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A CRITERION FOR KNOTS OF PERIOD 3

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A new criterion for knots of period 3 is introduced. It is used to fill in some blank positions as well as to correct some errors in existing tables of periods.

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knot periodic skein polynomial

We introduce a criterion for periodicity of knots, designed specially to deal with the case of suspected period 3—the one most obstinate to treatment by other methods. (Some criteria involving the new polynomials were introduced by Murasugi [5], Przytycki [6] and the author [7], but none of them works for period 3. Also some older criteria, due to Murasugi do not work too well for period 3.) We will use the two-variable polynomial P as described in [4], that is defined by the fundamental relation

$$lP(L_+)+l^{-1}P(L_-)+mP(L_0)=0 \tag{1}$$

and the assumption $P(\text{trivial knot})=1$. Following Lickorish and Millett we denote by P_0 the part of P gathering all the terms of degree zero in m. Throughout this paper we will compute P_0 and P with coefficients in Z_3 rather than in Z. To avoid complicating the notation, *we will still denote the reduced polynomials by P_0 and P.*

Theorem 1. *Assume a knot K has period 3. Let $P_0(K)=\sum a_k l^{2k}$. Then for every k, $a_{3k+1}+a_{3k+2}=0$.*

Application. This simple observation yields a very handy criterion for knots of period 3. By applying this criterion we can see immediately that 10_3, 10_{10}, 10_{20}, 10_{32}, 10_{59}, 10_{69}, and 10_{164} have no period 3. Those are 7 out of 15 cases declared doubtful in the tables in [1]. The two cases of suspected period 7 were excluded earlier by Murasugi [5] and the author [7]. There are some discrepancies betweeen our results and the data in [1]: our criterion rules out the possibility of knots 10_4,

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10_{29}, 10_{34}, 10_{87}, 10_{91}, 10_{98}, 10_{99}, and 10_{135} being 3-periodic, while they are classified as such in [1].

The range of the introduced criterion may be expanded by applying it to appropriate cables of the considered knots. Computing the P_0 polynomial for 2-cables we can show that 10_{61}, and 10_{162} have no period 3—another two of the doubtful cases in the tables of Burde and Zieschang. The polynomials of the above mentioned knots are given in the Appendix.

It should be mentioned that 10_3, 10_4, 10_{10}, 10_{14}, 10_{20}, 10_{29}, 10_{32} and 10_{34} were in fact earlier excluded by Gordon, Litherland and Murasugi, because these knots are not 2-bridge knots; see [3].

The polynomial P_0 takes values in $Z_3[l^2, l^{-2}]$, which may be considered as a $Z_3[l^6, l^{-6}]$-module. It will be convenient to denote the set of polynomials satisfying the condition given in Theorem 1 by M. Then M is a $Z_3[l^6, l^{-6}]$-submodule of $Z_3[l^2, l^{-2}]$. The proof of Theorem 1 is divided into 3 parts. First we reduce the general case of an arbitrary periodic diagram to that of a closure of a periodic braid. Then this case is reduced to that of the standard closed braid presentation of the torus knot $t(k, 3)$. Finally torus knots are dealt with separately.

The first two reductions both utilize the following observations, which will be summarized as a proposition.

Let D be a 3-periodic diagram with a positive (negative) crossing c, so that we can write $D = D_{+++}$ $(D = D_{---})$, the subscripts referring to the sign of c and its images under the considered rotation. Then

$$l^3 P(D_{+++}) + l^{-3} P(D_{---}) + m^3 P(D_{000}) = 0. \tag{2}$$

This is easily proved by applying formula (1) and observing that $D_{-00} = D_{0-0} = D_{00-}$ and $D_{--0} = D_{-0-} = D_{--0}$ (because D is periodic), for an illustration see Fig. 1.

