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On Ideals of Tensor Products*

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In the first part of this paper we study prime ideals of the tensor product of the central closures of prime algebras. It is shown that some of the prime ideals are centrally generated. This extends a result of Nicholson and Watters [5], and is applied to obtain a simpler and more "visible" proof of Krempa's Lemma 3.4 in [2] which plays an important role in a description of radicals of tensor products. Using this lemma we extend, in the second part of the paper, the following recent result of Lawrence [3]: the tensor product of algebras, over an algebraically closed field, containing no nonzero algebraic ideals is Jacobson semisimple. Lawrence's proof is quite difficult; it is based on the "Units Theorem" [9] and several theorems of commutative algebra, in particular valuation theory. We present a simpler proof which is based only on the Units Theorem and avoids any use of the valuation machinery. Next we show the result remains valid for algebras over perfect fields. And then we prove that, for arbitrary fields, the Jacobson radical of the tensor product of algebras not containing nonzero algebraic ideals is equal to the prime radical.

Throughout this paper let F denote a field, and all algebras are F-algebras with identity. Unless stated specifically all tensor products are taken over the field F.

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1. Tensor Products and Extended Centroids

For a prime algebra A we denote by $M(A)$ the left Martindale ring of quotients of A, by $C(A)$ the extended centroid of A and by $Q(A) = AC(A)$ the central closure of A. Recall that $C(A)$ is the center of $M(A)$ and that $C(A)$ is a field. The main aim of this section is to show a close relationship between some prime ideals of $Q(A) \otimes B$ and $C(A) \otimes B$, where B is also a prime algebra. Although our results are stated for the central closure, one easily verifies that our proofs are also valid for subrings of the left Martindale ring of quotients that contain the central closure (for example, the normal closure and the symmetric Martindale ring of quotients). We choose to work with central closure because it is more convenient in the proofs of our applications.

Let A and B be F-algebras, and let V be a fixed basis of B. Then every nonzero element x of $A \otimes B$ can be uniquely presented in the form $\sum_{s \in S} a_s \otimes s$, where S is a finite subset of V and each $0 \neq a_s \in A$. This set S, called the support of x, is denoted by $\operatorname{supp}(x)$.

Lemma 1. *Let A and B be prime algebras, and V be a basis of B. Assume P is a nonzero ideal of $Q(A) \otimes B$. Suppose $0 \neq x = \sum_{s \in S} a_s \otimes s \in P$, where $0 \neq a_s \in Q(A)$ and $s \in V$, is such that for every $0 \neq y \in P$, if $\operatorname{supp}(y) \subseteq \operatorname{supp}(x) = S$ then $\operatorname{supp}(y) = \operatorname{supp}(x)$. Then, for every $v \in S$, there exist $0 \neq c_s \in C(A)$ such that $x = (a_v \otimes 1)(\sum_{s \in S} c_s \otimes s)$.*

Proof. Since $x(qa_v \otimes 1) - (a_v q \otimes 1)x \in P$, for every $q \in Q(A)$, it follows from the assumption that $a_s q a_v - a_v q a_s = 0$, for every $s \in S$. Hence it is well known (cf. [4]) that, for every $s \in S$, $a_s = c_s a_v$ for some $c_s \in C(A)$. Consequently $x = \sum_{s \in S} c_s a_v \otimes s = (a_v \otimes 1)(\sum_{s \in S} c_s \otimes s)$. ∎

Theorem 1.2. *Let A and B be prime algebras. If P is a prime ideal of $Q(A) \otimes B$ without nonzero elements of the form $a \otimes b$, $a \in Q(A)$, $b \in B$, then*

$$P = (Q(A) \otimes B)[P \cap (C(A) \otimes B)].$$

Proof. Let $0 \neq x = \sum_{s \in S} a_s \otimes s \in P$, where S is a finite subset of a basis V in B, and $0 \neq a_s \in Q(A)$ for every $s \in S$. We prove by induction on $|S|$ that $x \in (Q(A) \otimes B)[P \cap (C(A) \otimes B)]$.

In case $\operatorname{supp}(x)$ satisfies the assumption of Lemma 1.1,

$$x = (a_v \otimes 1)\left(\sum_{s \in S} c_s \otimes s\right),$$

where v is a chosen element in S and every $c_s \in C(A)$. Consequently,

$$(a_v \otimes 1)(Q(A) \otimes B)\left(\sum_{s \in S} c_s \otimes s\right) \subseteq P.$$

Since P is a prime ideal, it follows from the assumption that $\sum_{s \in S} c_s \otimes s \in P$. So $x \in (Q(A) \otimes B)[P \cap (C(A) \otimes B)]$.

