Available online at www.sciencedirect.com

SCIENCE DIRECT

ELSEVIER

J. Math. Pures Appl. 83 (2004) 673–697

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

www.elsevier.com/locate/matpur

On the stability of global solutions to Navier–Stokes equations in the space

P. Auscher [a], S. Dubois [b], P. Tchamitchian [c,*]

[a] *Laboratoire de mathématiques, CNRS UMR 8628, Université Paris-Sud — bât 425, 91405 Orsay cedex, France*

[b] *Laboratoire amiénois de mathématique fondamentale et appliquée, CNRS UMR 6140, faculté de mathématiques et d'informatique, Université de Picardie Jules-Verne, 33, rue Saint-Leu, 89039 Amiens cedex, France*

[c] *Laboratoire d'analyse, topologie et probabilités, CNRS UMR 6632, faculté des sciences et techniques de Saint-Jérôme, case cour A, avenue Escadrille Normandie–Niemen, 13397 Marseille cedex, France*

Received 26 November 2003

Available online 21 February 2004

Abstract

We show that the global solutions to the Navier–Stokes equations in $\mathbb{R}^3$ with data in VMO^{-1} which belong to the space defined by Koch and Tataru are stable, in the sense that they vanish at infinity (in time), that they depend analytically on their data, and that the set of Cauchy data giving rise to such a solution is open in the BMO^{-1} topology. We then study the case of more regular data.

Résumé

Nous étudions les solutions globales des équations de Navier–Stokes qui appartiennent à l'espace de Koch et Tataru et qui sont associées à une donnée initiale dans VMO^{-1}. Nous démontrons qu'elles s'annulent à l'infini et qu'elles dépendent de façon analytique de leur donnée de Cauchy. Nous prouvons également que l'ensemble des distributions de VMO^{-1} qui donnent naissance à une telle solution est ouvert dans la topologie de BMO^{-1}. Enfin, nous étudions le cas des données initiales plus régulières.

Keywords: Navier–Stokes equations; Asymptotics; Stability

* Corresponding author.

E-mail addresses: pascal.auscher@math.u-psud.fr (P. Auscher), sandrine.dubois@u-picardie.fr (S. Dubois), philippe.tchamitchian@univ.u-3mrs.fr (P. Tchamitchian).

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2004.01.003

1. Introduction

In the absence, at the present time, of any satisfactory result on the existence of global, unique and regular solutions to homogeneous incompressible Navier–Stokes equations in the space, for a large enough class of initial data, a simpler question naturally arises: what can be said about the topology of the set of those initial data leading to such a good solution? In particular, is it *open*? Answering affirmatively means proving a stability result, with respect to perturbations on the Cauchy data, of the kind we are interested in here.

Of course, this question must be stated in a precise functional setting to make sense. It has already been done in the past twenty years by different authors, giving a variety of results on apparently different solutions. Following the illuminating description given by Chemin in [6], there are indeed two main streams in the history of the study of the solutions to Navier–Stokes equations, which have their origins in the works of Leray [24] on one side (appearing in the 1930s), of Fujita and Kato [10,11], Weissler [30] and Kato alone [17] on the other side (several decades later). It turns out that the study of the stability problem followed a parallel development—a fact which is not coincidental.

The first results on stability dealt with Leray weak solutions, thus in the setting of the energy space, or rather some appropriate subspaces. We mention contributions by Beirão da Veiga and Secchi [2], by Wiegner [31], and by Ponce, Racke, Sideris and Titi [26]. For example, let us extract from the latter this result: *the set of those $u_0 \in H^1(\mathbb{R}^3)$ leading to a global weak solution u which, in addition, belongs to some space $L^q(]0,+\infty[; L^p(\mathbb{R}^3))$, with $3/p + 2/q = 1$ and $3 < p < \infty$, is open in the H^1-topology*. Here the most important hypothesis is the global integrability of u. Its purpose is twofold, since it means that u have some decay property at infinity (in time), and that it satisfies what could be called an invariant estimate. We will comment on the first property later, and concentrate now on the second one.

An invariant estimate on a solution u is an estimate involving a set of norms or semi-norms which is invariant under translation in the space, and under the rescaling law $u(t,x) \mapsto \lambda u(\lambda^2 t, \lambda x)$. These transformations leave invariant the equations themselves. That invariant estimates are fundamental in studying Navier–Stokes equations is nothing new, and has been emphasized by Leray himself (he speaks of "formules homogènes" in his 1934 paper), as well as by many others (e.g., Caffarelli, Kohn and Nirenberg, in their celebrated paper, insisting on the role of "dimensionless quantities"). In particular, most (partial) results on the uniqueness or on the regularity of weak solutions are based on such estimates: well-known examples are in Serrin [27], Chemin [6] ... The same phenomenon appears for stability results, and the above-mentioned one is very representative.

One can say that the development initiated by Fujita and Kato rests on a more radical point of view, consisting of leaving aside the energy space and deliberately working in a fully invariant functional setting. This has led to many results on various classes of (always regular) solutions: the uniqueness problem has been settled by Furioli, Lemarié and Terraneo in [12], which is certainly the main reference. The existence of solutions has been treated by Fujita, Kato, Weissler, Giga, Miyakawa, Taylor, Kozono, Yamazaki, Cannone, Planchon, Lemarié, Barraza, etc. (see the bibliography), and this series of papers culminated in the article [19] by Koch and Tataru, which contains an optimal result. Optimality is meant here in a precise sense, that we recall in Section 4. Let us point out, however, that all these results do not use the cancellation property of the nonlinear term in Navier–Stokes equations, and are instead valid for a general class of equations.

