On models for propositional dynamic logic

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Abstract


In this paper we study some foundational aspects of the theory of PDL. We prove a claim made by Parikh (1981), namely, the existence of a Kripke model $\mathcal{M}$ that is universal in the sense that every other Kripke model $\mathcal{N}$ can be isomorphically embedded in it. Using this model we give different and particularly easy proofs of the completeness theorem for the Segerberg axiomatization of PDL and the small model theorem. We also give an infinitary axiomatization for PDL and prove it to be complete using a syntax model $\mathcal{A}$, by a technique that is well-known from modal logic. We prove that $\mathcal{M}$ and $\mathcal{A}$ are isomorphic. Finally, we briefly turn to dynamic algebras and show that the characteristic algebra of $\mathcal{M}$ is initial in the class of $*$-continuous dynamic algebras.

1. Introduction

Logics of programs are formal systems for reasoning about the behavior of computer programs. In these formal systems, computer programs are viewed as a means to enable certain logical formulae. The formulae may be propositional or first order, giving rise to propositional and first-order program logics, respectively. Pratt [13] recognized the possibility of modeling program logics by means of modal logic. His idea was fully developed by Fischer and Ladner [3] and many other authors; see Harel [6] and Kozen and Tiuryn [11] for a survey of results. If we view a program to be defined by its input/output (before/after) behavior, then modal logic provides a natural framework in which we can develop a program logic. Each program $\alpha$ is associated with its "own" modal operator $\Diamond\alpha$, or $\langle\alpha\rangle$. For a propositional program logic we can take a set of primitive programs and rules that determine how more
complex programs can be built. With each rule we can define how the modal operator for the more complex program relates to the modal operators of the building blocks. In this approach the modal operators for the primitive programs are parameters. See Goldblatt [4] for an introduction to modal logic and its connection with logics of programs.

In this paper we focus attention on a propositional program logic, namely, propositional dynamic logic (PDL). In PDL programs are regular expressions over a set of primitive programs; in particular, there is a nondeterministic looping operator $*$ for programs. In the PDL framework, programs can enable propositions by means of a possibility operator $\Diamond$. Thus, when $z$ is a program and $\phi$ is a proposition, $z \Diamond \phi$ states “program $z$ can terminate with $\phi$ holding upon termination”. We will write $\langle z \rangle \phi$ instead of $z \Diamond \phi$, as is common in PDL. In this paper we study some foundational aspects of the syntax and semantics of PDL and focus attention on the consequences of introducing the looping operator $*$. In a way, we argue that looping is inherently infinitary, thus giving rise to an infinitary axiomatization. The argument is split into two major parts outlined below.

The logic is interpreted over Kripke models and we prove the existence of a Kripke model $\mathcal{U}$ that is universal in the sense that every other Kripke model $\mathcal{M}$ can be isomorphically embedded in it. This proves a claim of Parikh [12]. The model $\mathcal{U}$ also appears to be a powerful tool in the study of the logic. We give two applications. First, Segerberg gave an axiomatization for the logic that is sound and complete, i.e. validity and derivability coincide (c.f. [10]). We give another proof of the completeness of the system using the model $\mathcal{U}$, which is particularly easy. Secondly, we prove the correctness of the construction of a small model satisfying a formula $\phi$ iff $\mathcal{M}$ is satisfiable as given by Sherman and Harel [6, 17]. Again, the proof uses the model $\mathcal{U}$ and is particularly straightforward.

Next, we define an infinitary axiomatization for PDL and prove it to be complete using a technique that is well-known from modal logic (see [4]), namely, by constructing a syntax model $\mathcal{A}$ for the logic. The state space of $\mathcal{A}$ consists precisely of the set of all maximal consistent sets of formulae. As a rather immediate consequence we deduce that $\mathcal{U} \cong \mathcal{A}$. This infinitary system can be viewed as the propositional variant of the infinitary axiomatization for first-order dynamic logic [4, 5, 6, 11]. We also show that we can use this technique to define a syntax model from the finitary Segerberg system, which is universal in the class of nonstandard Kripke models.

In the last section we briefly introduce dynamic algebras and $*$-continuous dynamic algebras. Each Kripke model $\mathcal{M}$ is associated with a characteristic dynamic algebra $\mathcal{A}_\mathcal{M}$. We show the algebra $\mathcal{U}$ to be initial in the class of $*$-continuous dynamic algebras.

2. Preliminaries

In this section we review the syntax and semantics of PDL. For a more detailed treatment, see [6, 11].
2.1. Syntax

The syntax of PDL is based on two disjoint sets of primitive symbols, namely, the set

\[ \Phi_0 = \{ p_0, p_1, \ldots \} \]

of primitive predicate symbols, and the set

\[ \Pi_0 = \{ a_0, a_1, \ldots \} \]

of primitive program symbols. From these base sets we inductively define the sets of PDL propositions \( \Phi \) and programs \( \Pi \) as the smallest set satisfying:

1. \( \Phi_0 \subseteq \Phi \);
2. if \( \phi, \psi \in \Phi \), then \( \phi \lor \psi, \neg \phi \in \Phi \);
3. if \( \alpha \in \Pi \) and \( \phi \in \Phi \), then \( <\alpha> \phi \in \Phi \);
4. \( \Pi_0 \subseteq \Pi \);
5. if \( \alpha, \beta \in \Pi \), then \( \alpha \lor \beta, \alpha; \beta, \alpha^* \in \Pi \);
6. if \( \phi \in \Phi \), then \( ? \phi \in \Pi \).

We abbreviate \( \neg(\neg \phi \lor \neg \psi) \) to \( \phi \land \psi \); \( \neg \phi \lor \psi \) to \( \phi \rightarrow \psi \); \( (\phi \rightarrow \psi) \land (\psi \rightarrow \phi) \) to \( \phi \leftrightarrow \psi \).

We further abbreviate \( \alpha \) to \( [\alpha] \phi \).

2.2. Semantics

First we give an informal semantics for the above constructs. The meaning of the propositional connectives is exactly like in ordinary, classical propositional logic CPC. Therefore, PDL can be seen as an extension of CPC, i.e. all tautologies of CPC are valid PDL formulae. Primitive programs are exactly what their name suggests: uninterpreted programs or input/output relations, which is essentially the way we view programs in general. That is, programs are black boxes and their input/output behavior completely characterizes their relevant aspects; two programs are equivalent if and only if they constitute the same input/output relation. The meaning of the operator \( \; \) is program concatenation; thus, \( \alpha; \beta \) means “first execute program \( \alpha \) and then execute \( \beta \)”. \( \lor \) means nondeterministic choice; \( \alpha \lor \beta \) means “choose nondeterministically program \( \alpha \) or \( \beta \) and execute it”. The \* operator is a nondeterministic looping operator and \( \alpha^* \) means “execute \( \alpha \) a nondeterministically chosen number of times”. In the sequel we often abbreviate \( \alpha; \alpha; \ldots; \alpha \) \( (n \times) \) to \( \alpha^n \). Thus, \( \alpha^* \) can be viewed as “choose \( n \) nondeterministically and execute \( \alpha^n \)”. The operator \( ? \) is a testing operator and \( ? \phi \) means “test \( \phi \) and proceed if true”. The operator \( \Diamond \) is the usual modal operator and the meaning of \( <\alpha> \phi \) is “program \( \alpha \) can be executed with \( \phi \) holding upon termination”. Its dual, \( [\alpha] \phi \), therefore means “whenever program \( \alpha \) terminates, \( \phi \) holds”.

Formally, PDL formulae are interpreted over Kripke models.
Definition 2.1. A Kripke model is a triple $\mathcal{A} = (W, \pi, \rho)$, where
- $W$ is a set of states,
- $\pi: \Phi_{0} \rightarrow 2^{W}$ is an interpretation function for the primitive predicate symbols;
- $\rho: \Pi_{0} \rightarrow 2^{W \times W}$ is an interpretation function for the primitive program symbols.

