ELSEVIER

Topology and its Applications 127 (2003) 393–408

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Local subgroupoids II: Examples and properties

Ronald Brown [a,*], İlhan İçen [b], Osman Mucuk [c]

[a] School of Informatics, Mathematics Division, University of Wales, Bangor, Gwynedd LL57 1UT, UK
[b] University of İnönü, Faculty of Science and Art, Department of Mathematics, Malatya, Turkey
[c] University of Erciyes, Faculty of Science and Art, Department of Mathematics, Kayseri, Turkey

Received 20 September 2000; accepted 19 March 2002

Abstract

The notion of local subgroupoid as a generalisation of a local equivalence relation was defined in a previous paper by the first two authors. Here we use the notion of star path connectivity for a Lie groupoid to give an important new class of examples, generalising the local equivalence relation of a foliation, and develop in this new context basic properties of coherence, due earlier to Rosenthal in the special case. These results are required for further applications to holonomy and monodromy.

MSC: 18F20; 18F05; 22E99; 22A22; 58H05

Keywords: Local equivalence relation; Local subgroupoid; Coherence; Holonomy; Monodromy

Introduction

Any foliation gives rise to a local equivalence relation, defined by the path components of local intersections of small open sets with the leaves. Local equivalence relations were generalised to local subgroupoids in a previous paper by the first two authors [2], referred to hereafter as paper I. In this paper we show that a basic topological groupoid notion, that of *identity star path component*, leads easily to a local subgroupoid of a wide class of Lie groupoids and this generalises the local equivalence relation of a foliation. We define local subgroupoids $c_1(Q, \mathcal{U})$ for certain open covers $\mathcal{U}$ of the object space of a Lie groupoid Q. Further, we show that the theory of coherence, which is prominent in the papers of

* Corresponding author.
E-mail addresses: r.brown@bangor.ac.uk (R. Brown), iicen@inonu.edu.tr (İ. İçen), mucuk@erciyes.edu.tr (O. Mucuk).

0166-8641/02/$ – see front matter
PII: S0166-8641(02)00101-3

Rosenthal [14,15], generalises nicely to the local subgroupoid case. The application of paper I to the holonomy and monodromy Lie groupoids of local subgroupoids (to be dealt with elsewhere) requires a condition on the local subgroupoid of having a 'globally adapted atlas'. We develop the 'coherence' theory by giving conditions on $\mathcal{U}$ for it to be globally adapted to the local subgroupoid $c_1(Q,\mathcal{U})$ (Corollary 7.9). This is related to the construction of a locally Lie groupoid from a foliation [3,12].

We also develop a similar theory for Lie groupoids Q with a 'path connection' Γ [4,13, 16] leading to a local subgroupoid $c_\Gamma(Q,\mathcal{U})$ for certain open covers $\mathcal{U}$.

This paper is strongly influenced by papers of Rosenthal [14,15] on local equivalence relations, a concept due originally to Grothendieck and Verdier [6] in a series of exercises presented as open problems concerning the construction of a certain kind of topos. The concept was investigated more recently by Kock and Moerdijk [9,10]. The main aims of the papers [6,9,10,14,15] are towards the connections with sheaf theory and topos theory.

The starting point of our investigation was to notice that an equivalence relation on X is a wide subgroupoid of the groupoid $X \times X$. However an equivalence relation is just one of the standard examples of a groupoid, and so it is natural to consider the theory corresponding to local equivalence relations but now for subgroupoids of a given groupoid Q, rather than just of the groupoid $X \times X$. The expectation is that this extended theory will allow applications to combinations of foliation and bundle theory, since a standard example of a Lie groupoid is the Ehresmann symmetry groupoid of a principal bundle [13]. For more information on abstract groupoids, see [1,7], and for the relationship with sheaves, see [8].

In the case Q is the indiscrete groupoid $X \times X$, we recover the well-known concept of *local equivalence relation*, which is related in [9,15] to foliation theory [5].

The remarkable fact is that the theory goes very smoothly, and so suggests it is a natural generalisation of the foliation case, and one which illuminates some constructions in that area.

1. Local subgroupoids: Definitions and examples

We first recall some definitions from [2].

Consider a groupoid Q on a set X of objects, and suppose also X has a topology. For any open subset U of X we write $Q|U$ for the full subgroupoid of Q on the object set U. Let $L_Q(U)$ denote the set of all subgroupoids of $Q|U$ with object set U (these are called *wide* subgroupoids of $Q|U$). For $V \subseteq U$, there is a restriction map $L_{UV}: L_Q(U) \to L_Q(V)$ sending H in $L_Q(U)$ to $H|V$. This gives L_Q the structure of presheaf on X.

We now interpret the sheaf $p_Q: \mathcal{L}_Q \to X$ constructed in the usual way from the presheaf L_Q.

For $x \in X$, the stalk ${p_Q}^{-1}(x)$ of $\mathcal{L}_Q$ has elements the germs $[U, H_U]_x$ where U is open in X, $x \in U$, H_U is a wide subgroupoid of $Q|U$, and the equivalence relation $\sim_x$ yielding the germs at x is that $H_U \sim_x K_V$, where K_V is wide subgroupoid of $Q|V$, if and only if there is a neighbourhood W of x such that $W \subseteq U \cap V$ and $H_U|W = K_V|W$. The topology on $\mathcal{L}_Q$ is the usual sheaf topology, with a sub-base of sets $\{[U,H]_x\colon x \in U\}$ for all open U of X and wide subgroupoids H of $G|U$.

Definition 1.1. A *local subgroupoid* of Q on the topological space X is a continuous global section of the sheaf $p_Q: \mathcal{L}_Q \to X$ associated to the presheaf L_Q.

Two standard examples of Q are $Q = X$, $Q = X \times X$, where $X \times X$ has the multiplication $(x, y)(y, z) = (x, z)$. In the first case, L_X is already a sheaf and $\mathcal{L}_X \to X$ is a bijection. More generally, we have:

Proposition 1.2. *If Q is a bundle of groups, then L_Q is a sheaf.*

Proof. By our assumption, if U is a subset of X then a wide subgroupoid $H|U$ of $Q|U$ is uniquely defined by the values $H(x)$ for all $x \in U$. This easily implies the usual two compatibility conditions for a sheaf. □

In the case Q is the indiscrete groupoid $X \times X$ the local subgroupoids of Q are the local equivalence relations on X, as mentioned in the Introduction. It is known that $L_{X \times X}$ is in general not a sheaf [14].

Definition 1.3. If G is a wide subgroupoid of the groupoid Q on X, then $\mathrm{loc}(G)$ is the local subgroupoid defined by

$$\mathrm{loc}(G)(x) = [X, G]_x.$$

This gives a wide and important class of local subgroupoids, but we are more interested in those which derive from connectivity considerations on a topological groupoid. For this we need to discuss the major way of specifying a local subgroupoid, namely in terms of atlases.