An obvious specialisation of (2) is the following

$$l^3 P_0(D_{+++}) + l^{-3} P_0(D_{---}) + m^3 P_{-3}(D_{000}) = 0, \tag{3}$$

where P_{-3} is the sum of all terms of m-degree equal to -3 in P. However, by [4, Proposition 22] the exponent of the lowest power of m which appears in $P(D_{000})$ is precisely $1-m$, where m is the number of components of D_{000}. It follows that

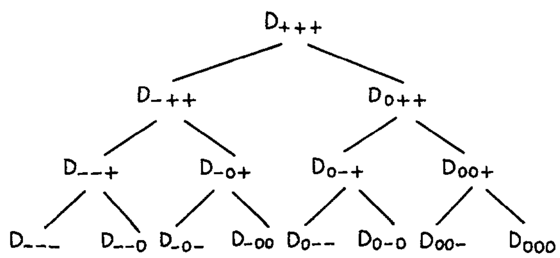

Fig. 1.

the last term in (3) is nontrivial only when D_{000} has four components. In this case three components of D_{000} form an orbit of the considered rotation and the fourth component is itself invariant (it is obvious that D_{000} is 3-periodic, one could however think that all four components may be themselves invariant; this cannot happen in fact, because in such case each of them would have to contain at least three of the arcs produced by smoothing the three crossings and there are altogether only six such arcs). Assume that D_{000} *has* four components. Let D_i denote the one that is invariant, D_p one of the remaining three and λ the total linking number of D_{000}. Using again [4, Proposition 22] we obtain the following.

Proposition 2. *If D_{000} has two components, then*

$$l^3 P_0(D_{+++}) + l^{-3} P_0(D_{---}) = 0. \tag{4}$$

If D_{000} has four components, then

$$l^3 P_0(D_{+++}) + l^{-3} P_0(D_{---}) + (-l^2)^{-\lambda}(-(l+l^{-1}))^3 P_0(D_i)(P_0(D_p))^3 = 0. \tag{5}$$

We will now reduce the proof of Theorem 1 to the case of periodic braids. That is we assume that the theorem holds for periodic braids and we will prove that it holds for arbitrary periodic diagrams, by induction on the number of crossings. We assume that the theorem holds for diagrams having less crossings than $D = D_{+++}$ (or D_{---}). In particular if D_{000} has four components then the theorem holds for D_i.

We now consider the two cases of Proposition 2.

Case 1. If D_{000} has two components, then formula (4) implies immediately that $P_0(D_{+++}) \in M$ if and only if $P_0(D_{---}) \in M$.

Case 2. If D_{000} has four components, then

$$(-l^2)^{-\lambda} = \pm l^{6k} \in Z_3[l^6, l^{-6}],$$

$$(-(l+l^{-1}))^3 = -l^3 - l^{-3},$$

$$(P_0(D_p))^3 \in Z_3[l^6, l^{-6}],$$

$$P_0(D_i) \in M.$$

It follows that the D_{000} contribution to D_{+++} (or D_{---}) computed from (5) is of the form $wP_0(D_i)$, where $w \in Z_3[l^6, l^{-6}]$, whence again $P_0(D_{+++}) \in M$ if and only if $P_0(D_{---}) \in M$.

To complete the reduction to the case of periodic braids it remains to prove the following.

Lemma 3. *Every periodic diagram D may be transformed into a periodic closed braid by a series of operations $D_{---} \to D_{+++}$, $D_{+++} \to D_{---}$ and equivariant isotopies in such a way that the number of crossings of the diagrams obtained in the process never surpasses the number of crossings of the initial diagram D.*

Proof. Let us consider a fundamental domain U of the considered $\frac{2}{3}\pi$ rotation (that is a $\frac{2}{3}\pi$ angle). We may assume that D meets the sides of the considered angle transversally. Assume that there is an arc A in D with its two ends X and Y lying on one side of the angle U, which does not meet the other side of U, see Fig. 2. (A is allowed to self-intersect.) We will say that such an arc is a *one-side arc.*

Assume that there is no other one-side arc with the ends lying *both between* X *and* Y; this amounts to choosing an appropriate arc from many one-side arcs that may exist. By reversing some of the crossings involving A, A may be transformed into an arc lying above everything else in U. By reversing some of the self-crossings of A it may be additionally arranged that A be a descending arc, see Fig. 3.