Usually we write a Kripke model as $\mathcal{A} = (W, \pi)$ when no confusion can arise. We further use the terms “Kripke model” and “model” interchangeably. The interpretation functions extend to the whole sets $\Phi$ and $\Pi$: 
- $\rho(x \cup \beta) = \rho(x) \cup \rho(\beta)$;
- $\rho(x; \beta) = \rho(x) \cup \rho(\beta)$, where $\circ$ is relation composition;
- $\rho(x^*) = \bigcup_{s \in W} \rho(x^s)$, the reflexive transitive closure of $\rho(x)$;
- $\pi(\phi ?) = \{(s, s) \in W \times W | s \in \pi(\phi)\}$;
- $\pi(\sigma \sqcup \psi) = \pi(\sigma) \cup \pi(\psi)$;
- $\pi(\neg \phi) = W - \pi(\phi)$;
- $\pi(\langle x \rangle \phi) = \{s \in W | \exists t \in W . ((s, t) \in \rho(x) \land t \in \pi(\phi))\}$.

We say that a proposition $\phi$ is satisfiable in a model $\mathcal{A}$ if and only if there exists a state $s$ in $\mathcal{A}$ such that $s \pi(\phi)$ and we write $\mathcal{A}, s \models \phi$. We omit $\mathcal{A}$ when it is clear from the context. We say that $\phi$ is $\mathcal{A}$-valid and write $\mathcal{A} \models \phi$ if $\mathcal{A}, s \models \phi$ for each $s \in W$. We say that $\phi$ is valid and write $\models \phi$ if $\phi$ is $\mathcal{A}$-valid for every model $\mathcal{A}$. Clearly, $\phi$ is valid if and only if $\neg \phi$ is not satisfiable.

In the sequel we use $\phi, \psi, \ldots$ to denote propositions and $\alpha, \beta, \ldots$ to denote programs.

2.3. Axiomatization

We now present an axiomatization for PDL as proposed by Segerberg [16].

Definition 2.2. The set of axioms $AX$ for PDL contains
1. the axioms for propositional logic;
2. $\langle x \rangle \phi \land [x] \psi \rightarrow \langle x \rangle (\phi \lor \psi)$;
3. $\langle x \rangle (\phi \lor \psi) \leftrightarrow \langle x \rangle \phi \lor \langle x \rangle \psi$;
4. $\langle x \cup \beta \rangle \phi \leftrightarrow \langle x \rangle \phi \lor \langle \beta \rangle \phi$;
5. $\langle x; \beta \rangle \phi \leftrightarrow \langle x \rangle \langle \beta \rangle \phi$;
6. $[\psi] \phi \leftrightarrow \psi \land \phi$;
7. $\phi \lor \langle x \rangle \langle x^* \rangle \phi \rightarrow \langle x^* \rangle \phi$;
8. $\langle x^* \rangle \phi \rightarrow \phi \lor \langle x^* \rangle (\neg \phi \land \langle x \rangle \phi)$.

In addition we have the following inference rules.
1. modus ponens: from $\phi, \phi \rightarrow \psi$, infer $\psi$;
2. modal generalization: from $\phi$, infer $[x] \phi$, for any $x \in \Pi$.

As usual, we define a derivation to be a finite sequence of well-formed formulae, each of which is an instance of an axiom or the conclusion of an inference rule, whose premisses occur earlier in the derivation. The last formula occurring in the derivation
is called the conclusion of the derivation. If, for any formula \( \phi \), there exists a derivation of which \( \phi \) is the conclusion, we say that \( \phi \) is derivable and write \( \vdash \phi \).

Axioms 1–3 are not particular for PDL, but hold in most modal systems. The dual of axiom 2 reads

\[ [x](\phi \rightarrow \psi) \rightarrow ([x]\phi \rightarrow [x]\psi), \]

which states that the logic is normal in the terminology of modal logic. Axiom 8 is called the induction axiom, and is better known in its dual form

\[ \phi \land [x^*](\phi \rightarrow [x]\phi) \rightarrow [x^*]\phi. \]

Note the resemblance between this axiom and the induction axiom in arithmetic. The intuition behind axiom 8 is that if a program \( x^* \) enables a proposition \( \phi \), then the proposition is always true or there is a point in the looping of the program, where the proposition becomes true for the first time.

Inspection of the system AX immediately gives us Theorem 2.3.

**Theorem 2.3** (soundness theorem). If \( \vdash \phi \), then \( \models \phi \).

A familiar fact of PDL is its lack of compactness. For an easy example, consider the infinite set \( \Gamma \):

\[ \Gamma = \{ \neg \phi, \neg (x)\phi, \neg (x^2)\phi, \ldots \} \cup \{ (x^*)\phi \} \]

\[ = A \cup \{ (x^*)\phi \}. \]

Every finite subset \( \Gamma' \subseteq \Gamma \) has a model. Suppose \( (x^*)\phi \in \Gamma' \) and let \( i \) be the largest integer such that \( \neg (x^i)\phi \in \Gamma' \). Then each model \( \mathcal{M} \) that satisfies \( \neg (x^j)\phi \) for \( j \leq i \) and \( (x^{i+1})\phi \), satisfies \( \Gamma' \). Yet the whole set \( \Gamma \) cannot have a model, for \( A \) is precisely the definition of \( \neg (x^*)\phi \). Note that this noncompactness property is essentially caused by the *-operator.

3. A universal model theorem for Kripke models

In this section we establish a nontrivial property of Kripke models, namely, the existence of a model \( \mathcal{U} \) that is universal in the sense that every other model can be isomorphically embedded in it. In this, we prove a claim of Parikh [12], which seems not to have been developed in the literature before. We also exhibit some immediate corollaries. We first establish some facts about models for PDL.

**Definition 3.1.** For each model \( \mathcal{M} \) the relation \( \equiv \) on the state space \( W^\mathcal{M} \) is defined by

\[ s \equiv t \text{ iff } \forall \phi. (\mathcal{M}, s \models \phi \Leftrightarrow \mathcal{M}, t \models \phi). \]
For each model \( \mathcal{M} \) we now define the collapse of \( \mathcal{M} \) to be the model \( \mathcal{M}_c = \mathcal{M}/\equiv: \)

\[
s_c = \{ t | s \equiv t \},
\]

\[
W_c = \{ s | s \in W^# \},
\]

\[
\pi^c(p_i) = \{ s_c | s \in \pi^#(p_i) \},
\]

\[
\rho^c(a_j) = \{ (s_c, t_c) | (s, t) \in \rho^#(a_j) \}.
\]

The following lemma is immediate.

**Lemma 3.2.** For each proposition \( \phi \),

\( \mathcal{M}, s \models \phi \) iff \( \mathcal{M}_c, s_c \models \phi \).

The lemma implies that we only need to consider models of cardinality at most \( \aleph_1 \), that is, the cardinality of the power set of \( \Phi \).

**Lemma 3.3.** For every model \( \mathcal{M} \) and program \( \pi \),

1. if \( (s, t) \in \rho(\pi) \), then \( \forall \phi(\mathcal{M}, t \models [x] \phi \Rightarrow \mathcal{M}, s \models \langle x \rangle \phi) \);
2. if \( (s, t) \in \rho(\pi) \), then \( \forall \phi(\mathcal{M}, s \models [x] \phi \Rightarrow \mathcal{M}, t \models \phi) \);
3. \( \forall \phi(\mathcal{M}, t \models \phi \Rightarrow \mathcal{M}, s \models \langle x \rangle \phi) \) iff \( \forall \phi(\mathcal{M}, s \models [x] \phi \Rightarrow \mathcal{M}, t \models \phi) \).

**Proof.** Clauses (1) and (2) follow immediately from the definition of \( \models \). For clause (3) \( \forall \phi(\mathcal{M}, s \models [x] \phi \Rightarrow \mathcal{M}, t \models \phi) \) iff \( \forall \phi(\mathcal{M}, s \not\models [x] \phi \Rightarrow \mathcal{M}, s \models \langle x \rangle \neg \phi) \) iff \( \forall \psi(\mathcal{M}, t \models \phi \Rightarrow \mathcal{M}, s \models \langle x \rangle \psi) \). \( \square \)

In the light of Lemma 3.3 we can define for each model \( \mathcal{M} \) another model \( \mathcal{M}_{ex} \), called the extension of \( \mathcal{M} \), by

\[
W_{ex}^# = W^#,
\]

\[
\pi^e_x = \pi^#,
\]

\[
\rho^{ex}(a) = \{ (s, t) \models \forall \phi(\mathcal{M}, s \models [a] \phi \Rightarrow \mathcal{M}, t \models \phi) \} \text{ for a primitive.}
\]

By Lemma 3.3, \( \rho^#(a) \subseteq \rho^{ex}(a) \) for each primitive program \( a \). Note that \( \rho^{ex}(a) \) need not equal \( \rho^#(a) \). Consider for example the case in which \( \mathcal{M}, s \models [a] \phi \) only if \( \phi \) is valid. Then, for every \( t \in W^#, (s, t) \in \rho^{ex}(a) \). Obviously, \( \rho^{ex}(a) \) can be substantially larger than \( \rho^#(a) \). We extend \( \rho^{ex} \) to the whole set \( \Pi \) in the usual way.