For an arbitrary $0 \neq x \in P$ there exists a nonzero element $x' = \sum_{s \in S'} a'_s \otimes s \in P$, where every $a'_s \in Q(A)$, such that $\operatorname{supp}(x') = S' \subseteq \operatorname{supp}(x)$ satisfies the assumption of Lemma 1.1. Hence if $v \in S'$ then

$$x' = (a'_v \otimes 1)\left(\sum_{s \in S'} c_s \otimes s\right),$$

for some $0 \neq c_s \in C(A)$. By the first part of this proof, $\sum_{s \in S'} c_s \otimes s \in P \cap (C(A) \otimes B)$. Now $y = x - (a_v c_v^{-1} \otimes 1)(\sum_{s \in S'} c_s \otimes s) = \sum_{s \in S' \setminus \{v\}} (a_s - a_v c_v^{-1} c_s) \otimes s + \sum_{s \in S \setminus S'} a_s \otimes s$. The induction hypothesis yields $y \in (Q(A) \otimes B)[P \cap (C(A) \otimes B)]$ and thus $x \in (Q(A) \otimes B)[P \cap (C(A) \otimes B)]$. ∎

COROLLARY 1.3. *Let A and B be prime algebras and P be a prime ideal of $Q(A) \otimes Q(B)$ not containing nonzero elements of the form $a \otimes b$, $a \in Q(A)$ and $b \in Q(B)$. Then $P = (Q(A) \otimes Q(B))(P \cap Z)$, where $Z = C(A) \otimes C(B)$ is the center of $Q(A) \otimes Q(B)$.*

Proof. Note that $Z = C(A) \otimes C(B)$ (see, for example, [1]). One easily checks that $P \cap (C(A) \otimes Q(B))$ is a prime ideal of $C(A) \otimes Q(B)$. Now the result follows directly from Theorem 1.2. ∎

We now give some applications of the last theorem.

COROLLARY 1.4 (Nicholson and Watters [5]). *Let A and B be prime algebras. The following are equivalent:*

(1) *$C(A) \otimes C(B)$ is a field;*

(2) *each nonzero ideal of $A \otimes B$ contains a nonzero element of the form $a \otimes b$, $a \in A$, $b \in B$.*

Proof. We first show that (1) implies (2). Suppose I is a nonzero ideal of $A \otimes B$ without nonzero elements of the form $a \otimes b$, $a \in A$, $b \in B$. Let $J = (Q(A) \otimes Q(B))\, I(Q(A) \otimes Q(B))$. One easily verifies that also J does not contain nonzero elements of the form $x \otimes y$, $x \in Q(A)$, and $y \in Q(B)$ (here it is useful that $Q(A)$ is the central closure and not the left Martindale ring of quotients). Let P be an ideal containing J and maximal for the condition not to contain elements of the form $x \otimes y$. It follows that P is a prime ideal.

So, by Corollary 1.3, $P=(Q(A)\otimes Q(B))[P\cap(C(A)\otimes C(B))]$. Therefore, by the assumption, $P=Q(A)\otimes Q(B)$, a contradiction.

The converse is proved similarly as in [5]. ▮

COROLLARY 1.5 (Krempa [2]). *Suppose F is of prime characteristic p and L is a purely inseparable field extension of F of degree p^n. If A is a prime F-algebra and I is an ideal of $L\otimes A$ such that $I\cap A=0$, then $I^{p^n}=0$.*

Proof. A simple induction argument allows us to assume that the field extension $F\subseteq L$ is simple, i.e., $L=F(\varepsilon)$, where ε is a root of an irreducible polynomial $X^{p^n}-a$ of $F[X]$.

As in the proof of Corollary 1.4 we first extend I to an ideal of $L\otimes Q(A)$. This ideal, denoted I^e, satisfies $I^e\cap Q(A)=0$. Let P be an ideal containing I^e and which is maximal for not intersecting $Q(A)$. Since $Q(A)$ is a prime algebra one easily verifies that P is a prime ideal, and clearly it does not contain nonzero elements of the form $l\otimes a$, $l\in L$ and $a\in Q(A)$. Because of Theorem 1.2 we obtain $P=(L\otimes Q(A))[P\cap(L\otimes C(A)]$. Now $L\otimes C(A)\cong F[X]/(X^{p^n}-a)\otimes_F C(A)\cong C(A)[X]/(X^{p^n}-a)$. If $X^{p^n}-a$ is an irreducible polynomial in $C(A)[X]$, then $C(A)[X]/(X^{p^n}-a)$ is a field, so $P\cap(L\otimes C(A))=0$. If $\alpha\in C(A)$ is a root of the polynomial $X^{p^n}-a$ and J is the image of the ideal of $C(A)[X]$ generated by $X-\alpha$ in $C(A)[X]/(X^{p^n}-a)$, then $J^{p^n}=0$. Now $(C(A)[X]/(X^{p^n}-a))/J\cong C(A)$ implies that J is a maximal ideal of $L\otimes C(A)$. Since $P\cap(L\otimes C(A))$ is a prime ideal of $L\otimes C(A)$, we obtain that $0=J^{p^n}\subseteq(P\cap(L\otimes C(A))$, and thus $J=P\cap(L\otimes C(A))$. The result follows. ▮

2. ON THE JACOBSON RADICAL OF TENSOR PRODUCTS

For a ring R we denoty by $\mathcal{J}(R)$ the Jacobson radical of R. Furthermore, the centralizer of a subset A of R is denoted $C_R(A)$. We need the following two elementary lemmas.