Definition 1.4. An *atlas* $\mathcal{U}_H = \{(U_i, H_i)\colon\ i \in I\}$ for a local subgroupoid consists of an open cover $\mathcal{U} = \{U_i\colon\ i \in I\}$ of X, and for each $i \in I$ a wide subgroupoid H_i of $Q|U_i$ such that the following compatibility condition holds:

> **Comp**(H): for all $i, j \in I$, $x \in U_i \cap U_j$ there is an open set W such that $x \in W \subseteq U_i \cap U_j$ and $H_i|W = H_j|W$.

The *local subgroupoid s of the atlas* is then well defined by $s(x) = [U_i, H_i]_x$, $x \in X$.

The above atlas is *compatible* with an atlas $\mathcal{U}'_{H'} = \{(U'_j, H'_j)\colon\ j \in J\}$ if for all $i \in I$, $j \in J$ and $x \in U_i \cap U'_j$ there is an open set W such that $x \in W \subseteq U_i \cap U'_j$ and $H_i|W = H'_j|W$. Clearly, two compatible atlases define the same local subgroupoid.

It is well known from general sheaf theory that any local subgroupoid has a compatible atlas. Note also that the atlas $\{(X, H)\}$ with a single element determines the local subgroupoid $\mathrm{loc}(H)$. So the atlas is a crucial part of the construction of a local subgroupoid s.

2. The star path component of a topological groupoid

A key concept for topological groups is the path component of the identity. The analogue for topological groupoids is the star identity path component.

Definition 2.1. Let Q be a topological groupoid. If $x \in \mathsf{Ob}(Q)$ we write $\mathrm{St}_Q\, x$ for the star of Q at x, namely the union of all the $Q(x, y)$ for $y \in \mathsf{Ob}(Q)$. The *star identity path component* $C_1(Q)$ of Q consists of all $g \in Q$ such that if $x = \alpha(g)$ then there is a path in $\mathrm{St}_Q\, x$ joining g to the identity 1_x. Such a path is called a *star path*. We say Q is *star path connected* if $Q = C_1(Q)$.

Proposition 2.2. *The star identity path component of Q is a subgroupoid of Q.*

Proof. Write C for this star identity component. Let $g \in Q(x, y)$, $h \in Q(y, z)$ and suppose also $g, h \in C$. Then there are paths l_t in $\mathrm{St}_Q\, x$, m_t in $\mathrm{St}_Q\, y$ such that $l_0 = g$, $l_1 = 1_x$, $m_0 = h$, $m_1 = 1_y$. Hence $g.m_t$ is a path in $\mathrm{St}_Q\, x$ joining gh to g, and this composed with l_t joins gh to 1_x. So C is closed under composition.

If $g \in C(x, y)$ and l_t joins g to 1_x then $g^{-1}l_{1-t}$ joins g^{-1} to 1_y. So C is a subgroupoid of Q. □

Note that [13, Example II.3.7, p. 46] gives an example where $C_1(Q)$ is not normal in Q. We will later need the following.

Proposition 2.3. *Let Q be star path connected and let $\mathcal{U}$ be an open cover of X. Then Q is generated by the subgroupoids $C_1(Q|U)$ for all $U \in \mathcal{U}$.*

Proof. Let $g \in \mathrm{St}_Q\, x$. Then there is a path λ in $\mathrm{St}_Q\, x$ from g to 1_x. Let $\mu = \beta\lambda$. By the Lebesgue covering lemma, we can write $\mu = \mu_1 + \cdots + \mu_n$ where each μ_r lie in an open set U_r of $\mathcal{U}$ for $r = 1, \ldots, n$. Then we can write $\lambda = \lambda_1 + \cdots + \lambda_n$ where $\beta\lambda_r = \mu_r$, $r = 1, \ldots, n$, and λ_r is a path in $\mathrm{St}_Q\, x \cap U_r$ for $r = 1, \ldots, n$. Let $g_r = -\lambda_r(0) + \lambda_r(1)$. Then $g_r \in C_1(Q|U_r)$ and $g = g_1 + \cdots + g_n$. □

Now we recall some major examples of Lie groupoids.

Example 2.4. Let $\mathcal{E}$ be a principal bundle $p : E \to B$ with group Ω. Then $E \times E$ is certainly a topological groupoid, and so also is its quotient $Q = E \times_\Omega E$ by the diagonal action of the topological group Ω. If $b, b' \in B$, we can by choosing a point in $p^{-1}(b)$, identify $Q(b, b')$ with the Ω-maps $p^{-1}(b) \to p^{-1}(b')$. For this reason, we also write $\mathsf{Sym}(\mathcal{E})$ or $\mathsf{Sym}(p)$ for Q. In the case $\mathcal{E}$ is locally trivial, and assuming X is a manifold, the topology may also be constructed from this alternative description, since the Ω-maps $p^{-1}(b) \to p^{-1}(b')$ may, again by choosing a point in $p^{-1}(b)$, be identified with the elements of Ω. It is this description we now use. Note also that the stars of Q are all homeomorphic to the original E. Thus if E is path connected, that Q is star path connected follows immediately.

Consider in particular the double cover of the circle $p : S^1 \to S^1$ given by $z \mapsto z^2$. In this case Ω is the cyclic group of order 2. Let $Q = \mathsf{Sym}(p)$.

This groupoid Q is star path connected. For suppose $g \in Q(z, w)$. Let λ be a path of shortest length in S^1 from z to w (if $z = -w$ then there are two such paths). Let $u \in S^1$ satisfy $u^2 = z$. Since p is a covering map, there are unique paths λ^+, λ^- starting at $u, -u$ and covering λ. Let $v = \lambda^+(1)$. Then g is a bijection $\{u, -u\} \to \{v, -v\}$. If $g(u) = v$, then the pair of paths λ^+, λ^- define a star path from the identity on $\{u, -u\}$ to g. If $g(u) = -v$, then such a path is determined by λ', the shortest path joining z to w in the opposite direction round S^1, and its corresponding lifts.

However, if $U = S^1$ with a single point removed, then $Q|U$ is not star-connected, since if λ is a path joining z to w in U, then λ' is not a path in U.

Example 2.5. Let Ω be a Lie group acting smoothly on the right of a $\mathcal{C}^r$-manifold X. Form the Lie action groupoid $Q = X \rtimes \Omega$. Even if Q is star path connected, this is not necessarily so for $Q|U$ for all open subsets U of X.

It would also be interesting to develop analogous concepts for connectivity rather than path-connectivity.

3. Local subgroupoids and star path connectivity

The previous notions give us our major examples of new and interesting local subgroupoids.