**Lemma 3.4.** For each proposition \( \phi \),

\( \mathcal{M}_{ex}, s \models \phi \) iff \( \mathcal{M}, s \models \phi \).

**Proof.** (\( \Rightarrow \)): Since \( \rho^#(a) \subseteq \rho^{ex}(a) \) for each primitive program \( a \), it is easy to see that
for each $\alpha \in \Pi$, $\rho^{\#}(\alpha) \subseteq \rho^{\#\#}(\alpha)$. The proof proceeds by induction on the complexity of $\phi$. The only nontrivial case is $\phi = \langle x \rangle \psi$, which follows from the inclusion given above.

$(\Rightarrow)$: Let $\mathcal{M}_{\alpha \gamma}, s \models \phi$. We define the mapping $R : \Pi \rightarrow 2^{w, a \times w, a}$ by

$$R(\alpha) = \{ (s, t) | \forall \psi. (\mathcal{M}, s \models [x] \psi \Rightarrow \mathcal{M}, t \models \psi) \}$$

$$= \{ (s, t) | \forall \psi. (\mathcal{M}, t \models \psi \Rightarrow \mathcal{M}, s \models [x] \psi) \}$$

for $\alpha \in \Pi$. Note that, by Lemma 3.3(3), we may use both conditions interchangeably in the definition of $R$.

Claim 1. $\mathcal{M}, s \models \phi$ iff $\mathcal{M}, s \models R(\phi)$, where $\models R$ is defined as the relation $\models$ except that we use $R(\alpha)$ instead of $\rho(\alpha)$.

Proof of claim. Induction on the structure of $\phi$. The only nontrivial case is $\phi = \langle x \rangle \psi$. Let $\mathcal{M}, s \models \langle x \rangle \psi$. Then there exists a state $t$ such that $\mathcal{M}, t \models \psi$ and $(s, t) \in \rho(\alpha)$. But then $(s, t) \in R(\alpha)$ by the construction of $R$ and $\mathcal{M}, s \models \langle x \rangle \psi$. Conversely, let $\mathcal{M}, s \models R(\phi)$; then there is a state $t$ such that $(s, t) \in \rho(\alpha)$ and $t \models \psi$. Suppose that there exists no state $t$ such that $(s, t) \in \rho(\alpha)$ and $\mathcal{M}, t \models \psi$. Then $\mathcal{M}, s \models [x] \neg \psi$ and, by the definition of $R$, if $(s, t) \in R(\alpha)$, then $t \models \neg \psi$. Contradiction.

Claim 2. For each $\alpha \in \Pi$, $\rho^{\#\#}(\alpha) \subseteq R(\alpha)$.

Proof of claim. Induction on the complexity of $\alpha$. For $\alpha$ primitive, the claim holds by definition. Next we consider more complex programs $\alpha$.

Case 1: $\alpha = \beta \cup \gamma$.

Clearly, $\rho(\beta \cup \gamma) = \rho(\beta) \cup \rho(\gamma) \subseteq R(\beta) \cup R(\gamma)$. The last union equals

$$\{ (s, t) | \forall \phi. (t \models \phi \Rightarrow s \models \langle \beta \rangle \phi) \lor \forall \phi. (t \models \phi \Rightarrow s \models \langle \gamma \rangle \phi) \}.$$
hence, \((s, t) \in R(\beta; \gamma)\) and \(R(\beta) \circ R(\gamma) \subseteq R(\beta; \gamma)\).

**Case 3:** \(\alpha = \beta^*\).

By the former argument we get
\[
\rho(\beta^n) \subseteq R(\beta^n)
\]
for each \(n < \omega\). We further have, for each \(n < \omega\),
\[
R(\beta^n) \subseteq R(\beta^*).
\]
Suppose \((s, t) \in R(\beta^n)\); then \(t \models \psi \Rightarrow s \models \langle \beta^n \rangle \psi\) for all \(\psi\). Surely \(t \models \psi \Rightarrow s \models \langle \beta^* \rangle \psi\) for all \(\psi\), by the definition of \(\rho(\beta^*)\). Hence, \((s, t) \in R(\beta^*)\). Hence, by induction on \(n\),
\[
\rho(\beta^*) = \bigcup_{n < \omega} \rho(\beta^n) \subseteq \bigcup_{n < \omega} R(\beta^n) \subseteq R(\beta^*).
\]

Note that this is the place where we use the infinitary properties of \(\beta^*\).

**Case 4:** \(\alpha = \psi^?\).

Clearly, \(\rho(\psi^?) = R(\psi^?)\) follows immediately by the definitions of \(\rho\) and \(R\).

The proof of the lemma now follows by induction on the structure of \(\phi\). Again, the only nontrivial case is \(\phi = \langle \alpha \rangle \psi\). If \(\mathcal{M}_\alpha, s \models \langle \alpha \rangle \psi\), then by claim 2 \(\mathcal{M}, s \models \langle \alpha \rangle \psi\) and, hence, by claim 1, \(\mathcal{M}, s \models \langle \alpha \rangle \psi\).

Next we define, for each model \(\mathcal{M}\), the model \(\mathcal{M}'\) by replacing every state in \(W^\mathcal{M}\) by the set of propositions that hold at that state. We denote the state in \(W'^\mathcal{M}\) corresponding to \(s\) by \(\widehat{s}\). It is easy to see that
\[
\mathcal{M}, s \models \phi \iff \mathcal{M}', \widehat{s} \models \phi \iff \phi \in \widehat{s}
\]
for each proposition \(\phi \in \Phi\).

**Definition 3.5.** For each model \(\mathcal{M}\), the canonical model for \(\mathcal{M}\) is \([\mathcal{M}] = (\mathcal{M})_{can}\).

**Theorem 3.6.** For each proposition \(\phi\) and each model \(\mathcal{M}\), \(\mathcal{M}, s \models \phi\) iff \([\mathcal{M}], [s] \models \phi\).

**Proof.** Immediate from Lemmas 3.2 and 3.4. \(\square\)

We can now define a universal Kripke model \(\mathcal{K}\). Consider the class \(\mathcal{X}\) of all Kripke models. For each \(\mathcal{M} \in \mathcal{X}\) we define the mapping \(\theta_{\mathcal{M}}: W^\mathcal{K} \mapsto W^\mathcal{K}\) by
\[
\theta_{\mathcal{M}}(s) = \{\phi \mid \mathcal{M}, s \models \phi\}.
\]
We let the set of states \(W^\mathcal{K}\) of the universal model be exactly the set of all subsets of \(\Phi\) that can be obtained this way (when \(\mathcal{M}\) ranges over all Kripke models). That is, for \(\psi \in \Phi, \psi \in W^\mathcal{K}\) iff \(\psi = \theta_{\mathcal{M}}(s)\) for some model \(\mathcal{M}\) and state \(s \in W^\mathcal{K}\). We define \(\pi^\mathcal{K}\) by
\[
\pi^\mathcal{K}(p_i) = \{s \in W^\mathcal{K} \mid p_i \in s\}\]
for $0 \leq i < \omega$. The interpretation for the primitive programs is defined as

$$\rho^*(a_j) = \{(s, t) \in W^* \times W^* | \forall \phi, ([a_j] \phi \in s \Rightarrow \phi \in t)\}$$

for $0 \leq j < \omega$. Note that the states of $\mathcal{U}$ are precisely the semantically consistent complete sets of formulae.

We can also describe the universal model as the model which results from “pasting together” all canonical models $[\mathcal{M}]$ for all Kripke models $\mathcal{M}$. All states in $\mathcal{U}$ are “copies” of states in some canonical model $[\mathcal{M}]$.

**Lemma 3.7.** For each canonical model $[\mathcal{M}]$ and $\alpha \in \Pi$, $\rho^{[\mathcal{M}]}(\alpha) \subseteq \rho^*(\alpha)$.

