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Generalized independence and domination in graphs

Mieczysław Borowiecki *, Danuta Michalak

Institute of Mathematics, Technical University, Podgórna 50, 65-246 Zielona Góra, Poland

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Abstract

The purpose of this paper is to introduce various concepts of $\mathscr{P}$-domination, which generalize and unify different well-known kinds of domination in graphs. We generalize a result of Lovász concerning the existence of a partition of a set of vertices of G into independent subsets and a result of Favaron concerning a property of S_k-dominating sets. Gallai-type equalities for the strong $\mathscr{P}$-domination number are proved, which generalize Nieminen's result.

Keywords: Dominating set; Independent set; Hereditary property of graphs; Vertex partition

1.

In this paper we will consider finite undirected graphs with no multiple edges, and with no loops. For a graph G we will refer to $V(G)$ (or V) and $E(G)$ (or E) as the vertex and edge set, respectively.

A nonempty subset D of the vertex set V of a graph G is a *dominating* set if every vertex in $V - D$ is adjacent with a member of D. If $u \in D$ and $v \in V - D$, and $uv \in E$, we say that u *dominates* v and v is *dominated* by u. The minimum of the cardinalities of the minimal dominating sets in G is called the *domination number* of G and it is denoted $\gamma(G)$.

The study of domination in graphs was initiated by Ore [11], for a survey see the special volume Discrete Math. 86 (1990). Applications of minimum dominating sets have been suggested by many authors, but the determination of the domination number is an NP-complete problem, see [6]. It should be noted that bounds on $\gamma(G)$ do exist, though the parameters values on which these bounds depend may also be diffucult to determine.

* Corresponding author. E-mail: m.borowiecki@im.pz.zgora.pl.

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We write $H \leqslant G$ if H is an induced subgraph of G. We use the notation $\langle A \rangle_G$ for the subgraph of G induced by $A \subseteq V(G)$, by $m(A)$ we denote the number of edges of $\langle A \rangle_G$ and by $d_A(v)$ the number of neighbours in A of a vertex $v \in V$.

A set $S \subseteq V(G)$ is said to be *independent* if $\langle S \rangle_G$ is totally disconnected, i.e., has no edge. Obviously, each maximal independent set is a minimal dominating set. If S is a maximal independent set of G, then $\langle S \cup \{v\} \rangle_G$ contains K_2, as a subgraph for any $v \in V - S$ i.e., the subgraph which is forbidden for the property '*to be totally disconnected*'. This observation leads us to the various concepts of $\mathscr{P}$-domination in graphs with respect to any hereditary property $\mathscr{P}$.

Let $\mathscr{I}$ denote the set of all mutually nonisomorphic graphs.

If $\mathscr{P}$ is a nonempty subset of $\mathscr{I}$, then $\mathscr{P}$ will also denote the property that a graph is a member of the set $\mathscr{P}$.

A property $\mathscr{P}$ of graphs is said to be *induced hereditary* if whenever $G \in \mathscr{P}$ and $H \leqslant G$, then also $H \in \mathscr{P}$. For hereditary properties with respect to a partial order see [1].

Any induced hereditary property $\mathscr{P}$ of graphs is uniquely determined by the set of its forbidden subgraphs, which is defined as follows:

$$C(\mathscr{P}) = \{H \in \mathscr{I}: H \notin \mathscr{P} \text{ but } (H - v) \in \mathscr{P} \text{ for any } v \in V(H)\}.$$

Let us denote by $\mathbf{M}$ the set of all induced hereditary properties of graphs.

A property $\mathscr{P}$ is said to be *additive*, if for each graph G all of whose components have the property $\mathscr{P}$ it follows that $G \in \mathscr{P}$. Obviously, $\mathscr{P}$ is additive if and only if the following holds: if H and G have property $\mathscr{P}$, then so does their disjoint union $H \cup G$. Denote by $\mathbf{M}^{\mathrm{a}}$ the set of all additive-induced hereditary properties of graphs.