LEMMA 2.1. *For every subring A of a ring R, $\mathcal{J}(R)\cap C_R(A)\subseteq\mathcal{J}(C_R(A))$.*

LEMMA 2.2. *Let A and B be algebras and I an ideal of $A\otimes B$. If $e_1,...,e_n\in B$, then $\{x\in A\mid x\otimes e_1+a_2\otimes e_2+\cdots a_n\otimes e_n\in I$ for some $a_2,...,a_n\in A\}$ is an ideal of A.*

We also need the following "Units Theorem."

THEOREM 2.3 (Sweedler [9]). *Suppose F is algebraically closed. If A and B are commutative algebras in which F is algebraically closed, then every unit of $A \otimes B$ is of the form $a \otimes b$ for some $a \in A$ and $b \in B$.*

Using the three previous results we now give a simple proof of Lawrence's theorem [3]. Recall that an ideal I of an algebra A is called algebraic if every element of I is algebraic over F.

THEOREM 2.4. *Suppose F is an algebraically closed field. If A and B are algebras both different from F and without nonzero algebraic ideals, then $\mathcal{J}(A \otimes B) = 0$.*

Proof. We first reduce the problem to the commutative case. For this assume that $\mathcal{J}(A \otimes B) \neq 0$. Fix a basis E of B and let $0 \neq x = a_1 \otimes e_1 + \cdots + a_n \otimes e_n \in \mathcal{J}(A \otimes B)$, where the e_i's are distinct elements of E, the a_i's are nonzero elements of A and n is minimal. By Lemma 2.2 we may assume that a_1 is not algebraic over F. Moreover, the minimality of x implies that for every i, $x(a_i \otimes 1) - (a_i \otimes 1)x = 0$, so the a_i's are pairwise commuting. Let R be the subalgebra of A generated by $a_1, ..., a_n$. Because of Lemma 2.1, $x \in \mathcal{J}(A \otimes B) \cap C_{(A \otimes B)}(R \otimes 1) = \mathcal{J}(A \otimes B) \cap (C_A(R) \otimes B) \subseteq \mathcal{J}(C_A(R) \otimes B)$. Let Z be the center of $C_A(R)$. By an argument using bases over F, one easily verifies that $C_{A \otimes B}(C_A(R) \otimes 1) = Z \otimes B$. Applying Lemma 2.1 once more, one obtains that $x \in \mathcal{J}(C_A(R) \otimes B) \cap C_{A \otimes B}(C_A(R) \otimes 1) \subseteq \mathcal{J}(Z \otimes B)$. Hence, there exists a commutative subalgebra Z of A such that $x \in \mathcal{J}(Z \otimes B)$. Now, as a_1 is transcendental over F, the set $S = F[a_1] \backslash 0$ is multiplicatively closed. Let P be an ideal of Z maximal for not intersecting S. Clearly P is prime and therefore $D = Z/P$ is a commutative domain. Consequently, there exists a commutative domain D different from F such that $\mathcal{J}(D \otimes B) \neq 0$. A similar reduction applied to B allows us to assume that B is a commutative domain different from F. Hence we have reduced the problem to tensor products of commutative domains.

So assume that A and B are commutative domains, both different from F. Suppose $0 \neq j \in \mathcal{J}(A \otimes B)$. Then because of Theorem 2.3, $j = 1 \otimes 1 + a \otimes b$, for some $a \in A$, $b \in B$. Now if both a and b are algebraic over F, then, because F is algebraically closed in both algebras, it follows that $a, b \in F$ and thus $j = 1 \otimes (1 + ab)$ is invertible; a contradiction. So say b is transcendental over F. Because $F \neq A$ and F is algebraically closed, there exists a transcendental element t in A. Consider the element $j + (t \otimes b^2) j \in \mathcal{J}(A \otimes B)$. Again by Theorem 2.3, there exists $x \in A$ and $y \in B$ such that $j + j(1 \otimes b^2) = 1 \otimes 1 + x \otimes y$. This implies, $a \otimes b + t \otimes b^2 + ta \otimes b^3 = x \otimes y$. Extending the independent set $\{b, b^2, b^3\}$ to an F-basis of B, we can write $y = \alpha b + \beta b^2 + \gamma b^3 + v$, where v is in the linear complement of the subspace generated by $\{b, b^2, b^3\}$. Substituting the latter in the previous equation, one obtains $\alpha x = a$ and $\gamma x = ta$. However, these two equations yield that $\alpha\gamma x = \alpha ta$ and thus $\gamma = \alpha t$. But then (since $\alpha \neq 0$) we obtain $t \in F$, a contradiction. This finishes the proof. ▮