**Proof.** It follows immediately from the definitions of $\rho^{[\mathcal{M}]}$ and $\rho^*$ that, for primitive $a$, $\rho^{[\mathcal{M}]}(a) \subseteq \rho^*(a)$. The lemma follows. $\square$

**Lemma 3.8.** Consider the universal model $\mathcal{U}$.

1. For each $\phi \in \Phi$ and $\alpha \in \Pi$,

   $$\langle \alpha \rangle \phi \in s \iff \exists t. (s, t) \in \rho(\alpha) \land \phi \in t.$$  

2. For each $\phi \in \Phi$,

   $$\mathcal{U}, s \models \phi \iff \phi \in s.$$

**Proof.** (1) ($\Rightarrow$): Let $\langle \alpha \rangle \phi \in s$. Then there exists a canonical model $[\mathcal{M}]$ and a state $[s] \in W^{[\mathcal{M}]}$ such that $\langle \alpha \rangle \phi \in [s]$. Then there exists a $[t] \in W^{[\mathcal{M}]}$ such that $([s], [t]) \in \rho^*(\alpha)$ and $\phi \in [t]$. Hence, by Lemma 3.7, $(s, t) \in \rho^*(\alpha)$ and $\phi \in t$.

   ($\Leftarrow$): Again define the function $R : \Pi \rightarrow 2^{W \times W}$ as in Theorem 3.4 except that we use $\in$ instead of $\models$. By the proof of that theorem, $\rho(\alpha) \subseteq R(\alpha)$. Hence, if $(s, t) \in \rho(\alpha)$ and $\phi \in t$, then $(s, t) \in R(\alpha)$ and by the definition of $R$, $\langle \alpha \rangle \phi \in s$.

(2) The proof is by induction on the structure of $\phi$. For $\phi$ primitive, the lemma holds by definition. Otherwise, we consider three cases.

   *Case 1:* ($\phi = \psi \lor \chi$)

   $s \models \psi \lor \chi$ iff $s \models \psi$ or $s \models \chi$, by the induction hypothesis, $\psi \in s$ or $\chi \in s$ iff $\psi \lor \chi \in s$ by the maximality of $s$.

   *Case 2:* ($\phi = \neg \psi$)

   Similar.

   *Case 3:* ($\phi = \langle \alpha \rangle \psi$)

   $s \models \langle \alpha \rangle \psi$ iff there is a state $t \in W$ such that $(s, t) \in \rho(\alpha)$ and $t \models \psi$ iff $\psi \models t$ by the induction hypothesis and $\langle \alpha \rangle \phi \in s$ by the first part of the lemma. $\square$

The following theorem is an immediate consequence of the lemma.

**Theorem 3.9.** There exists a universal Kripke model $\mathcal{U} = (W^*, \pi^*, \rho^*)$ such that for each
Kripke model \( \mathcal{M} = (W^*, \pi^*, \rho^*) \) there exists an embedding \( \theta : W^* \rightarrow W^* \) such that 
\[ \mathcal{M}, s \models \phi \iff \mathcal{M}, \theta(s) \models \phi \] for each well-formed formula \( \phi \).

**Proof.** The model \( \mathcal{U} \) constructed above and mappings \( \theta \) for each \( \mathcal{M} \) are the required model and mappings. \( \Box \)

We give two immediate consequences of Theorem 3.9 which will be instrumental for obtaining the results of the next section.

**Lemma 3.10.** (1) For all propositions \( \phi, \psi \) is satisfiable if and only if \( \psi \) is \( \mathcal{U} \)-satisfiable.

(2) For all propositions \( \phi, \psi \) is valid if and only if \( \phi \) is \( \mathcal{U} \)-valid.

### 4. Applications

In this section we prove the completeness of the system \( AX \) and the correctness of a construction for a small model using the universal model \( \mathcal{U} \).

#### 4.1. Completeness of \( AX \)

To prove the completeness of \( AX \) we adapt the Lindenbaum construction [1] to PDL. We impose a Boolean algebra structure on the state space \( W^* \) of \( \mathcal{U} \). With each proposition \( \phi \) we associate the set of states that satisfy \( \phi \):

\[ |\phi| = \{ s \in W^* | s \models \phi \} . \]

Let \( P \) be the set of all such \( |\phi| \). We define a partial ordering \( \leq \) on \( P \):

\[ |\phi| \leq |\psi| \text{ iff } \vdash \phi \rightarrow \psi . \]

**Lemma 4.1.** \( \mathcal{B} = \langle P, \leq \rangle \) is a complemented distributive lattice, i.e. a Boolean algebra.

**Proof.** By propositional reasoning we have

\[ \vdash \psi \rightarrow \text{true}, \]
\[ \vdash \text{false} \rightarrow \psi , \]

for all propositions \( \psi \). Hence, we can take \( |\text{true}| = 1 \) and \( |\text{false}| = 0 \) in \( \mathcal{B} \).

Let \( |\phi| \in P \). Then its complement, \( |\phi|^c \), is defined as

\[ |\phi|^c = \{ s | s \models \phi \}^c \]
\[ = \{ s | s \not\models \phi \} \]
\[ = \{ s | s \models \neg \phi \} \]
\[ = |\neg \phi| \]
and \( \neg \phi \in P \).

Let \( \phi, \psi \in P \). Then
\[
|\phi| \land |\psi| = \{s | s \models \phi \} \cap \{s | s \models \psi\} \\
= \{s | s \models \phi \land s \models \psi\} \\
= \{s | s \models \phi \land \psi\} \\
= |\phi \land \psi|.
\]

Hence, \( |\phi| \land |\psi| \in P \). By propositional reasoning,
\[
\vdash (\phi \land \psi) \rightarrow \phi \quad \text{and} \quad \vdash (\phi \land \psi) \rightarrow \psi.
\]

Hence, \( |\phi \land \psi| \) is a lower bound for \( \{|\phi|, |\psi|\} \). Suppose \( |\chi| \) is a lower bound too. Then \( \vdash \chi \rightarrow \phi \) and \( \vdash \chi \rightarrow \psi \). Hence, \( \vdash \chi \rightarrow (\phi \land \psi) \). This shows that \( |\phi \land \psi| \) is the greatest lower bound, i.e. the infimum of \( \{|\phi|, |\psi|\} \). Similarly, \( |\phi \lor \psi| \) is the supremum of \( \{|\phi|, |\psi|\} \). Thus, \( \mathcal{B} \) is a lattice.

Let \( \phi, \psi, |\chi| \in P \). Then \( |(\phi \land \psi) \lor | \in P \) and because
\[
\vdash ((\phi \land \psi) \lor \chi) \leftrightarrow ((\phi \lor \chi) \land (\psi \lor \chi)),
\]
we get from the soundness theorem,
\[
|((\phi \land \psi) \lor \chi)| = |(\phi \lor \chi) \land (\psi \lor \chi)|.
\]

This shows that \( \mathcal{B} \) is a complemented distributive lattice. \( \square \)

**Lemma 4.2.** In the Boolean algebra \( \mathcal{B} \),

1. \( |\phi| = 1 \) if and only if \( \vdash \phi \);
2. \( |\psi| = 0 \) if and only if \( \vdash \neg \psi \).

**Proof.** (1) Let \( |\phi| = 1 \). Then for each \( |\psi| \in P \), \( |\psi| \leq |\phi| \). Hence, for each \( |\psi| \), \( \vdash \psi \rightarrow \phi \).

Choose \( \psi \) so that \( \vdash \psi \), then, by modus ponens, \( \vdash \phi \). Conversely, suppose \( \vdash \phi \). Then, for each \( \psi \), \( \vdash \psi \rightarrow \phi \). Hence, for each \( \psi \), \( |\psi| \leq |\phi| \), so \( |\phi| = 1 \) in \( \mathcal{B} \).

(2) Similar. \( \square \)

**Lemma 4.3.** For all proposition \( \phi \), if \( \mathcal{U} \models \phi \), then \( \vdash \phi \).

**Proof.** Suppose that \( \phi \) is not provable in the system \( AX \). Then, by Lemma 4.2, in the Lindenbaum algebra \( \mathcal{B} \), \( |\phi| \neq 1 \) and so \( |\neg \phi| \neq 0 \). Hence, there exists a state \( s \in |\neg \phi| \) such that \( \mathcal{U}, s \models \neg \phi \). Hence, \( \phi \) is not \( \mathcal{U} \)-valid. \( \square \)

**Theorem 4.4** (completeness theorem). \( \vdash \phi \) if and only if \( \vdash \phi \).

**Proof.** One direction is the soundness theorem. The other direction follows from Lemmas 3.10 and 4.3. \( \square \)
4.2. The small model theorem

We find another application of Theorem 3.9 in a different proof of the small model theorem. This theorem is one of the basic results of the theory of PDL and was first discovered by Fischer and Ladner [3]. It states that every proposition $\phi$ that is satisfiable, is satisfiable in a model with $2^{2^{n}}$ states. This fact immediately gives rise to a naïve doubly exponential time decision procedure for the validity problem for PDL: to check whether $\phi$ is valid, generate all models with $2^{2^{n}}$ states and cycle through them in search of a model that satisfies $\neg\phi$. If such a model does not exist, then $\phi$ is valid. Sherman and Harel [6, 17] proved the existence of a singly exponential time procedure by constructing a model $\mathcal{M}_{\phi}$ that satisfies $\phi$ iff $\phi$ is satisfiable, following an idea of Pratt [14]. Thus, one can construct a model in polynomial time and check whether this model satisfies $\neg\phi$ in exponential time.