According to [1] we list some induced hereditary properties in order to introduce the notion which will be used in the paper.

$$\mathscr{O} = \{G \in I: G \text{ is totally disconnected}\},\ C(\mathscr{O}) = \{K_2\};$$

$$\mathscr{S}_k = \{G \in I: \Delta(G) \leqslant k\},\ C(\mathscr{S}_k) = \{H: |V(H)| = k + 2 = \Delta(H) + 1\};$$

$$\mathscr{I}_k = \{G \in I: G \text{ does not contain } K_{k+2}\},\ C(\mathscr{I}_k) = \{K_{k+2}\}.$$

Let $\mathscr{P} \in \mathbf{M}$ and $G = (V, E)$ be a graph. Two vertices u and v of G are called *$\mathscr{P}$-adjacent* if there is a subgraph H' of G isomorphic to $H \in C(\mathscr{P})$ containing u and v. For a vertex $v \in V$ by $N_{\mathscr{P}}(v)$ we denote the *$\mathscr{P}$-neighbourhood* of v, i.e., $N_{\mathscr{P}}(v) = \{u \in V: u \text{ is } \mathscr{P}\text{-adjacent to } v\}$. For a set $X \subseteq V$, let $N_{\mathscr{P}}(X) = \bigcup_{v \in X} N_{\mathscr{P}}(v)$. Especially $N(v) = N_{\mathscr{O}}(v)$.

Next, for a vertex $v \in V(G)$ we denote the set of all forbidden subgraphs containing v by $C_{G,\mathscr{P}}(v) = \{H' \leqslant G: v \in V(H'),\ H' \simeq H \in C(\mathscr{P})\}$.

The number $|C_{G,\mathscr{P}}(v)|$ is called *$\mathscr{P}$-degree* of v in G and it is denoted $\deg_{G,\mathscr{P}}(v)$. If $\deg_{G,\mathscr{P}}(v) = 1$, then v is said to be *$\mathscr{P}$-pendant*. If $\deg_{G,\mathscr{P}}(v) = 0$, then v is said to be *$\mathscr{P}$-isolated*.

For a property $\mathscr{P}$, let $\Delta(\mathscr{P}) = \min\{\Delta(H): H \in C(\mathscr{P})\}$.

A set $S \subseteq V(G)$ is *$\mathscr{P}$-independent* in G if $\langle S \rangle_G \in \mathscr{P}$.

A set $D \subseteq V$ is said to be *$\mathscr{P}$-dominating* in G if $N_{\mathscr{P}}(v) \cap D \neq \emptyset$ for any $v \in V - D$.

A set $D \subseteq V$ is said to be *strongly $\mathscr{P}$-dominating* in G if for every $v \in V - D$ there is $H' \leqslant G$ containing v such that $H' \simeq H \in C(\mathscr{P})$ and $V(H') - \{v\} \subseteq D$.

The minimum of the cardinalities of the (strongly) $\mathscr{P}$-dominating sets of G is called the (*strong*) *$\mathscr{P}$-domination number* of G and is denoted by $\gamma_{\mathscr{P}}(G)$ $(\gamma'_{\mathscr{P}}(G))$, respectively.[1]

Notice, that if $\mathscr{P} = \mathcal{O}$, then $\mathscr{P}$-dominating and strongly $\mathscr{P}$-dominating sets in G are dominating sets in the ordinary sense.

Next, if $\mathscr{P} = \mathscr{I}_{n-2}$, then the $\mathscr{I}_{n-2}$-dominating set in G is the K_n-dominating set in G, see [8].

2.

Lemma. *Let $\mathscr{P} \in \mathbf{M}$. For any graph G and every $\mathscr{P}$-independent set D of G such that $w(D) = \Delta(P)|D| - m(D)$ is maximum, every vertex of $V - D$ is dominated by at least $\Delta(\mathscr{P})$ vertices of D.*

Proof. Let D be a $\mathscr{P}$-independent set such that $w(D)$ is maximum. Let us assume that there is a vertex $v \in V - D$ which is not dominated by at least $\Delta(\mathscr{P})$ vertices of D. Let $\mathscr{H}_v = \{H_v^i\colon i = 1, \ldots, r\}$ be the family of all forbidden subgraphs in G with $v \in V(H_v^i)$ and $(V(H_v^i) - \{v\}) \subseteq D$. It is clear that $\mathscr{H}_v \neq \emptyset$, for otherwise, $\langle D \cup \{v\} \rangle \in \mathscr{P}$ and $w(D \cup \{v\}) > w(D)$, a contradiction.