If analogous changes are simultaneously introduced in the other two domains, then the whole operation may be written as a series of operations $D_{---} \to D_{+++}$ and $D_{+++} \to D_{---}$. Now, an equivariant isotopy transforms the modified diagram into one in which the original arc A is replaced with a new arc, without self-intersections now, with the same ends X and Y but passing very closely to the side of U, and lying above everything else. It may also be easily arranged that the modified arc be involved in no more crossings than the original one (here we use the special choice of A from among other one-side arcs: A must have been involved in at least as many crossings as the number of points belonging to D and lying between X and Y, the modified arc may be arranged to have exactly that number of crossing points). We can now push our arc a little further, thus removing it altogether from U into the next domain, see Fig. 4.

Again, the number of crossings is not increased and the operation may be extended to an equivariant isotopy. Since the number of intersections of D with the side of

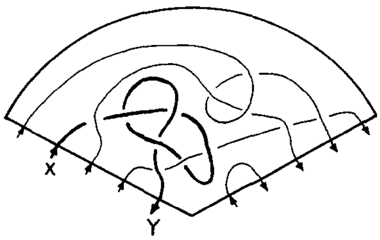

Fig. 2.

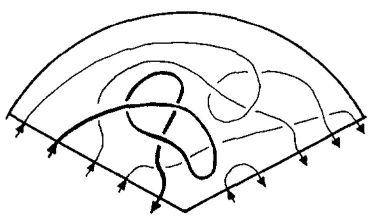

Fig. 3.

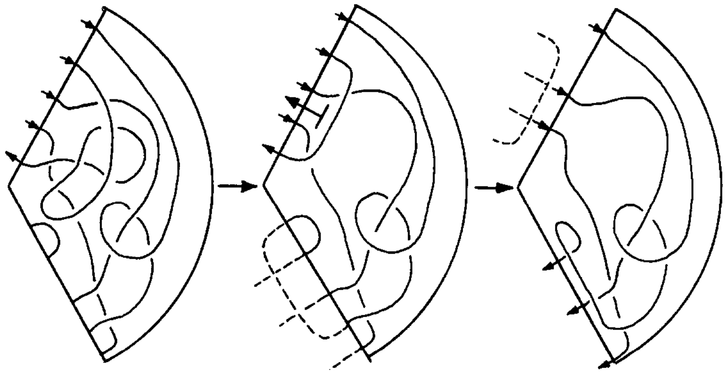

Fig. 4.

U was decreased, it follows that by repeating this procedure we can reduce the general case to one when every arc entering U from one side must leave it by the other side. It is easily observed that in this situation passing from one domain into the next one is possible only in one direction, see Fig. 5 (remember that we assumed D to represent a knot, not a multi-component link).

A similar procedure transforms such a diagram into a periodic braid.

We are now reduced to showing that $P(\gamma^3) \in M$, where γ^3 is a braid whose closure represents a knot. We may as well assume that the permutation induced by γ is the cycle $(n, n-1, \ldots, 1)$ (if not we can replace γ with $\alpha\gamma\alpha^{-1}$, where the latter braid induces the required permutation). This will be reduced to showing that $P_0((\delta_1\delta_2 \cdots \delta_{n-1})^3) \in M$.

This second reduction is quite similar to the first, the main difference being that we apply the induction on the number of strings rather than on the number of crossings:

Theorem 1 is of course true for one-string periodic braids, and to make the inductive step let us first observe that D_i has less strings than D (D_i being the invariant component of the four component D_{000}) which reduces the inductive step to showing that γ^3 may be transformed into $(\delta_1\delta_2 \cdots \delta_{n-1})^3$ by a series of operations $D_{---} \to D_{+++}$, $D_{+++} \to D_{---}$ and equivariant isotopies, this time without increasing the number of strings at any stage. To do this it is enough to rearrange the

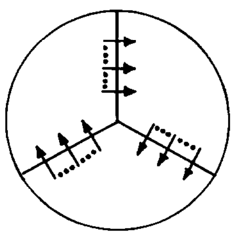

Fig. 5.

under/over-crossing information of the diagram represented by γ so that the first string would lie above all the other, the second above all but the first and so on. The modified braid is then isotopic (as a braid, that is without moving the ends) to $\delta_1\delta_2\cdots\delta_{n-1}$. Since the whole operation was performed without changing the situation on the side of our fundamental domain, it may be performed simultaneously in all three domains.