We first need a notion of the “subformulae” of a PDL formula $\phi$. This concept is captured by the Fischer–Ladner closure of $\phi$ [3].

**Definition 4.5.** Let $\phi \in \Phi$ be a PDL formula. The Fischer–Ladner closure of $\phi$, denoted by $FL(\phi)$, is the smallest set $S$ of formulae containing $\phi$ and satisfying the following closure rules for all $a \in \Pi_{0}, \alpha, \beta \in \Pi$ and $\psi, \chi \in \Phi$.

- $\neg\psi \in S \Rightarrow \psi \in S$,
- $\psi \lor \chi \in S \Rightarrow \psi, \chi \in S$,
- $\langle a \rangle \psi \in S \Rightarrow \psi \in S$,
- $\langle \alpha \rangle \beta \psi \in S \Rightarrow \langle \alpha \rangle \langle \beta \rangle \psi \in S$,
- $\langle \alpha \land \beta \rangle \psi \in S \Rightarrow \langle \alpha \rangle \psi, \langle \beta \rangle \psi \in S$,
- $\langle \alpha \ast \rangle \psi \in S \Rightarrow \psi, \langle \alpha \rangle \langle \alpha \ast \rangle \psi \in S$,
- $\langle \psi ? \rangle \chi \in S \Rightarrow \psi, \langle \chi \rangle \langle \psi ? \rangle \psi \in S$.

The Fischer–Ladner closure of $\phi$ is the set of all “subformulae” that are relevant for the meaning of $\phi$. The set $FL(\phi)$ induces an equivalence relation $\equiv_{\phi}$ on the state space $W$ of any model $\mathcal{M}$:

$s \equiv_{\phi} t$ iff $\forall \psi \in FL(\phi) (s \models \psi \iff t \models \psi)$.

In other words, we “collapse” $s$ and $t$ if they are not distinguishable by any formula of $FL(\phi)$. We now define the quotient model $\mathcal{M}/FL(\phi)$:

$[s] = \{ t | s \equiv_{\phi} t \}$,

$W^{\equiv_{FL(\phi)}} = \{ [s] | s \in W^{\equiv_{\phi}} \}$,

$\pi^{\equiv_{FL(\phi)}} (p_{i}) = \{ [s] | s \in \pi^{\equiv_{\phi}} (p_{i}) \}$ for all $p_{i} \in \Pi_{0}$,

$\rho^{\equiv_{FL(\phi)}} (a_{j}) = \{ ([s], [t]) | (s, t) \in \rho^{\equiv_{\phi}} (a_{j}) \}$ for all $a_{j} \in \Pi_{0}$.
The sets of formulae $\tilde{s}$ are called atoms of $FL(\phi)$ and play a crucial role in the definition of the model $\mathcal{A}_\phi$. For the definition of $\mathcal{A}_\phi$ we follow the exposition in [6].
Definition 4.10. Let $Z$ be the set of PDL formulae in which all formulae of $FL(\phi)$ and their negations occur. Then an atom of $FL(\phi)$ is defined to be a subset $A \subseteq Z$ such that for every $\alpha, \beta \in \Pi$ and $\psi, \chi \in \Phi$

- if $\neg \psi \in Z$, then $\psi \in A$ iff $\psi \notin A$;
- if $\psi \lor \chi \in Z$, then $\psi \lor \chi \in A$ iff $\psi \in A$ or $\chi \in A$;
- if $\langle x\beta \rangle \psi \in Z$, then $\langle x\beta \rangle \psi \in A$ iff $\langle x \chi \rangle \psi \in A$;
- if $\langle x \lor \beta \rangle \psi \in Z$, then $\langle x \lor \beta \rangle \psi \in A$ iff $\langle x \rangle \psi \in A$ or $\langle \beta \rangle \psi \in A$;
- if $\langle x^* \rangle \psi \in Z$, then $\langle x^* \rangle \psi \in A$ iff $\psi \in A$ or $\langle x \rangle \langle x^* \rangle \psi \in A$;
- if $\langle \psi \chi \rangle \psi \in Z$, then $\langle \psi \chi \rangle \psi \in A$ iff $\psi \in A$ and $\chi \in A$.

Note that for all $\psi \in FL(\phi)$, either $\psi$ or $\neg \psi$ is contained in each atom. Denote the set of all atoms of $FL(\phi)$ by $At(\phi)$. From the definition of atoms it follows that an $A \in At(\phi)$ is free of "obvious" or internal contradictions. In the construction of the model $\mathcal{A}_\phi$ we will eliminate the "nonobvious" or external contradictions also. This model will be constructed in phases. For the definition of the interpretation functions $\pi$ and $\rho$ we limit ourselves, without loss of generality, to the primitive predicate and program symbols occurring in $\phi$.

$\mathcal{A}_0 = (W_0, \pi_0, \rho_0)$ is defined by

- $W_0 = At(\phi)$;
- $\pi_0 : \mathcal{A}_0 \rightarrow 2^{W_0}$ by $A \in \pi_0(p)$ iff $p \in A$;
- $\rho_0 : \Pi_0 \rightarrow 2^{W_0 \times W_0}$ by $(A, B) \in \rho_0(a)$ iff
  1. there is an $\langle a \rangle \psi \in A$ with $\psi \in B$, and
  2. for every $[a] \psi \in A, \psi \in B$.

For $i > 0$, $\mathcal{A}_{i+1} = (W_{i+1}, \pi_{i+1}, \rho_{i+1})$ is defined by

- $W_{i+1} = \{ A \mid A \in W_i \text{ and for every } \langle a \rangle \psi \in A, \text{ there is } B \in W_i \text{ with } (A, B) \in \rho_i(a) \text{ and } \psi \in B \}$;
- $\pi_{i+1}(p) = \pi_i(p) \cap W_{i+1}$;
- $\rho_{i+1}(a) = \rho_i(a) \cap (W_{i+1} \times W_{i+1})$.

Here $\rho_i$ is the ordinary extension of $\rho_i$ to $\Pi$, except that for $\psi \in Z$ we define $\rho_i(\psi) := \{ (A, A) \mid \psi \in A \}$. The unprimed $\rho$ is the usual extension.

It follows from the finiteness of $At(\phi)$ and the fact that $W_{i+1} \subseteq W_i$ that there is a $j$ for which the construction closes up, i.e., $\mathcal{A}_i = \mathcal{A}_j$ for each $i > j$. Accordingly, set $\mathcal{A}_\phi = \mathcal{A}_j$.

The following lemma is the main technical lemma we need for our final result.

Lemma 4.11. For every $A \in W^{\rho_0}$,

1. for each $\langle x \rangle \psi \in FL(\phi)$, $\langle x \rangle \psi \in A$ iff there exists a $B \in W^{\rho_0}$ with $(A, B) \in \rho(x)$ and $\psi \in B$;
2. for each $\psi \in FL(\phi), \psi \in A$ iff $\mathcal{A}_{\phi}, A \models \psi$.

Proof. The proof proceeds by simultaneous induction on the structure of $x$ in (1) and the structure of $\psi$ in (2). See [17] for details. \qed
Theorem 4.12 (small model theorem). For all $\psi \in FL(\phi), \psi$ is satisfiable iff $\psi \in A$ for some $A \in W^{*\phi}$.

Proof. In the light of Theorem 4.9, we only need to prove that $W^{*\phi} = W^{df}$, from which the theorem follows.