Let

$$U = \bigcup_{i=1}^{r} (V(H_v^i) - \{v\}) \text{ and } N(v) \cap D = B \text{ with } |B| = b.$$

Let A be a subset of U defined as follows:

$$A = \{x\colon d_U(x) \geqslant \Delta(\mathscr{P}) - 1, \text{ if } x \in N(v)\} \cup \{x\colon d_U(x) \geqslant \Delta(\mathscr{P}), \text{ if } x \notin N(v)\}.$$

Let $T \subseteq A$ be a minimal transversal of $\{V(H_v^i) - \{v\}\colon i = 1, \ldots, r\}$. Such a transversal exists since $A \cap (V(H_v^i) - \{v\}) \neq \emptyset$ for $i = 1, \ldots, r$. Now we consider the set $C = (D - T) \cup \{v\}$. Obviously, the set C is $\mathscr{P}$-independent. Let $|T| = t$ and $|T \cap B| = s$. Since T is minimal, for each $x \in B \cap T$ there is $H_x \in \mathscr{H}_v$ such that $d_{H_x}(x) \geqslant \Delta(\mathscr{P}) - 1$ and for each $y \in T - B$ there is $H_y \in \mathscr{H}_v$ such that $d_{H_y}(y) \geqslant \Delta(\mathscr{P})$. Now, we can estimate the number of edges in $\langle C \rangle_G$ as follows:

$$\begin{aligned} m(C) &\leqslant m(D) + b - s - (t - s)\Delta(\mathscr{P}) - s(\Delta(\mathscr{P}) - 1) - m(T) \\ &= m(D) + b - t\Delta(\mathscr{P}) - m(T). \end{aligned}$$

Hence, $w(C) = \Delta(\mathscr{P})|C| - m(C) \geqslant \Delta(\mathscr{P})(|D| - t + 1) - (m(D) + b - t\Delta(\mathscr{P}) - m(T)) = \Delta(\mathscr{P})|D| - m(D) + \Delta(\mathscr{P}) - b + m(T) > w(D)$, a contradiction. □

[1] $V(G)$ is $\mathscr{P}$-dominating set and also strongly $\mathscr{P}$-dominating set. Every (strongly) $\mathscr{P}$-dominating set contains a minimal (strongly) $\mathscr{P}$-dominating subset, therefore $\gamma_{\mathscr{P}}(G)$ and $(\gamma'_{\mathscr{P}}(G))$ are defined for every G.

Using induction on $\Delta(G)$ and the above lemma we get the following result concerning the existence of a partition of $V(G)$ into a 'small' number of $\mathscr{P}$-independent subsets.

Theorem 1. *Let $\mathscr{P} \in \mathbf{M}$. Then for every graph G there is a partition $(V_1, \ldots, V_t)$ of V into $t = \lfloor \Delta(G)/\Delta(\mathscr{P}) \rfloor + 1$ vertex disjoint subsets with $\langle V_i \rangle \in \mathscr{P}$ for $i = 1, \ldots, t$.*

For $\mathscr{P} = \mathscr{S}_k$ Theorem 1 implies a result of Lovász [9].

Some other results of this type are presented in [1].

Since every maximal $\mathscr{P}$-independent set of G is a minimal strongly $\mathscr{P}$-dominating set, Lemma implies the following result.

Theorem 2. *Let $\mathscr{P} \in \mathbf{M}$. In every graph G there exists a minimal strongly $\mathscr{P}$-dominating set D of G such that every vertex of $V - D$ is dominated by at least $\Delta(\mathscr{P})$ vertices of D.*

Theorem 2 implies Favaron's Theorem [2] in the case $\mathscr{P} = \mathscr{S}_k$.

3.

In 1959 Gallai presented his, now classical, theorem, involving the vertex covering number α_0, the vertex independence number β_0, the edge covering number α_1 and the edge independence number β_1.

Theorem (Gallai [5]). *For every nontrivial connected graph G with p vertices we have*

$$\alpha_0 + \beta_0 = p \quad \text{and} \quad \alpha_1 + \beta_1 = p.$$

A large number of similar results and generalizations of this theorem have been obtained in subsequent years; they are called Gallai-type equalities.

Theorem (Nieminen [10]). *Let $\gamma(G)$ be the domination number and $\varepsilon(G)$ be the maximum number of pendant edges in a spanning forest of a graph G with p vertices. Then $\gamma(G) + \varepsilon(G) = p$.*

Let $\mathscr{P} \in \mathbf{M}$ and G be a graph. Let S be a spanning subgraph of G. A family $X_{\mathscr{P}}(S) = \{G_1, G_2, \ldots, G_k\}$ of induced subgraphs of S such that

(1) $G_i \simeq H \in C(\mathscr{P})$ and

(2) For any G_i there is a vertex $v_i \in V(G_i)$ such that $v_i \notin V(G_j)$, $j \neq i$, $1 \leqslant i, j \leqslant k$ is called a *family of $\mathscr{P}$-pendant subgraphs of S*.

A vertex $v_i \in V(G_i)$ satisfying (2) is called a *$\mathscr{P}$-pendant vertex in the family $X_{\mathscr{P}}(S)$*.