Example 3.1. Consider an equivalence relation E on the space X. Then for each open set U of X we have an equivalence relation $E|U$ on U and we can consider the partition of U given by the path components of the equivalence classes of $E|U$. In general, this will not give us a local equivalence relation. Instead we need to assume given an open cover $\mathcal{U} = \{U_i\colon i \in I\}$ of X satisfying the compatibility condition that for all $i, j \in I$, $x \in U_i \cap U_j$ there is an open set W such that $x \in W \subseteq U_i \cap U_j$ and the path components of $E|W$ are the intersections with W of the path components of the equivalence classes of $E|W$ the classes of each of $E|U_i$, $E|U_j$. The resulting local equivalence relation will be written $c_1(E, \mathcal{U})$. The compatibility condition is satisfied in for example equivalence relations given by the leaves of a foliation on a manifold, and is the standard example of the local equivalence relation defined by a foliation.

We now consider similar questions for topological groupoids.

Of course if G is a wide subgroupoid of Q, then so also is $C_1(G)$ and then $\mathrm{loc}(C_1(G))$ is a local subgroupoid of Q.

Suppose Q is *star path connected*, that is $Q = C_1(Q)$. Let $X = \mathsf{Ob}(Q)$ and let U be a subset of X. In general $Q|U$ need not be star path connected, as we show below. Further, while $C_1(Q|U) \subseteq C_1(Q)|U$, in general we do not have equality here. Such a condition is needed locally to obtain the local subgroupoid $c_1(Q, \mathcal{U})$ defined below.

Definition 3.2. An open cover $\mathcal{U} = \{U_i\colon i \in I\}$ of X is said to be *path compatible* with a topological groupoid Q on X if for all $i, j \in I$, $x \in U_i \cap U_j$ there is an open set W such that $x \in W \subseteq U_i \cap U_j$ and

$$C_1(Q|U_i)|W = C_1(Q|U_j)|W.$$

In this case, the local subgroupoid $c_1(Q, \mathcal{U})$ is defined to have value $[U_i, C_1(Q|U_i)]_x$ at $x \in U_i$.

The next proposition gives useful sufficient conditions for $c_1(Q,\mathcal{U})$ to be defined.

Proposition 3.3. *Let Q be a topological groupoid on X and suppose there is an open cover $\mathcal{U} = \{U_i\colon\ i \in I\}$ of X such that for all $i, j \in I$ and $x \in U_i \cap U_j$ there is an open set W_x such that $x \in W_x \subseteq U_i \cap U_j$ and there are groupoid retractions $r_{i,W_x} : Q|U_i \to Q|W_x$, $r_{j,W_x} : Q|U_j \to Q|W_x$ over retractions $U_i \to W_x$, $U_j \to W_x$. Then a local subgroupoid $c_1(Q,\mathcal{U})$ is well defined by for $i \in I$, $x \in U_i$, $x \mapsto [U_i, C_1(Q|U_i)]_x$.*

Proof. The retractions ensure the compatibility condition, since if $x, y \in W$ and if λ is a path in $\mathrm{St}_{Q|U_i}$ joining 1_x to the element $g : x \to y$ of $Q|W$, then $r_{i,W}\lambda$ is a path in $\mathrm{St}_{Q|W}$ joining 1_x to g. So $C_1(Q|U_i)|W = C_1(Q|W)$, and similarly for j. □

Let Q be a topological groupoid on X. Then Q is called *locally trivial* if for all $x \in X$ there is an open set U containing x and a section $s : U \to \mathrm{St}_G\, x$ of β. Thus $\beta s = 1_U$ and for each $y \in U$, $\alpha(s(y)) = x$, i.e., $s(y) : x \to y$ in Q. We recall the following standard result (see for example [13]).

Proposition 3.4. *Let Q be a topological groupoid on X and U be an open subset of X. If $s : U \to \mathrm{St}_Q\, x$ is a continuous section of β for some $x \in U$, then the topological groupoid $Q|U$ is topologically isomorphic to the product groupoid $Q(x) \times (U \times U)$, and if $x \in W \subseteq U$, then any retraction $U \to W$ is covered by a retraction $Q|U \to Q|W$.*

Proof. Remark that the groupoid multiplication on $Q(x) \times (U \times U)$ is defined by

$$\big(g, (y, z)\big)\big(h, (z, w)\big) = \big(gh, (y, w)\big).$$

Define

$$\phi : Q|U \to Q(x) \times (U \times U), \qquad g \mapsto \big(s(y)gs(z)^{-1}, (y, z)\big)$$

where $y = \alpha(g)$ and $z = \beta(g)$. Since s is continuous, ϕ is clearly an isomorphism of topological groupoids.

The last part follows easily. □

We have emphasised these results, despite their simple proofs, because they have useful applications for example to manifold and bundle theory.

If s is a local subgroupoid of Q defined by an atlas $\mathcal{U} = \{(U_i, H_i)\colon\ i \in I\}$ and U is an open subset of X then $s|U$ is the local subgroupoid of $Q|U$ defined by the atlas $\mathcal{U} \cap U = \{(U_i \cap U, H_i|(U_i \cap U))\colon\ i \in I\}$. It is easy to verify this is an atlas, and as a section $s|U$ is just the restriction of s to the open subset U.

Suppose now that we have the local subgroupoid $c_1(Q,\mathcal{U})$ defined by the open cover $\mathcal{U}$, and U is an open subset of X. We will later need a result which follows easily from compatibility:

Proposition and Definition 3.5. *The equality*

$$c_1(Q,\mathcal{U})|U = c_1(Q|U, \mathcal{U} \cap U)$$

holds if for any $i, j \in I$ and $x \in U_i \cap U_j \cap U$ there is an open set W such that $x \in W \subseteq U_i \cap U_j \cap U$ and $C_1(Q|U_i \cap U)|W = C_1(Q|U_j \cap U)|W$. If this condition holds for all open sets U of X, then we say that the cover $\mathcal{U}$ is path local for $c_1(Q, \mathcal{U})$.

Remark 3.6. There is a variation of the local subgroupoid $c_1(Q, \mathcal{U})$ in which the paths in Q which are used are controlled, for example to belong to a given class, or to derive from the paths in X in a specified way. We give an example of this in the next section.

4. Path connections

The purpose of this section is to give new examples of local subgroupoids with a possibility of working towards relating the concepts of holonomy in foliation theory and in bundle theory.