- $W^{df} \subseteq W^{*\phi}$: immediate from the construction of $W_{\phi}$;
- suppose there exists an atom $A \in W^{*\phi}$ and $A \notin W^{df}$. As we have started from the set of all atoms in $W_0$, there exists a phase $i$ in which the first such atom is removed from $W_{i+1}$. Inspection of the algorithm shows that this can happen only if there exists a formula $\langle x^i \rangle \psi \in A$ such that there exists no $B \in W_i$ with $(A, B) \in \rho_i(x)$ and $\psi \in B$. But $A \in W^{*\phi}$; hence, there exists a state $B \in W^{*\phi}$ with $(A, B) \in \rho_i(x)$ and $\psi \in B$. Because $A$ is the first state to be removed, $B \notin W_i$; a contradiction. 0

5. An infinitary axiom system

Intuitively, the nature of the *-operator requires an infinitary axiom system. We define the system $AX_{\omega}$ as such an infinitary system. The induction axiom is replaced by an inference rule with an infinite set of premisses.

Definition 5.1. The infinitary axiom system $AX_{\omega}$ contains the following axioms.

1. All PDL axioms, except the induction axiom;
2. $[x^i]^\phi \rightarrow [x^i]^\phi$, for each $i < \omega$;

In addition, we have the following inference rules:
1. modus ponens: from $\phi, \phi \rightarrow \psi$, infer $\psi$;
2. modal generalization: from $\phi$, infer $[\alpha]^\phi$, for any $\alpha \in \Pi$;
3. $\infty$-rule: from $\{\psi \rightarrow [\beta; x^i]^\phi\}_{i < \omega}$, infer $\psi \rightarrow [\beta; x^\omega]^\phi$.

In a way, we treat $[x^\omega]^\phi$ as an "abbreviation" for $\bigwedge_{i < \omega} [x^i]^\phi$. By contraposition, we have, for each $i < \omega$,

$\langle x^i \rangle^\phi \rightarrow \langle x^\omega \rangle^\phi$

We define a derivation in $AX_{\omega}$ to be a countable sequence of well-formed formulae, each of which is either an instance of an axiom or the conclusion of an inference rule whose premisses occur earlier in the sequence. The last formula in the sequence is called the conclusion of the derivation and any formula $\phi$ for which such a derivation exists is called derivable or provable and we write $\vdash_{\omega} \phi$.

From the soundness theorem for $AX$, we immediately get a soundness theorem for $AX_{\omega}$.

Theorem 5.2 (soundness theorem). If $\vdash_{\omega} \phi$, then $\models \phi$.

In both systems, $AX$ and $AX_{\omega}$, derivability of formulae of the form $[x^\omega]^\phi$ is closely related, as the following theorem shows; a proof of the theorem can be found in [7].
Theorem 5.3. (1) In the infinitary system $AX_\infty$, the induction axiom is derivable.
(2) In the Segerberg system $AX$, $\vdash [x^*] (\phi \rightarrow [x] \phi)$ for all $n < \omega$.

We next give some definitions. Let $Pr(AX_\infty) = \{ \phi \mid \vdash_\infty \phi \}$ be the set of all provable formulae of the axiom system $AX_\infty$. For any subset $\Sigma \subseteq \Phi$, let $\bar{\Sigma}$ be the union $\Sigma \cup Pr(AX_\infty)$ closed under modus ponens and $\infty$-rule. $\Sigma$ is a theory if $\Sigma = \bar{\Sigma}$. Intuitively, $\bar{\Sigma}$ contains all immediate consequences of $\Sigma$; in particular, if $\{ \{x^i\} \mid i < \omega \} \subseteq \Sigma$, then $\{x^*\} \phi \in \bar{\Sigma}$.

Definition 5.4. Let $\Sigma$ be a set of formulae and $\phi$ a formula.
(1) $\Sigma \vdash_\infty \phi$ if and only if $\phi$ belongs to every theory that contains $\Sigma$.
(2) We say that $\Sigma$ is inconsistent iff $\Sigma \vdash_\infty \bot$.
(3) We say that $\Sigma$ is consistent iff $\Sigma$ is not inconsistent.
(4) $\Sigma$ is maximally consistent iff $\Sigma$ is consistent and for each $\phi \in \Phi$, either $\phi$ or $\neg \phi \in \Sigma$.

We give some useful lemmas.

Lemma 5.5. Let $\Sigma$ be a maximally consistent theory. Then $\{x^*\} \phi \in \Sigma$ implies $\{x^m\} \phi \in \Sigma$ for some $m < \omega$.

Lemma 5.6. Let $\Sigma$ be a theory. Then $\Sigma \vdash_\infty \phi$ iff $\phi \in \Sigma$.

Theorem 5.7 (deduction theorem). $\Sigma \cup \{ \phi \} \vdash_\infty \psi$ if and only if $\Sigma \vdash_\infty \phi \rightarrow \psi$.

Proof. Suppose that $\Sigma \cup \{ \phi \} \vdash_\infty \psi$. Let $\Delta = \{ \psi \mid \Sigma \vdash_\infty \phi \rightarrow \psi \}$.

We show that $\Delta$ is a theory containing $\Sigma \cup \{ \phi \}$. Since $\psi \rightarrow (\phi \rightarrow \psi)$ is a tautology, $\psi \in \Delta$ in case $\psi \in \Sigma$ or $\vdash_\infty \psi$. Since $\phi \rightarrow \phi$ is a tautology, $\phi \in \Delta$. Hence, $\Sigma \cup \{ \phi \} \subseteq \Delta$.

From the tautology
$$((\phi \rightarrow (\phi_1 \rightarrow \phi_2)) \rightarrow (\phi \rightarrow \phi_2))$$

we deduce that $\Delta$ is closed under modus ponens.

Finally, suppose that $\{ \phi_1 \rightarrow [x; \alpha^n] \phi_2 \mid n < \omega \} \subseteq \Delta$.

From the assumption we can deduce, using the $\infty$-rule and propositional reasoning, that
$$\Sigma \vdash_\infty (\phi \wedge \phi_1) \rightarrow [x; \alpha^*] \phi_2$$
Hence, \( \Delta \) is closed under the \( \infty \)-rule. This proves one direction; the other direction is trivial. \( \square \)

**Corollary 5.8.** \( \Sigma \cup \{ \phi \} \) is consistent iff \( \Sigma \not\vdash \neg \phi \).

It is interesting to note that a semantic counterpart of the deduction theorem does not hold. For an easy example, consider a nontautology \( \phi \). Then we have \( \{ \phi \} \models [\alpha]\phi \) for all programs \( \alpha \). But clearly \( \not\models \phi \rightarrow [\alpha]\phi \). This observation also prevents us from deriving a "strong" completeness theorem, i.e. \( \Gamma \vdash_{\infty} \phi \) iff \( \Gamma \models \phi \): the deduction theorem together with the completeness theorem for \( AX_{\infty} \) would imply a semantical deduction theorem. See, however, Corollary 5.11. We now define a model \( \mathcal{A} \) by

- \( W^\mathcal{A} = \{ s \in \Phi \mid \Pr(AX_{\infty}) \subseteq s \text{ and } s \text{ is maximally consistent} \} \),
- \( \pi^\mathcal{A}(p) = \{ s \mid p \in s \} \) for primitive predicates \( p \),
- \( \rho^\mathcal{A}(a) = \{ (s, t) \mid \forall \psi . ([a] \psi \in s \Rightarrow \psi \in t) \} \) for primitive programs \( a \).

**Lemma 5.9.** For each proposition \( \phi \),

\[ \mathcal{A}, s \models \phi \text{ iff } \phi \in s. \]

**Proof.** We proceed by induction on the complexity of \( \phi \). For \( \phi \) a primitive predicate, the theorem holds by definition.

- \( \phi = \psi \lor \chi \): \( \mathcal{A}, s \models \psi \lor \chi \) iff \( \mathcal{A}, s \models \psi \) or \( \mathcal{A}, s \models \chi \) iff, by induction hypothesis, \( \psi \in s \) or \( \chi \in s \) iff \( \psi \lor \chi \in s \), by construction.

- \( \phi = \neg \psi \): \( \mathcal{A}, s \models \neg \psi \) iff \( \mathcal{A}, s \models \psi \) iff \( \psi \notin s \) iff \( \neg \psi \in s \).

- \( \phi = [\alpha] \psi \): The only nontrivial case. We prove this case by induction on the structure of \( \alpha \).