Let $\varepsilon_{\mathscr{P}}(G)$ be the maximum number of $\mathscr{P}$-pendant subgraphs in a spanning subgraph of the graph G.

Notice, that if $\mathscr{P} = \mathcal{O}$, then $\varepsilon_{\mathscr{P}}(G) = \varepsilon(G)$.

Theorem 3. *Let $\mathscr{P} \in \mathbf{M}$. For every graph G of order p, we have*

$$\gamma'_{\mathscr{P}}(G) + \varepsilon_{\mathscr{P}}(G) = p.$$

Proof. Let D be a minimal strongly $\mathscr{P}$-dominating set with $|D| = \gamma'_{\mathscr{P}}(G)$. Then for every $x \in V - D$ there is $H' \leqslant G$, $H' \simeq H \in C(\mathscr{P})$, such that $x \in V(H')$ and $V(H') \cap D = V(H') - \{x\}$. For every $x \in V - D$ we choose exactly one such subgraph and denote it by H_x; in this way, we have a family of $\mathscr{P}$-pendant subgraphs in a spanning subgraph S of G with the edge set $E(S) = \bigcup_{x \in V - D} E(H_x)$. Hence, $\varepsilon_{\mathscr{P}}(G) \geqslant |V - D| = p - \gamma'_{\mathscr{P}}(G)$.

On the other hand, let S be a spanning subgraph of G with $\mathscr{P}$-pendant subgraphs $G_1, G_2, \ldots, G_\varepsilon$ in S, where $\varepsilon = \varepsilon_{\mathscr{P}}(G)$. By X_i we denote the set of all $\mathscr{P}$-pendant vertices of the subgraph G_i, $1 \leqslant i \leqslant \varepsilon$. The family of sets $\{X_1, X_2, \ldots, X_\varepsilon\}$ has a system of different representatives. Denote one of them by Y. It is obvious that $|Y| = \varepsilon_{\mathscr{P}}(G)$. The set $V - Y$ is a strongly $\mathscr{P}$-dominating set of G. Hence $\gamma'_{\mathscr{P}}(G) \leqslant |V - Y| = p - \varepsilon_{\mathscr{P}}(G)$. This completes the proof. □

Hedetniemi and Laskar proved a similar equality as in Nieminen's Theorem, involving connectivity.

A set $D \subseteq V$ is called *connected dominating* in G, if D is a dominating set and $\langle D \rangle_G$ is a connected graph. By $\gamma_c(G)$ is denoted the cardinality of a minimum connected dominating set in G. Let $\varepsilon_c(G)$ equal the maximum number of pendant edges in a spanning tree of G.

Theorem (Hedetniemi and Laskar [7]). *Let G be a connected graph of order p. Then $\gamma_c(G) + \varepsilon_c(G) = p$.*

Let G be a connected graph and $\mathscr{P} \in \mathbf{M}^a$. If a set $D \subseteq V(G)$ is strongly $\mathscr{P}$-dominating and $\langle D \rangle_G$ is a connected graph, then D is said to be a *connected strongly $\mathscr{P}$-dominating* set. The minimum of the cardinalities of the connected strongly $\mathscr{P}$-dominating sets is called the *connected strong $\mathscr{P}$-domination number* and denoted by $\gamma'_{c,\mathscr{P}}(G)$.

Now, we introduce the corresponding number to $\varepsilon_c(G)$.

Let G be a connected graph and $\mathscr{P} \in \mathbf{M}^a$ and S be a connected spanning subgraph of G with a family $X_{\mathscr{P}}$ of $\mathscr{P}$-pendant subgraphs. Let $Y = \{v_1, v_2, \ldots, v_k\}$, $v_i \in V(G_i)$, $1 \leqslant i \leqslant k$ be a set of $\mathscr{P}$-pendant vertices in $X_{\mathscr{P}}(S)$.

If $\langle V - Y \rangle_G$ is a connected graph, then we denote this family by $X_{c,\mathscr{P}}(S)$.

Let $\varepsilon_{c,\mathscr{P}}(G)$ equal the maximum number of elements in an $X_{c,\mathscr{P}}(S)$.

Theorem 4. *For every connected graph G of order p and $\mathscr{P} \in \mathbf{M}^a$ we have*

$$\gamma'_{c,\mathscr{P}}(G) + \varepsilon_{c,\mathscr{P}}(G) = p.$$

Proof. To prove the above theorem it is enough to notice that forbidden subgraphs of an additive property are connected and to proceed analogously to the proof of the previous theorem. □

Note that the results of this paper can be extended to the hereditary properties with respect to a partial order as well.

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