cdot, \lambda) \in L^2((a, b); w)$, that satisfy the boundary condition

$$[y(\cdot, \lambda), \beta](b^-) - [y(\cdot, \lambda), \beta](a^+) = 0. \tag{5.2}$$

The solution of this problem depends upon the nature of the spectrum $\sigma(T)$ of the self-adjoint operator T determined by the choice of the boundary condition element β.

If $\lambda \in \mathbb{C}$ can be found such that both (5.1) and (5.2) are satisfied then λ is an eigenvalue of the boundary value problem and the solution $y(\cdot, \lambda)$ is the corresponding eigenfunction. The point λ is then an eigenvalue of the self-adjoint operator T and hence $\lambda \in \mathbb{R}$. We note that this case can provide examples of interest for sampling and interpolation theories; for more information on this matter we refer to [8].

In this case (ii) of Proposition 3.1 it is shown in [8, Theorem 5.1] that the spectrum $\sigma(T)$ of any self-adjoint extension T, of T_0, is discrete, simple and has equally spaced eigenvalues on the real line of the complex spectral plane.

As an example of this general result we have the following boundary value problem which leads to the famous Whittaker–Shanon sampling and interpolation theorem; see [8, Theorem 7.1].

For $\sigma > 0$ consider the boundary value problem

$$iy'(x) = \lambda y(x) \quad (x \in [-\sigma, \sigma]), \tag{5.3}$$

$$y(-\sigma) = y(\sigma). \tag{5.4}$$

For Eq. (5.3) the general solution has a basis $\{\exp(-\mathrm{i}x\lambda) : x \in [-\sigma, \sigma] \text{ and } \lambda \in \mathbb{C}\}$; all solutions are in $L^2(I; w) \equiv L^2(-\sigma, \sigma)$ and thus from a combination of Propositions 2.1 and 3.1 we have $d^+ = d^- = 1$. The bilinear form for (5.3) is given by $[f, g](x) = \mathrm{i} f(x)\bar{g}(x)$ $(x \in [-\sigma, \sigma])$ and the symmetric boundary condition (5.4) which generates a self-adjoint operator T in $L^2(-\sigma, \sigma)$ can be rewritten in the form $[y, 1] = [y, 1](+\sigma) - [y, 1](-\sigma) = 0$ where 1 represents the unit function on $\mathbb{R}$ and is the boundary condition function β, i.e., $\beta(x) = 1$ for all $x \in [-\sigma, \sigma]$. The self-adjoint operator T for this example, is determined by (on using (4.2) and(4.3))

$$D(T) := \{f : [-\sigma, \sigma] \to \mathbb{C} : f \in \mathrm{AC}[-\sigma, \sigma],\ f' \in L^2(-\sigma, \sigma),\ [f, 1](+\sigma) - [f, 1](-\sigma) = 0\},$$

$$Tf := \mathrm{i}f' \quad \text{for all } f \in D(T).$$

A direct calculation from the classical formulation ((5.3) and (5.4)) of the problem, shows

$$\sigma(T) = \{\lambda_n = n\pi/\sigma : n \in \mathbb{Z}\};$$

this spectrum is discrete, simple and satisfies $\lim_{n\to\pm\infty}\lambda_n=\pm\infty$. The corresponding eigenfunctions are given by $\psi_n(x)=\exp(-\mathrm{i}xn\pi/\sigma)$ $(x\in[-\sigma,\sigma],\ n\in\mathbb{Z})$.

This example is considered again in [8].

6. A general example of a boundary condition function

As an example of a boundary condition function β, let $\mu\in\mathbb{R}$; then from the general form of solutions of Eq. (2.1) as given by (2.4) define, for some $\mu\in\mathbb{R}$,

$$\beta(x):=\frac{1}{\sqrt{\rho(x)}}\exp\left(\int_c^x\frac{\mu w-q}{\mathrm{i}\rho}\right)\quad\text{for all } x\in(a,b). \tag{6.1}$$

Then $\beta\in D(T_1)$ and from (3.4)

$$[\beta,\beta](x)=\mathrm{i}\rho(x)\frac{1}{\sqrt{\rho(x)}}\exp\left(\int_c^x\frac{\mu w-q}{\mathrm{i}\rho}\right)\frac{1}{\sqrt{\rho(x)}}\exp\left(\int_c^x\frac{\mu w-q}{-\mathrm{i}\rho}\right)=\mathrm{i}\quad\text{for all } x\in(a,b).$$

Thus

$$\lim_{x\to b^-}[\beta,\beta](x)=\mathrm{i}=\lim_{x\to a^+}[\beta,\beta](x)$$

and the symmetric condition (4.1) holds; also it follows that $\beta\notin D(T_0)$ so that β arises from a non-null member of the quotient space $D(T_1)/D(T_0)$, as required; it is shown in [8] that all symmetric boundary condition functions can be determined in the form (6.1). In particular for the boundary value problem (5.3) and (5.4) we have that $\beta(x)=1$ for all $x\in[-\sigma,\sigma]$.

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