First let \( \alpha = a \) be a primitive program. \( \mathcal{A}, s \models [a] \psi \) iff there exists a state \( t \) such that \( (s, t) \in \rho(a) \) and \( \mathcal{A}, t \models \psi \). By induction hypothesis, \( \psi \in t \) and by the definition of \( \rho(a), [a] \psi \in s \). Conversely, suppose \( [a] \psi \in s \). Consider the set

\[ \Gamma = \{ [\alpha] \phi \mid \phi \in s \}. \]

**Claim 1.** \( \Gamma \) is a theory.

**Proof of Claim 1.** \( \Pr(AX_{\infty}) \subseteq \Gamma \), by the definition of \( s \). Suppose \( \phi, \phi \rightarrow \psi \in \Gamma \), then \( \psi \in \Gamma \) since the logic is normal. Suppose \( \psi \rightarrow [\delta; \beta^*] \phi \in \Gamma \) for all \( i < \omega \); then \( [a] \psi \rightarrow [\delta; \beta^*] \phi \in s \) for all \( i < \omega \). Hence, \( [a] \psi \rightarrow [a] [\delta; \beta^*] \phi \in s \) by the maximality of \( s \). We argue that \( [a] (\psi \rightarrow [\delta; \beta^*] \phi) \in s \). Suppose not. Then \( \neg[a] (\psi \rightarrow [\delta; \beta^*] \phi) \in s \) or

\[ [a] (\psi \land [\delta; \beta^*] \neg \phi) \in s \]

and, by Lemma 5.5,

\[ [a] (\psi \land [\delta; \beta^*] \neg \phi) \in s \]
for some \( m < \omega \); a contradiction. Hence, \( \Gamma \) is closed under the \( \infty \)-rule and is a theory. \( \square \)

Extend \( \Gamma \) to the set \( \Gamma' = \Gamma \cup \{ \psi \} \).

**Claim 2.** \( \Gamma' \) is consistent.

**Proof of claim 2.** Suppose \( \Gamma' \) is inconsistent. Then, by Corollary 5.8, \( \Gamma \vdash_\infty \neg \psi \). By Lemma 5.6, \( \neg \psi \in \Gamma \) or \( [a] \neg \psi \in \Gamma \). But \( \langle a \rangle \psi \in \Gamma \) by assumption; a contradiction. \( \square \)

**Proof of Lemma 5.9 (conclusion).** Hence, \( \Gamma' \) can be extended to a maximally consistent set \( t \). By the definition of \( \rho, (s, t) \in \rho(a) \) and by induction hypothesis, \( \mathcal{A}, t \models \psi \). Hence, \( \mathcal{A}, s \models \langle a \rangle \psi \). The case \( x \) is primitive, is proved. The other cases follow easily.

\( \mathcal{A}, s \models \langle \chi^? \rangle \psi \iff \mathcal{A}, s \models \chi \land \psi \iff \), by induction hypothesis, \( \chi \land \psi \in \Gamma \) iff \( \langle \chi^? \rangle \psi \in \Gamma \).

\( \mathcal{A}, s \models \langle x \cup \beta \rangle \psi \iff \mathcal{A}, s \models \langle x \rangle \psi \lor \langle \beta \rangle \psi \iff \langle x \rangle \psi \lor \langle \beta \rangle \psi \in \Gamma \) iff \( \langle x \cup \beta \rangle \psi \in \Gamma \).

\( \mathcal{A}, s \models \langle x \rangle \beta \psi \iff \mathcal{A}, s \models \langle x \rangle \langle \beta \rangle \psi \iff \langle x \rangle \langle \beta \rangle \psi \in \Gamma \) iff \( \langle x; \beta \rangle \psi \in \Gamma \).

Dually we prove \( [\chi^*] \psi \in \Gamma \) iff \( \mathcal{A}, s \models [\chi^*] \psi \). \( \mathcal{A}, s \models [\chi^*] \psi \iff \), by definition of Kripke models, \( \mathcal{A}, s \models [\chi^*] \psi \) for each \( n < \omega \), iff, by induction hypothesis, \( [\chi^*] \psi \in \Gamma \) for each \( n < \omega \), iff, by the \( \infty \)-rule, \( [\chi^*] \psi \in \Gamma \).

With Lemma 5.9 we can easily prove the completeness of the system \( AX_\infty \).

**Theorem 5.10** (completeness theorem). For each PDL formula \( \phi \), \( \vdash_\infty \phi \iff \models \phi \).

**Proof.** One direction is the soundness theorem; for the other direction, let \( \phi \) be such that \( \vdash_\infty \phi \). Then \( \Pr(AX_\infty \cup \{ \neg \phi \}) \) is consistent and can be extended to a maximally consistent set \( s \) by Lindembaum's theorem. Hence, \( s \in W^{\mathcal{A}} \) and \( \mathcal{A}, s \models \neg \phi \) by Lemma 5.9, which implies that \( \phi \) is not valid or \( \not\models \phi \). \( \square \)

From the completeness theorem we immediately deduce the following weaker forms of the deduction and strong completeness theorems.

**Corollary 5.11.** (1) \( \mathcal{A}, \Gamma \models \phi \iff \Gamma \vdash_\infty \phi \).

(2) \( \mathcal{A}, \Gamma \cup \phi \models \psi \iff \mathcal{A}, \Gamma \models \phi \iff \psi \),

where \( \mathcal{A}, \Gamma \models \phi \) holds if for all \( s \in W^{\mathcal{A}} \), if \( s \models \phi \), for all \( \phi_1 \in \Gamma \), then \( s \models \phi \).

Since the Segerberg axiomatization is complete for PDL, we have the following corollary.

**Corollary 5.12.** For all \( \phi \in \Phi \),

\( \vdash \phi \iff \models \phi \iff \models \neg_\infty \phi \).
Let $\mathcal{U}$ be the model as defined in the previous section. An immediate observation leads to the next lemma.

**Lemma 5.13.** $W^* = W^\#$.

**Proof.** By soundness, each $s \in W^\#$ is maximally consistent and $\mathsf{Pr}(AX_n) \subseteq s$ so $W^* \subseteq W^\#$. Conversely, $W^\# \subseteq W^*$ by completeness. □

By the lemma and the constructions of $\mathcal{U}$ and $\mathcal{A}$ we get Theorem 5.14.

**Theorem 5.14.** $\mathcal{U} \cong \mathcal{A}$.

In fact we may say that $\mathcal{U}$ and $\mathcal{A}$ are only two different names for the same model and conclude that $\mathcal{U} = \mathcal{A}$.

### 6. Nonstandard Models

We have introduced a completeness technique for PDL which is based on an infinitary axiom system. One might ask whether this technique is applicable to the “normal” axiomatization as well. The answer to this question is “No”. The difficulty in proving a lemma such as Lemma 5.9 lies in the case $J = [\star]$. Let us see what happens when we try to prove the case. We can prove that $\mathcal{A}, s \models [\star] \psi$ implies $[\star^n] \psi \in s$ for each $n < \omega$, but we may not infer then that $[\star^*] \psi \in s$. In fact, we can prove the following theorem.

**Theorem 6.1.** Let

$$\Gamma = \mathsf{Pr}(AX) \cup \{ \phi, [a^1] \phi, [a^2] \phi, \ldots \} \cup \{ \neg [a^*] \phi \}$$

Then $\Gamma$ is consistent.

**Proof.** Suppose $\Gamma$ is inconsistent. Then for some finite subset $\Gamma' = \{ \phi_0, \phi_1, \ldots, \phi_n \} \subseteq \Gamma$,

$$\Gamma' \not\models \text{false},$$

or

$$\not\models \phi_0 \land \cdots \land \phi_n \rightarrow \text{false}.$$ 

Without loss of generality, we may assume that $\phi_n = \neg [a^*] \psi$ and the other $\phi_j \in \Delta$. By soundness, then, for all models $\mathcal{M}$ and states $s \in W^\#,\mathcal{M}, s \models \phi_0 \land \cdots \land \phi_{n-1} \rightarrow [\star] \phi$. But counterexamples are easily found. Hence, $\Gamma$ is consistent. □
Essentially, this is the same argument as we used for proving noncompactness. There we saw that an infinite, semantically inconsistent set could not be proved to be inconsistent by proving inconsistency of each of its finite subsets. In fact, each of its finite subsets was consistent. For exactly the same reason, namely, syntactic consistency of each of the finite subsets of \( \Gamma \), we must conclude that \( \Gamma \) itself is syntactically consistent. Yet it surely is not semantically consistent in standard Kripke models. We therefore conclude that syntactic and semantic consequences are two different notions in the case of the axiom system \( AX \) and standard models.