It is known that [7] if a field K is a Galois extension of F, then every F-invariant ideal I of $A \otimes K$ is of the form $I = (I \cap A) \otimes K$. Hence, if A has no nonzero algebraic ideals and F is a perfect field with algebraic closure $\bar{F}$, then $A \otimes \bar{F}$ is also without nonzero algebraic ideals. Similarly $B \otimes \bar{F}$ is without algebraic ideals when F is perfect. Hence by Theorem 2.4 $\mathscr{J}(A \otimes B \otimes \bar{F}) \cong \mathscr{J}((A \otimes \bar{F}) \otimes_{\bar{F}} (B \otimes \bar{F})) = 0$. Consequently, since $\mathscr{J}(A \otimes B \otimes \bar{F}) \cap (A \otimes B) = \mathscr{J}(A \otimes B)$ (cf. [6, Theorem 7.2.11]), we obtain that $\mathscr{J}(A \otimes B) = 0$. So we have extended the theorem as follows.

Corollary 2.5. *Suppose F is a perfect field. If A and B are algebras both different from F and without nonzero algebraic ideals, then $\mathscr{J}(A \otimes B) = 0$.*

For an arbitrary field F of characteristic $p > 0$, let $\tilde{F}$ denote the purely inseparable closure of F, that is the field of all $x \in \bar{F}$ such that $x^{p^n} \in F$ for some $n > 0$. By $\mathcal{N}(R)$ we denote the sum of all nilpotent ideals of a ring R.

Corollary 2.6. *If A and B are algebras both different from F and without nonzero algebraic ideals, then $\mathscr{J}(A \otimes B) = \mathcal{N}(A \otimes B)$.*

Proof. If F has characteristic zero, then F is a perfect field and the result follows from Corollary 2.5. So we may assume $\operatorname{char}(F) = p > 0$. Since $\tilde{F}$ is a purely inseparable field extension it follows from Corollary 1.5 that both $A \otimes \tilde{F}/\mathcal{N}(A \otimes \tilde{F})$ and $B \otimes \tilde{F}/\mathcal{N}(B \otimes \tilde{F})$ are without nonzero algebraic ideals. Since $\tilde{F}$ is a perfect field, Corollary 2.5 implies that $\mathscr{J}\{(A \otimes \tilde{F}/\mathcal{N}(A \otimes \tilde{F})) \otimes_{\tilde{F}} (B \otimes \tilde{F}/\mathcal{N}(B \otimes \tilde{F}))\} = 0$. Consequently $\mathscr{J}((A \otimes \tilde{F}) \otimes_{\tilde{F}} (B \otimes \tilde{F})) = \mathcal{N}((A \otimes \tilde{F}) \otimes_{\tilde{F}} (B \otimes \tilde{F}))$; or equivalently $\mathscr{J}(A \otimes B \otimes \tilde{F}) = \mathcal{N}(A \otimes B \otimes \tilde{F})$. The result now follows from $\mathscr{J}(A \otimes B \otimes \tilde{F}) \cap (A \otimes B) = \mathscr{J}(A \otimes B)$. ▮

The following example shows that in the previous corollary the Jacobson radical is not necessarily trivial. Indeed, let $\mathbf{Z}_p$ be the field with p elements, $K = \mathbf{Z}_p(x, y)$ the field of rational functions in two variables, and $F = \mathbf{Z}_p(x^p)$. Clearly K is without nonzero algebraic ideals, but $0 \neq (x \otimes 1) - (1 \otimes x) \in \mathscr{J}(K \otimes_F K)$.

Remark. (a) In [2, 6] it was shown that if $\mathscr{J}(A \otimes K) = 0$ for every field extension $F \subseteq K$ and $\mathscr{J}(B) = 0$, then $\mathscr{J}(A \otimes B) = 0$. It is also true that if B and $A \otimes K$ have no nonzero algebraic ideals, for every field extension K of F, then $\mathscr{J}(A \otimes B) = 0$. Indeed, because of the assumptions, A and

$A \otimes K$ are semiprime for every field extension K of F. Thus by [2, 8] $A \otimes B$ is also semiprime. Hence by Corollary 2.6 it follows that $\mathscr{J}(A \otimes B) = 0$.

(b) As a special case of [8, Theorem 4] one obtains that if $A \otimes K$ has no nonzero algebraic ideals for every finite field extension K of F and B has no nonzero nilideal, then also $A \otimes B$ has no nonzero algebraic ideals.

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