Let $\Lambda(X)$ denote the path space of a topological space X. Let Q be a topological groupoid over X. A *path connection* [4,13,16] Γ in Q is a continuous map

$$\Gamma : \Lambda(X) \to \Lambda(Q), \qquad \lambda \mapsto \Gamma(\lambda)$$

satisfying the following conditions

(i) $\alpha(\Gamma(\lambda)(t)) = \lambda(0)$ and $\beta(\Gamma(\lambda)(t)) = \lambda(t)$, $t \in [0,1]$;

(ii) the transport condition: If

$$\psi : [0,1] \to [t_0, t_1] \subseteq [0,1]$$

is a homeomorphism, then

$$\Gamma(\lambda) \circ \psi = \Gamma(\lambda)\big(\psi(0)\big) \circ \Gamma(\lambda\psi).$$

The second condition means

$$\Gamma(\lambda)\big(\psi(t)\big) = \Gamma(\lambda)\big(\psi(0)\big) \circ \Gamma(\lambda\psi)(t)$$

for $t \in [0,1]$,

By taking the homeomorphism ψ to be the identity map $\psi : [0,1] \to [0,1]$ it follows from the condition (ii) that $\Gamma(\lambda)(0) = 1_{\lambda(0)}$. Let $\lambda, \mu \in \Lambda(X)$ and $\lambda(1) = \mu(0)$, that is the composition $\lambda + \mu$ is defined, then we have $\lambda = (\lambda + \mu) \circ \psi_0$ and $\mu = (\lambda + \mu) \circ \psi_1$ where $\psi_0(t) = \frac{1}{2}t$ and $\psi_1(t) = \frac{1}{2}t + \frac{1}{2}$. Moreover applying (ii) to the path $\lambda + \mu$ and ψ_0 and then applying to $\lambda + \mu$ and ψ_1 we obtain

$$\Gamma(\lambda + \mu)(t) = \begin{cases} \Gamma(\lambda)(2t) & 0 \leqslant t \leqslant \frac{1}{2}, \\ \Gamma(\lambda)(1) \circ \Gamma(\mu)(2t-1) & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

In particular

$$\Gamma(\lambda + \mu)(1) = \Gamma(\lambda)(1) \circ \Gamma(\mu)(1).$$

Let Q be a topological groupoid on X with a continuous path connection $\Gamma : \Lambda(X) \to \Lambda(Q)$. Let $C_\Gamma(Q)$ be the set of all $g \in Q$ such that if $\alpha(g) = x$ then there is a path λ in X such that $\Gamma(\lambda)$ joins g to the identity 1_x at x, that is, $\Gamma(\lambda)(0) = 1_x$ and $\Gamma(\lambda)(1) = g$. Then we prove the following proposition.

Proposition 4.1. *$C_\Gamma(Q)$ is a wide subgroupoid of Q.*

Proof. Let $g,h \in C_\Gamma(Q)$ such that gh is defined in Q. Then there are paths λ and μ joining g to $1_{\alpha(g)}$ and h to $1_{\alpha(h)}$, respectively. Here note that $\lambda(0)=\alpha(g)$, $\lambda(1)=\beta(g)$ and $\mu(0)=\alpha(h)$, $\mu(1)=\beta(h)$. So the composition $\lambda+\mu$ of the paths is defined and $\Gamma(\lambda+\mu)(0)=\Gamma(\lambda)(0)=1_{\alpha(g)}$ and $\Gamma(\lambda+\mu)(1)=\Gamma(\lambda)(1)\circ\Gamma(\mu)(1)=gh$. So $gh\in C_\Gamma(Q)$. That means $C_\Gamma(Q)$ is closed under the groupoid composition.

If $g\in C_\Gamma(Q)$ with $\alpha(g)=x$ then there is a path λ in X such that $\Gamma(\lambda)(0)=1_x$ and $\Gamma(\lambda)(1)=g$. Define a path $\bar\lambda$ in X such that $\bar\lambda(t)=\lambda(1-t)$. Then $\bar\lambda(t)=(\lambda\psi)(t)$ with $\psi(t)=1-t$. By the transport law we have $\Gamma(\lambda)(\psi(t))=\Gamma(\lambda)(1)\circ\Gamma(\bar\lambda)(t)$ where $\Gamma(\lambda)(1)=g$. So we have

$$\Gamma\big(\bar\lambda\big)(0)=g^{-1}\Gamma(\lambda)(1)=g^{-1}g=1_y$$

and

$$\Gamma\big(\bar\lambda\big)(1)=g^{-1}\circ\big(\Gamma(\lambda)(0)\big)=g^{-1}\circ 1_x=g^{-1}.$$

So $g^{-1}\in C_\Gamma(Q)$. Hence $C_\Gamma(Q)$ is a wide subgroupoid of Q. □

We also need an analogue of Proposition 2.3.

Proposition 4.2. *If Γ is a path connection on the topological groupoid Q and $\mathcal{U}$ is an open cover of X, then $C_\Gamma(Q)$ is generated by the family $C_\Gamma(Q|U)$ for all $U\in\mathcal{U}$.*

Proof. If $g\in C_\Gamma(Q)$ is joined to 1_x by the path $\Gamma(\mu)$, then we can write $\mu=\mu_1+\cdots+\mu_n$ where each μ_r lies in some set U_r of $\mathcal{U}$. Let $g_r=-\Gamma(\mu_r)(0)+\Gamma(\mu_r)(1)$. Then $g_r\in C_\Gamma(Q|U_r)$ and $g=g_1+\cdots+g_n$. □

If Q is a topological groupoid on X with a path connection $\Gamma:\Lambda(X)\to\Lambda(Q)$ then of course $\mathrm{loc}(C_\Gamma(Q))$ is a local subgroupoid. However we would like an analogue of $c_1(Q,\mathcal{U})$ and this needs extra conditions. In fact the existence of a smooth path connection for the groupoid $\mathsf{Sym}(p)$ of a principal bundle $p:E\to B$ relies on the existence of an infinitesimal connection (see [11,13]) which itself requires extra structure on the space involved.

We give some conditions which are sufficient for $c_\Gamma(Q,\mathcal{U})$ to be well defined.

We suppose given an open cover $\mathcal{U}=\{U_i\colon i\in I\}$ for X and for each $i\in I$ a collection $\mathrm{geod}(U_i)$ of paths in U_i – an element $\lambda\in\mathrm{geod}(U_i)$ with $\lambda(0)=x$, $\lambda(1)=y$ is called a "geodesic path" from x to y. We suppose

(i) if $x,y\in U_i$, then there is a unique geodesic path $\mathrm{geod}_i(x,y)$ from x to y;
(ii) if $x,y\in U_i\cap U_j$ then $\mathrm{geod}_i(x,y)=\mathrm{geod}_j(x,y)$.

We also need the connection to be 'flat' for this structure in the sense that $x,y,z\in U_i$ then

$$\Gamma\big(\mathrm{geod}_i(x,y)+\mathrm{geod}_i(y,z)\big)(1)=\Gamma\big(\mathrm{geod}_i(x,z)\big)(1).$$

Then we find that $c_\Gamma(q,\mathcal{U})$ is well defined by the atlas (see below). (We could strengthen the 'flat' condition by requiring that if $\lambda : x \to y$ is any path in U_i then $\Gamma(\lambda)(1) = \Gamma(\mathrm{geod}_i(x, y))(1)$, but we do note do this.)