As has been noted in [4, 11], we can construct a syntax model \( \mathcal{A}' \) from the Segerberg axiomatization that is a nonstandard model in the following sense.

**Definition 6.2.** A nonstandard Kripke model is any model \( \mathcal{M} \) that is a Kripke model according to Definition 2.1, except that \( \rho^*(x^*) \) need not be the reflexive transitive closure of \( \rho^*(x) \), but only a reflexive transitive relation containing \( \rho^*(x) \) and satisfying the induction axiom.

In a way we might view this relaxation as a means to “compactify” the logic: the set \( F \) from Theorem 6.1 is satisfiable in a nonstandard model. In nonstandard models the set \( \rho(x^*) \) is simply larger than in standard models.

The construction of \( \mathcal{A}' \) proceeds as follows. Let consistency for \( \vdash \) be defined in the usual way (cf. [11]).

- \( W^\mathcal{A}' = \{ s \models \Phi | \Pr(AX) \models s \text{ and } s \text{ is maximally consistent} \} \);
- \( \pi^\mathcal{A}'(p) = \{ s | p \in s \} \) for primitive \( p \);
- \( \rho^\mathcal{A}'(x) = \{ (s, t) | \forall \phi.([x] \phi \models s \Rightarrow \phi \models t) \} \).

Note the definition of \( \rho^\mathcal{A}' \) which is defined for all programs, rather than only for primitive one's.

**Theorem 6.3.** Let \( \mathcal{A}' \) be the syntax model constructed from the Segerberg axiom system as indicated above. Then

1. \( \mathcal{A}' \) is nonstandard;
2. \( \mathcal{A}' \) is universal in the class of nonstandard models.

**Proof.** For (1), see [4, 11]. For (2), it is sufficient to prove

\[ \mathcal{A}', s \models \phi \iff \phi \in s. \]

To prove this claim we can adapt the proof of Lemma 5.9, or see [2].

**Corollary 6.4.** The Segerberg system \( AX \) is complete for PDL with respect to nonstandard models.

Note that the infinitary system is not complete with respect to these models.
7. Dynamic algebras

In this section we recall the notion of dynamic algebras [8, 9, 15] and study the relationship between these algebras and Kripke models.

Dynamic algebras were introduced by Kozen [8, 9] and Pratt [15] to give PDL a more algebraic interpretation, in much the same way as Boolean algebras give an interpretation for propositional logic.

A dynamic algebra is a two-sorted algebra $D = (K, B, \Diamond)$, where $K$ is a Kleene or relational algebra and $B$ is a Boolean algebra, for which a scalar multiplication $\Diamond : K \times B \to B$ is defined. The basic operators for the Boolean algebra are $\land, \lor$ and $\neg$; the operators for the Kleene algebra are $;$, $\cup$ and $\ast$. The defining axioms for the Boolean algebra are standard. However, we do not have equality for the Kleene elements. Instead, we axiomatize the meaning of $\Diamond$. As we have seen, there exist two axiomatizations for PDL that are sound and complete; Pratt used the Segerberg system and Kozen the infinitary system to axiomatize their versions of dynamic algebras. We concentrate on the version of Kozen, which is called $\ast$-continuous. Hence, we have as axioms

1. the axioms for Boolean algebras;
2. $\Diamond \phi = 0$;
3. $\Diamond (\phi \lor \psi) = \Diamond \phi \lor \Diamond \psi$;
4. $\Diamond (\phi ; \beta) = \Diamond \phi \lor \Diamond \beta$;
5. $\Diamond (\phi \cup \beta) = \Diamond \phi \lor \Diamond \beta$;
6. $\Diamond (\phi \ast) = \bigvee \Diamond (\phi \ast, \beta) \phi$.

Here we have used $\phi, \psi$ to denote the Boolean elements and $\alpha, \beta$ to denote the Kleene elements. Let $\Phi_0$ and $\Pi_0$ be the sets of names for (primitive) propositions and programs as defined in Section 2. These names act as names for constants in these algebras. Let $\mathcal{D}$ be the class of all $\ast$-continuous dynamic algebras with sets of constants $\Phi_0$ and $\Pi_0$. $\mathcal{D}$ is equationally defined and, thus, has an initial algebra $\mathcal{I}$. We construct $\mathcal{I}$ as follows. Let $T$ be the term algebra generated over $\Phi_0$ and $\Pi_0$. Then $T = (\Phi, \Pi, \Diamond)$ and $\mathcal{I} = (\Phi / =, \Pi, \Diamond)$. Let $D$ be any member of $\mathcal{D}$; then every assignment of elements of $D$ to the sets $\Phi_0$ and $\Pi_0$ extends to a homomorphism of $\mathcal{I}$ into $D$. An immediate observation is

$\mathcal{I} = \phi = \psi$ if $\vdash_\mathcal{I} \phi \leftrightarrow \psi$.

With every Kripke model $\mathcal{M}$ we can easily associate a dynamic algebra $\mathcal{M}$. With every $\phi \in \Phi$ we associate the subset $|\phi| \subseteq W$, where $|\phi|$ is defined by

$|\phi| = \{ s \mid \mathcal{M}, s \models \phi \}$.

Denote the set of all such subsets $|\phi|$ by $|\Phi|^\ast$. Similarly, with every $\alpha \in \Pi$ we associate the function $|\alpha|$ defined by

$\langle |\alpha| \rangle |\phi| = |\alpha \phi|$. 

Denote the set of all such functions $|\alpha|$ by $|\Pi|^\pi$. We let $\tilde{\mathcal{M}} = \{ |\Phi|^\pi, |\Pi|^\pi, \Diamond \}$.

**Lemma 7.1.** For every Kripke model $\mathcal{M}$, $\tilde{\mathcal{M}}$ is well defined.

**Proof.** We have already proven that $|\Phi|^\pi$ is a Boolean algebra. For the Kleene part of $\tilde{\mathcal{M}}$, we define operators and prove the axioms.

$(\tilde{\alpha} = |\alpha|)$: axioms (2) and (3) obviously hold.

$(\tilde{\alpha} = |\alpha|; |\beta|)$:

\[
\langle |\alpha|; |\beta| \rangle \phi \overset{\text{def}}{=} \langle |\alpha| \rangle \langle |\beta| \rangle \phi
\]

\[
\overset{\text{(2)}}{=} \langle |\alpha| \rangle \langle |\beta| \rangle \phi
\]

\[
\overset{\text{(3)}}{=} \langle |\alpha| \rangle \langle |\beta| \rangle \phi.
\]

$(\tilde{\alpha} = |\alpha| \cup |\beta|)$: Similar.

$(\tilde{\alpha} = |\alpha|^*)$:

\[
\langle |\alpha|^* \rangle \phi \overset{\text{def}}{=} \langle |\alpha|^* \rangle \phi
\]

\[
\overset{\text{(2)}}{=} \bigvee_{\alpha} \langle |\alpha|^* \rangle \phi
\]

\[
\overset{\text{(3)}}{=} \bigvee_{\alpha} \langle |\alpha|^* \rangle \phi
\]

\[
\overset{\text{□}}{=} \bigvee_{\alpha} \langle |\alpha|^* \rangle \phi.
\]

Next we consider the algebra $\tilde{\mathcal{M}}$. A first observation is that every (associated) algebra $\tilde{\mathcal{M}}$ is a subalgebra of $\mathcal{M}$: the embedding $\theta_N$ of $W^\pi$ into $W^\pi$ extends to an embedding of $\mathcal{M}$ into $\mathcal{M}$. The main result of this section is now immediate.

**Theorem 7.2.** $\tilde{\mathcal{M}} \cong \mathcal{M}$.

**Proof.** Consider the mapping $f: \tilde{\mathcal{M}} \rightarrow \mathcal{M}$ defined by

- $f(|\Phi|) = \Phi$;
- $f(|\alpha|) = \alpha$.

$f$ is clearly surjective. $f$ is injective as well:

\[
\tilde{\mathcal{M}} \models |\Phi| \neq |\psi| \iff \mathcal{M} \not\models \Phi \leftrightarrow \psi
\]

\[
\iff \not\models \mathcal{M} \phi \leftrightarrow \psi
\]

\[
\iff \mathcal{M} \models \phi = \psi
\]

\[
\iff \mathcal{M} \not\models \phi \neq \psi.
\]

Finally, by Lemma 7.1, $f$ is a homomorphism. □
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