We are now reduced to proving that $P_0((\delta_1\delta_2\cdots\delta_{n-1})^3)\in M$ (for n not divisible by 3; otherwise the considered braid would represent a link of more than one component). This last stage of the argument is the only one in which the assumption that the considered period is equal to 3 is really used. The first two reductions still work for any odd prime p if M is replaced by a $Z_p[l^{2p}, l^{-2p}]$-submodule of $Z_p[l^2, l^{-2}]$ generated by the mod p polynomials of p-torus knots. It seems possible that these submodules may show enough regularity to provide working criteria for other odd primes.

To prove that $P_0((\delta_1\delta_2\cdots\delta_{n-1})^3)=P_0(t(n,3))\in M$ we will use a direct computational argument. In his paper [2] Jones gives an explicit formula for $X_{t(n,3)}(q,\lambda)$, where X is a certain version of the polynomial P, obtained from P by the substitution $l=i(\sqrt{\lambda}\sqrt{q})^{-1}$ and $m=-i(\sqrt{q}-1/\sqrt{q})$. Jones' formula reads as follows:

$$X(q,\lambda)=\frac{\lambda^{m-1}}{(1-q^3)(1-q^2)}((1-\lambda q^3)(1-\lambda q^2)-q^{m+1}(1+q)(1-\lambda q^2)(1-\lambda)$$

$$+q^{2m+2}(1-\lambda)(q-\lambda)).$$

We now want to return to our choice of variables. Since we are interested only in P_0 it is enough to evaluate X in $q=1$ (we *know* that X is well defined and continuous in $q=1$) and then substitute $l=i\lambda^{-1/2}$. The evaluation is probably best done by applying the de l'Hospital's rule twice. An easy computation shows that the P_0 polynomial with coefficients in Z is given by the following formula:

$$\pm P_0(t(n,3))=\frac{(n+1)(n+2)}{6}\,l^{2n-2}+\frac{(n-1)(n+1)}{3}\,l^{2n}+\frac{(n-2)(n-1)}{6}\,l^{2n+2}.$$

For the reduced polynomial $P_0\in M$ follows immediately. This completes the proof. □

Appendix

$P_0(10_3)=l^{-4}+l^2-l^6$, $\quad P_0(10_{10})=-l^{-4}-l^{-2}-1+l^2$,

$P_0(10_{20})=-1+l^2+l^6+l^8$, $\quad P_0(10_{32})=-l^{-2}-1-l^2$,

$P_0(10_{59})=-l^{-2}+1-l^2-l^6$, $\quad P_0(10_{69})=l^2+l^4+l^6-l^8$,

$P_0(10_{164})=l^{-2}+l^2$,

Let $\underline{10}_{61}$ $\underline{10}_{162}$ denote the 2-cables around 10_{61} and 10_{162} represented by the braids:

$$23124^{-1}5^{-1}3^{-1}4^{-1}(2^{-1}3^{-1}1^{-1}2^{-1})^2(6756)^3 4^{-1}5^{-1}3^{-1}4^{-1}(6756)^3 111$$

and

$$(2^{-1}3^{-1}1^{-1}2^{-1})^2 67564^{-1}5^{-1}3^{-1}4^{-1}(6756)^2 2312(4534)^2 67564^{-1}5^{-1}3^{-1}4^{-1}111.$$

Then

$$P_0(\underline{10}_{61}) = -l^8 - l^{12} + l^{16} + l^{18} + l^{20} + l^{22},$$
$$P_0(\underline{10}_{162}) = -l^8 - l^{10} - l^{12} - l^{16} - l^{18} + l^{22}.$$

Acknowledgement

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