Proposition 4.3. *Under the above atlas assumptions, there is a local subgroupoid $c_\Gamma(Q,\mathcal{U})$ defined by*

$$c_\Gamma(Q,\mathcal{U})(x) = \left[U_i, C_\Gamma(Q|U_i)\right]_x.$$

Proof. We have to prove that if $x \in U_i \cap U_j$ then

$$\left[U_i, C_\Gamma(Q|U_i)\right]_x = \left[U_j, C_\Gamma(Q|U_j)\right]_x.$$

This means there is an open neighbourhood W of x in $U_i \cap U_j$ such that

$$C_\Gamma(Q|U_i)|W = C_\Gamma(Q|U_j)|W.$$

Let W be an open neighbourhood of x in $U_i \cap U_j$. Let $g \in C_\Gamma(Q|U_i)|W$ with $\alpha(g) = x$ and $\beta(g) = y$. So there is a path $\lambda : x \to y$ in U_i such that $\Gamma(\lambda)(1) = g$. Let $\lambda_i : x \to y$ be the geodesic path. So $\Gamma(\lambda_i)(1) = g$, by the flat condition, and so $g \in C_\Gamma(Q|U_j)|W$. Hence $C_\Gamma(Q|U_i)|W \leqslant C_\Gamma(Q|U_j)|W$. Since the converse proof is similar we have $C_\Gamma(Q|U_i)|W = C_\Gamma(Q|U_j)|W$. □

Suppose now that we have the local subgroupoid $c_\Gamma(Q,\mathcal{U})$ defined by the open cover $\mathcal{U}$, and U is an open subset of X. We will later need a result which follows easily from compatibility:

Proposition and Definition 4.4. *The equality*

$$c_\Gamma(Q,\mathcal{U})|U = c_\Gamma(Q|U,\mathcal{U} \cap U)$$

holds if for any $i, j \in I$ and $x \in U_i \cap U_j \cap U$ there is an open set W such that $x \in W \subseteq U_i \cap U_j \cap U$ and $C_\Gamma(Q|U_i)|W = C_\Gamma(Q|U_j \cap U)|W$. If this condition holds for all open sets U of X, then we say that the cover $\mathcal{U}$ is Γ path local for $c_\Gamma(Q,\mathcal{U})$.

5. Partial orders and induced morphisms

We first establish some elementary but essential basic theory.

The set $L_Q(X)$ of wide subgroupoids of Q is a poset under inclusion. We write $\leqslant$ for this partial order. This poset has a top element namely Q and a bottom element namely the discrete groupoid X.

Let $\mathbf{Loc}(Q)$ be the set of local subgroupoids of Q. Let $x \in X$. We define a partial order on the stalks $p_Q{}^{-1}(x) = (\mathcal{L}_Q)_x$ by $[U', H']_x \leqslant [U, H]_x$ if there is an open neighbourhood W of x such that $W \subseteq U \cap U'$ and $H'|W$ is a subgroupoid of $H|W$. Clearly this partial order is well defined. Its bottom element is of the form $[U, H]_x$ where H is discrete, and its top element is of the form $[U, Q|U]_x$. This partial order induces a partial order on $\mathbf{Loc}(Q)$ by $s \leqslant t$ if and only if $s(x) \leqslant t(x)$ for all $x \in X$.

The major purpose of the next topic is to relate local subgroupoids of Q and local equivalence relations on $\mathsf{Ob}(Q)$. This seems an area requiring much more development, and we hope will be the start of new ways of relating bundle and foliation theory.

Suppose given two groupoids Q, H and a groupoid morphism $\phi : Q \to H$ on X, which is the identity on objects. Then we obtain morphisms of presheaves $\phi^* : L_Q \to L_H$, $\phi_* : L_H \to L_Q$ as follows.

Let U be open in X. Then $\phi_* : L_Q(U) \to L_H(U)$ is given by $\phi(K)$ is the image of $K \in L_Q(U)$ by ϕ. Here K is a wide subgroupoid of $Q|U$, and so its image $\phi(K)$ is a subgroupoid of $H|U$, since $\mathsf{Ob}(\phi)$ is injective and is clearly wide.

Further $\phi^* : L_H(U) \to L_Q(U)$ is given by $\phi^*(K') = \phi^{-1}(K')$, for $K' \in L_H(U)$.

Hence we get induced morphism of sheaves $\phi_* : \mathcal{L}_Q \to \mathcal{L}_H$, $\phi^* : \mathcal{L}_H \to \mathcal{L}_Q$.

In particular, we get for a groupoid Q an 'anchor' morphism of groupoids $A : Q \to X \times X$ and so sheaf morphisms

$$A_* : \mathcal{L}_Q \to \mathcal{L}_{X\times X}, \qquad A^* : \mathcal{L}_{X\times X} \to \mathcal{L}_Q.$$

Hence a local subgroupoid s of Q yields a local equivalence relation $A_*(s)$ on X, and a local equivalence relation r on X yields a local subgroupoid $A^*(r)$ of Q. This gives further examples of local subgroupoids.

Clearly also ϕ_*, ϕ^* are order preserving on stalks for any morphism $\phi : Q \to H$ of groupoids over X. Hence they induce morphism of posets

$$\phi_* : \mathbf{Loc}(Q) \to \mathbf{Loc}(H), \qquad \phi^* : \mathbf{Loc}(H) \to \mathbf{Loc}(Q).$$

Further, $s \leqslant \phi^* r$ if and only if $\phi_* s \leqslant r$. This can be expressed by saying that ϕ_* is left adjoint to ϕ^*.

6. Coherence for wide subgroupoids and local subgroupoids

We now fix a groupoid Q on X, so that $L_Q(X)$ is the set of wide subgroupoids of Q, with its inclusion partial order, which we shall write $\leqslant$.

Clearly loc_Q as defined in Definition 1.3 gives a poset morphism

$$\mathrm{loc}_Q : L_Q(X) \to \mathbf{Loc}(Q).$$

Definition 6.1. Let s be a local subgroupoid of Q. Then $\mathrm{glob}(s)$ is the wide subgroupoid of Q which is the intersection of all wide subgroupoids H of Q such that $s \leqslant \mathrm{loc}(H)$.

We think of $\mathrm{glob}(s)$ as an approximation to s by a global subgroupoid.

Proposition 6.2.

(i) loc *and* glob *are morphisms of posets.*
(ii) *For any wide subgroupoid H of Q,* $\mathrm{glob}(\mathrm{loc}(H)) \leqslant H$.

Proof. The proofs are clear. □

However, $s \leqslant \text{loc}(\text{glob}(s))$ need not hold. Rosenthal in [14] gives the example of the local equivalence relation $s = \text{loc}(E)$ where E is the equivalence relation aEb if and only if $a = \pm b$. Here is a similar example.

Example 6.3. Let Q be a groupoid on $\mathbb{R}$ such that all $x \in \mathbb{R}$ with $x \neq 0$ there is a neighbourhood U of x such that $Q|U$ is a bundle of groups, while no such neighbourhood of 0 exists. Let $s = \text{loc}(Q)$. Then $H = \text{glob}(s)$ on $\mathbb{R} \setminus \{0\}$ coincides with Q on this set, and in fact H is the bundle of groups $Q(x)$ for all $x \in \mathbb{R}$. It follows that $s(0) > \text{loc}(H)(0)$.

We therefore adapt from [14,15] some notions of coherence.

Definition 6.4. Let s be a local subgroupoid of Q on X.

(i) s is called *coherent* if $s \leqslant \text{loc}(\text{glob}(s))$.
(ii) s is called *globally coherent* if $s = \text{loc}(\text{glob}(s))$.
(iii) s is called *totally coherent* if for every open set U of X, $s|U$ is coherent.

Definition 6.5. Let $H \in L_Q(X)$, so that H is a wide subgroupoid of Q.

(i) H is called *locally coherent* if $\text{loc}(H)$ is coherent.
(ii) H is called *coherent* if $H = \text{glob}(\text{loc}(H))$.

Example 6.6. Let Q be a groupoid on the discrete space X. Then $\text{glob}(\text{loc}(Q)) = \text{Inn}(Q)$, the groupoid of vertex groups of Q. Thus in general, Q is not coherent.

At another extreme we have:

Proposition 6.7. *Let Q be a bundle of groups. Then any local subgroupoid of Q is globally coherent, and any wide subgroupoid of Q is coherent.*

Proof. Let s be a local subgroupoid of Q and let $\{(U_i, H_i)\colon\ i \in I\}$ be an atlas for s. Then if $x \in U_i$, we have $s(x) = [U_i, H_i]_x$. Let $H(x) = H_i(x)$. If $x \in U_i \cap U_j$, there is a neighbourhood W of x such that $W \subseteq U_i \cap U_j$ and $H_i|W = H_j|W$, and hence $H_i(x) = H_j(x)$. Thus H is independent of the choices. Also $H|U_i = H_i$. Hence $\text{loc}(H)(x) = [U_i, H_i]_x$, and so $\text{loc}(H) = s$, $H = \text{glob}(s)$. □

Coherence of s says that in passing between local and global information nothing is lost due to collapsing. Notice also that these definitions depend on the groupoid Q.

Proposition 6.8. loc *and* glob *induce morphisms of posets from the locally coherent subgroupoids of Q to the coherent local subgroupoids of Q, and on these posets* glob *is left adjoint to* loc*. Further,* glob *and* loc *give inverse isomorphisms between the posets of coherent subgroupoids and of globally coherent local subgroupoids.*

Proof. Let H be a locally coherent subgroupoid of Q and let s be a coherent local subgroupoid of Q. By the definition of locally coherent subgroupoid $\mathrm{loc}(H)$ is a coherent local subgroupoid of Q.

Conversely, let $K = \mathrm{glob}(s)$. Since s is coherent, $s \leqslant \mathrm{loc}(K)$. Since glob is a poset morphism, $\mathrm{glob}(s) \leqslant \mathrm{glob}(\mathrm{loc}(K))$, i.e., $K \leqslant \mathrm{glob}(\mathrm{loc}(K))$. Since loc is also a poset morphism,

$$\mathrm{loc}(K) \leqslant \mathrm{loc}\big(\mathrm{glob}\big(\mathrm{loc}(K)\big)\big).$$

So $\mathrm{loc}(K)$ is a coherent local subgroupoid, and K is locally coherent.

The adjointness relation is that $\mathrm{glob}(s) \leqslant H \iff s \leqslant \mathrm{loc}(H)$. The implication $\Leftarrow$ follows from the fact that for all H we have $\mathrm{glob}(\mathrm{loc}(H)) \leqslant H$. The implication $\Rightarrow$ follows from the coherence of s.

The final statement is obvious. □

Note in particular that coherence of H implies local coherence of H.

Proposition 6.9. *Let Q be a topological groupoid on X and G a star path connected wide subgroupoid of Q. Then G is coherent and* $\mathrm{loc}(G)$ *is globally coherent.*

Proof. We prove that $\mathrm{glob}(\mathrm{loc}(G)) = G$. By Proposition 6.2 $\mathrm{glob}(\mathrm{loc}(G)) \leqslant G$. To prove that $G \leqslant \mathrm{glob}(\mathrm{loc}(G))$ let H be a wide subgroupoid of Q such that $\mathrm{loc}(G) \leqslant \mathrm{loc}(H)$. Then for $x \in X$,

$$[X, G]_x \leqslant [X, H]_x$$

and so for some open neighbourhood U_x of x, $G|U_x \leqslant H|U_x$. These sets U_x form a cover $\mathcal{U}$ of X. By Proposition 2.3, G is generated by the $G|U$ for $U \in \mathcal{U}$. It follows that $G \leqslant H$. □

Corollary 6.10. *If X is a topological space then its fundamental groupoid $\pi_1 X$ is coherent and globally coherent.*

Corollary 6.11. *Let Q be a topological groupoid on X. Then the star identity component $C_1(Q)$ is coherent.*

Corollary 6.12. *Let Q be a topological groupoid with a path connection $\Gamma : \Lambda(X) \to \Lambda(Q)$. Then the wide subgroupoid $C_\Gamma(Q)$ is coherent.*

Proposition 6.13. *Let Q be a topological groupoid on X. Suppose that the local subgroupoid $c_1(Q,\mathcal{U})$ of Q is well defined by an open cover $\mathcal{U}$. Then*

(i) $\mathrm{glob}(c_1(Q,\mathcal{U})) = C_1(Q)$.
(ii) *$c_1(Q,\mathcal{U})$ is coherent.*
(iii) *If $\mathcal{U}$ is path local for Q, then $c_1(Q,\mathcal{U})$ is totally coherent.*

Proof. (i) Certainly $c_1(Q,\mathcal{U}) \leqslant \mathrm{loc}(C_1(Q))$ since for all open U in X, $C_1(Q|U) \leqslant C_1(Q)|U$ and so $[U_i, C_1(Q|U_i)]_x \leqslant [X, C_1(Q)]_x$ for all $x \in U$.

Now suppose H is a wide subgroupoid of Q and $c_1(Q,\mathcal{U}) \leqslant \mathrm{loc}(H)$. We have to prove $C_1(Q) \leqslant H$.

Let $i \in I$ and $x \in U_i$. We have $[U_i, C_1(Q|U_i)]_x \leqslant [X, H]_x$. Hence there is an open neighbourhood W_x of x contained in U_i and such that

$$C_1(Q|U_i)|W_x \leqslant H.$$

By Proposition 2.3, $C_1(Q|U_i)$ is generated by the $C_1(Q|U_i)|W_x$ for all $x \in U_i$ and by the same Proposition, $C_1(Q)$ is generated by the $C_1(Q|U_i)$ for all $i \in I$. Hence $C_1(Q) \leqslant H$.

Coherence of $c_1(Q,\mathcal{U})$ follows from (i) and $C_1(Q|U_i) \leqslant C_1(Q)|U_i$. Total coherence in the path local case follows by applying (ii) to $Q|U$, using Proposition and Definition 3.5. □

Proposition 6.14. *Let Q be a topological groupoid on X such that the local subgroupoid $c_\Gamma(Q,\mathcal{U})$ is well defined by the open cover $\mathcal{U}$. Then*:

(i) $\mathrm{glob}(c_\Gamma(Q,\mathcal{U})) = C_\Gamma(Q)$.
(ii) *$c_\Gamma(Q,\mathcal{U})$ is coherent.*
(iii) *If $\mathcal{U}$ is Γ path local for Q, then $c_\Gamma(Q,\mathcal{U})$ is totally coherent.*

Proof. (i) Note that $c_\Gamma(Q,\mathcal{U}) \leqslant \mathrm{loc}(C_\Gamma(Q))$ since for all U in X, $C_\Gamma(Q|U) \leqslant C_\Gamma(Q)|U$. So $\mathrm{glob}(c_\Gamma(Q)) \leqslant C_\Gamma(Q)$. To prove that $C_\Gamma(Q) \leqslant \mathrm{glob}(c_\Gamma(Q,\mathcal{U}))$ suppose that H is a wide subgroupoid of Q such that $c_\Gamma(Q,\mathcal{U}) \leqslant \mathrm{loc}(H)$. We have to prove that $C_\Gamma(Q) \leqslant H$. If $U \in \mathcal{U}$ and $x \in U$ then

$$\big[U, C_\Gamma(Q|U)\big]_x \leqslant [X, H]_x.$$

Hence U has a covering by open sets W_x such that $[U, C_\Gamma(Q|U)|W_x \leqslant H$. By Proposition 2.3, $C_\Gamma(Q|U)$ is generated by the groupoids $C_\Gamma(Q|U)|W_x$ and by Proposition 4.2 $C_\Gamma(Q)$ is generated by the $C_\Gamma(Q|U)$ for $U \in \mathcal{U}$. Hence $C_\Gamma(Q) \leqslant H$.

The proofs of (ii), (iii) are analogous to those in the previous proposition. □

7. Coherence and atlases

We lead up to conditions for an atlas for a local subgroupoid s to be *globally adapted* to s. This notion is important for considerations of holonomy (see [2]), and the applications will be developed elsewhere.

The next proposition gives an alternative description of glob.

Let $\mathcal{U}_s = \{(U_i, H_i)\colon\ i \in I\}$ be an atlas for the local subgroupoid s. Then $\mathrm{glob}(\mathcal{U}_s)$ is defined to be the subgroupoid of Q generated by all the H_i, $i \in I$.

An atlas $\mathcal{V}_s = \{(V_j, H'_j)\colon\ j \in J\}$ for s is said to refine $\mathcal{U}_s$ if for each index $j \in J$ there exists an index $i(j) \in I$ such that $V_j \subseteq U_{i(j)}$ and $H_{i(j)}|V_j = H'_j$.

Proposition 7.1. *Let s be a local subgroupoid of Q given by the atlas $\mathcal{U}_s = \{(U_i, H_i)\colon i \in I\}$. Then* $\mathrm{glob}(s)$ *is the intersection of the subgroupoids* $\mathrm{glob}(\mathcal{V}_s)$ *of Q for all refinements $\mathcal{V}_s$ of $\mathcal{U}_s$.*

Proof. Let K be the intersection given in the proposition.

Let Q be a subgroupoid of Q on X such that $s \leqslant \mathrm{loc}(Q)$. Then for all $x \in X$ there is a neighbourhood V of x and $i_x \in I$ such that $x \in U_{i_x}$ and $H_{i_x}|V_x \cap U_{i_x} \leqslant Q$. Then $\mathcal{W} = \{(V_x \cap U_{i_x}, H_{i_x}|V_x \cap U_{i_x})\colon x \in X\}$ refines $\mathcal{U}_s$ and $\mathrm{glob}(\mathcal{W}) \leqslant Q$. Hence $K \leqslant Q$, and so $K \leqslant \mathrm{glob}(s)$.

Conversely, let $\mathcal{V}_s = \{(V_j, H'_j)\colon j \in J\}$ be an atlas for s which refines $\mathcal{U}_s$. Then for each $j \in J$ there is an $i(j) \in I$ such that $V_j \subseteq U_{i(j)}$, $H'_j = H_{i(j)}|V_j$. Then $s \leqslant \mathrm{loc}(\mathrm{glob}(\mathcal{V}_s))$. Hence $\mathrm{glob}(s) \leqslant \mathrm{glob}(\mathcal{V}_s)$ and so $\mathrm{glob}(s) \leqslant K$. □

Corollary 7.2. *A wide subgroupoid H of Q is coherent if and only if for every open cover $\mathcal{V}$ of X, H is generated by the subgroupoids $H|V$, $V \in \mathcal{V}$.*

Proof. Note that $\{(X, H)\}$ is an atlas for $\mathrm{loc}(H)$, which is refined by $\mathcal{V}_H = \{(V, H|V)\colon V \in \mathcal{V}\}$ for any open cover $\mathcal{V}$.

Suppose the latter condition holds. Then Proposition 7.1 implies that $H = \mathrm{glob}(\mathrm{loc}(H))$, i.e., H is coherent. The converse holds since $\mathrm{glob}(\mathcal{V}_H) \leqslant H$. □

Let U be an open subset of X. Then we have notions of local subgroupoids of $Q|U$ and also of the restriction $s|U$ of a local subgroupoid s of Q. Clearly if H is a wide subgroupoid of Q then $\mathrm{loc}(H|U) = (\mathrm{loc}(H))|U$.

Proposition 7.3. *Let s be a local subgroupoid of Q and let U be open in X. Then* $\mathrm{glob}(s|U) \leqslant \mathrm{glob}(s)|U$.

Proof. Let H be a wide subgroupoid of Q such that $s \leqslant \mathrm{loc}(H)$. Then $s|U \leqslant \mathrm{loc}(H|U)$. So $\mathrm{glob}(s|U) \leqslant H|U$. The result follows.

Proposition 7.4. *Let s be a local subgroupoid of Q. Then*

(i) *If s is globally coherent, U is open in X, and $s|U$ is coherent, then $s|U$ is globally coherent.*
(ii) *If there is an open cover $\mathcal{V}$ of X such that $s|V$ is coherent for all $V \in \mathcal{V}$, then s is coherent.*
(iii) *If s is globally coherent then for any open cover $\mathcal{V}$ of X,* $\mathrm{glob}(s)$ *is generated by the groupoids* $\mathrm{glob}(s)|V$ *for all $V \in \mathcal{V}$.*
(iv) *If there is open cover $\mathcal{V}$ of X such that $s|V$ is globally and totally coherent for $V \in \mathcal{V}$, then s is totally coherent.*

Proof. (i) We are given $s = \mathrm{loc}(\mathrm{glob}(s))$. By Proposition 7.3

$$\mathrm{loc}\big(\mathrm{glob}(s|U)\big) \leqslant \mathrm{loc}\big(\mathrm{glob}(s)|U\big) = \mathrm{loc}\big(\mathrm{glob}(s)\big)|U = s|U.$$

Since $s|U$ is coherent, we have $s|U \leqslant \mathrm{loc}(\mathrm{glob}(s|U))$. So $s|U = \mathrm{loc}(\mathrm{glob}(s|U))$, i.e., $s|U$ is globally coherent.

(ii) We have

$$s|V \leqslant \mathrm{loc}\big(\mathrm{glob}(s|V)\big) \leqslant \mathrm{loc}\big(\mathrm{glob}(s)|V\big) \leqslant \big(\mathrm{loc}\big(\mathrm{glob}(s)\big)\big)|V.$$

Since this holds for all V of an open cover, we have $s \leqslant \mathrm{loc}(\mathrm{glob}(s))$.

(iii) This follows from Corollary 7.2.

(iv) Let U be open in X. Let $V \in \mathcal{V}$. Since $s|V$ is globally and totally coherent, then $s|V \cap U$ is globally coherent. Hence by (ii) $s|U$ is coherent, since the $V \cap U, V \in \mathcal{V}$, cover U. □

Proposition 7.5. *Let $\mathcal{U}_s = \{(U_i, H_i)\colon\ i \in I\}$ be an atlas for the local subgroupoid s. Then*:

(i) $s|U_i = \mathrm{loc}(H_i)$ *for all* $i \in I$;
(ii) $\mathrm{loc}(\mathrm{glob}(s|U_i)) \leqslant s|U_i$ *for all* $i \in I$;
(iii) *if* $s|U_i$ *is coherent for all* $i \in I$ *then* s *is globally coherent*;
(iv) *if* $s|U_i$ *is coherent for all* $i \in I$ *then* $\mathrm{glob}(s) = \mathrm{glob}(\mathcal{U}_s)$.

Proof. (i) This is clear.

(ii) We have by Proposition 6.2

$$\mathrm{loc}\big(\mathrm{glob}(s|U_i)\big) = \mathrm{loc}\big(\mathrm{glob}\big(\mathrm{loc}(H_i)\big)\big) \leqslant \mathrm{loc}(H_i) = s|U_i.$$

(iii) This is immediate from the definition of coherence and (ii).

(iv) Let $H = \mathrm{glob}(\mathcal{U}_s)$, i.e., H is the subgroupoid of Q generated by the H_i, $i \in I$. Then $\mathrm{glob}(s) \leqslant H$. Let K be a wide subgroupoid of Q such that $s \leqslant \mathrm{loc}(K)$. Then for all $i \in I$ and $x \in U_i$ there is a neighbourhood V_x^i of x such that $V_x^i \subseteq U_i$ and $H_i|V_x^i \leqslant K|V_x^i$. By global coherence of $s|U_i$ and Proposition 7.4(i) and (ii), H_i is generated by the $H_i|V_x^i$ for all $x \in U_i$. Hence $H_i \leqslant K|U_i \leqslant K$. Hence $H \leqslant K$. Hence $H \leqslant \mathrm{glob}(s)$. □

Definition 7.6. Let s be a local subgroupoid of the groupoid Q on X. An atlas $\mathcal{U}_s$ for s is called *globally adapted* if $\mathrm{glob}(s) = \mathrm{glob}(\mathcal{U}_s)$.

Remark 7.7. This is a variation on the notion of an r-adaptable family defined by Rosenthal in [15, Definition 4.4] for the case of a local equivalence relation r. He also imposes a connectivity condition on the local equivalence classes.

The construction of the holonomy groupoid of a local subgroupoid requires a globally adapted atlas (see Theorems 3.7 and 3.8 [2]). The following proposition is very useful to this end.

Proposition 7.8. *Let s be a totally coherent local subgroupoid of the groupoid Q on X. Then any atlas for s is globally adapted.*

Proof. This is immediate from the previous Proposition, since total coherence implies that each $s|U_i$ is coherent. □

Corollary 7.9. *Any path local atlas $\mathcal{U}$ of the local subgroupoid $c_1(Q,\mathcal{U})$ is globally adapted.*

Corollary 7.10. *Any Γ path local atlas of the local subgroupoid $c_\Gamma(Q,\mathcal{U})$ is globally adapted.*

References

[1] R. Brown, Topology; A Geometric Account of General Topology, Homotopy Types and the Fundamental Groupoid, Ellis Horwood, Chichester, 1988.

[2] R. Brown, İ. İçen, Lie local subgroupoids and their Lie holonomy and monodromy groupoids, Topology Appl. 15 (2001) 125–138.

[3] R. Brown, O. Mucuk, Foliations, locally Lie groupoids and holonomy, Cahiers Topologie Géom. Différentielle Catégoriques 37 (1996) 61–71.

[4] R. Brown, C.B. Spencer, Double groupoids and crossed modules, Cahiers Topologie Géom. Différentielle Catégoriques 17 (1976) 343–362.

[5] A.C. Ehresmann, Structures feuillétées, in: Proc. 5th Canad. Math. Cong. Montreal, 1961. Re-edited in: Charles Ehresmann, Oeuvres Complétes et Commentées, Partie II-2, Amiens, 1982, pp. 563–629.

[6] A. Grothendieck, J.L. Verdier, Théorie des Topos (SGA, Vol. 1), in: Lectures Notes in Math., Vol. 269, Springer-Verlag, Berlin, 1972.

[7] P.J. Higgins, Categories and Groupoids, Van Nostrand, New York, 1971.

[8] İ. İçen, Sheaves and local subgroupoids, UWB Maths Preprint 00.14 (65 p.), math.CT0009086.

[9] A. Kock, I. Moerdijk, Spaces with local equivalence relations, and their sheaves, Topology Appl. 72 (1996) 47–78.

[10] A. Kock, I. Moerdijk, Every étendue comes from a local equivalence relation, J. Pure Appl. Algebra 82 (1992) 155–174.

[11] S. Kobayashi, K. Nomizo, Foundation of Differential Geometry, Vol. 1, Interscience, New York, 1963.

[12] J. Kubarski, Some generalizations of Godement's theorem on division, in: Proceedings of the Winter School on Geometry and Physics (Srn, 1987), Rend. Circ. Palermo (2) Suppl. 16 (1987) 119–124.

[13] K.C.H. Mackenzie, Lie Groupoids and Lie Algebroids in Differential Geometry, in: London Math. Soc. Lecture Note Ser., Vol. 124, Cambridge University Press, Cambridge, 1987.

[14] K. Rosenthal, Local equivalence relations, Topology Appl. 13 (1982) 167–176.

[15] K. Rosenthal, Sheaves and local equivalence relations, Cahiers Topologie Géom. Différentielle Catégoriques 25 (1984) 1–31.

[16] J. Virsik, On the holonomity of higher order connections, Cahiers Topologie Géom. Différentielle Catégoriques 12 (1971) 197–212.