The same change occurred for the question of stability, starting from a recent paper by Kawanago [18], in which he proved that: *the set of those* $u_0 \in L^3(\mathbb{R}^3)$ *leading to a global solution* $u \in C([0,+\infty[; L^3(\mathbb{R}^3))$ *such that, in addition,* $\lim_{t\to+\infty} \|u(t,\cdot)\|_{L^3} = 0$ *is open in the* $L^3(\mathbb{R}^3)$*-topology*. This result highlights the importance of the decay property of u, which plays a crucial role in the proof. Also, Kawanago made the nice observation that if u fulfills an energy inequality then this decay property is true, and needs not to be assumed. With the help of some previously known results this implies that Kawanago's theorem encompasses that of Ponce, Racke, Sideris and Titi we have cited above.

Later, Gallagher, Iftimie and Planchon [15,14] extended Kawanago's result in two respects: they considered more general invariant functional settings, and more importantly, they discarded the hypothesis on the decay of the solution, *proving* that it always hold. What is remarkable in their proof is that, in the line of Kawanago's observation, it reintroduces the cancellation property of the nonlinear term as a key feature in the asymptotic analysis of the solutions, even though they are not considering Leray weak solutions. This is reminiscent of Calderón's and Lemarié's (independent) constructions of weak solutions in L^p spaces for $2 < p < 3$ [3,23]. At the end of this paper, we give a counterexample which strongly suggests that this is not an artefact of their method, but an essential argument.

Indeed, we here elaborate on Gallagher, Iftimie and Planchon paper in order to reach the main case which is out of the scope of their results, that of Koch and Tataru solutions. For this purpose, we abandon Littlewood–Paley techniques (a main tool in [14]) and instead use simple real variable estimates. Since Koch and Tataru construction is optimal, our result is optimal, too, implying all the stability results previously known for these equations.

2. Statement of the result

Spaces of scalar-valued and spaces of vector-valued functions or distributions will abusively be denoted the same way.

Let $\mathcal{C}$, or sometimes $\mathcal{C}_\infty$, be the space of functions $u(t,x)$ defined on $]0,+\infty[\times\mathbb{R}^3$ and valued in $\mathbb{C}^3$, such that

$$N_\infty(u) \overset{def}{=} \sup_{t>0} \sqrt{t}\,\big\|u(t)\big\|_\infty < +\infty, \tag{1}$$

$$N_c(u) \overset{def}{=} \left(\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|}\int_0^{l_Q^2}\int_Q \big|u(t,x)\big|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} < +\infty, \tag{2}$$

where $\mathcal{Q}$ denotes the set of all cubes Q in $\mathbb{R}^3$ with sides parallel to the axes, lengthside l_Q and measure $|Q|$.

Similarly, for $T \in\,]0,+\infty[$, let $\mathcal{C}_T$ be the space of functions $u(t,x)$ defined on $]0,T[\times\mathbb{R}^3$ and valued in $\mathbb{C}^3$, such that

$$N_{\infty,T}(u) \stackrel{def}{=} \sup_{0<t<T} \sqrt{t}\,\|u(t)\|_\infty < +\infty, \tag{3}$$

$$N_{c,T}(u) \stackrel{def}{=} \left(\sup_{Q\in\mathcal{Q},\, l_Q \leqslant \sqrt{T}} \frac{1}{|Q|} \int_0^{l_Q^2} \int_Q |u(t,x)|^2 \mathrm{d}x\,\mathrm{d}t \right)^{1/2} < +\infty. \tag{4}$$

Let finally $\mathcal{C}_0$ (respectively $\mathcal{C}_{0,T}$) be the closed subspace in $\mathcal{C}$ (respectively $\mathcal{C}_T$) of the functions $u(t,x)$ such that

$$\lim_{T'\to 0} \|u\|_{\mathcal{C}_{T'}} = 0. \tag{5}$$

We will sometimes write $\mathcal{C}_{0,\infty}$ instead of $\mathcal{C}_0$, too.

In a recent paper [19], Koch and Tataru showed the following two statements.

(1) If $u_0 \in BMO^{-1}(\mathbb{R}^3)$ is small enough and divergence-free, there exists a solution $u \in \mathcal{C}$ to the integral Navier–Stokes equations in $\mathbb{R}^3$ with initial data u_0 (that we will denote $(NSI)_{u_0}$ from now on). This means that

$$u(t) = \mathrm{e}^{t\Delta}u_0 - \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\big(u(s)\otimes u(s)\big)\,\mathrm{d}s, \tag{$(NSI)_{u_0}$}$$

where $\mathbb{P}$ is the Leray projector onto the divergence-free vector fields.

(2) Also, if u_0 is any divergence-free vector-valued distribution in the closure of the Schwartz class in $BMO^{-1}(\mathbb{R}^3)$, that we will denote VMO^{-1} in the sequel, then there exist $T>0$ and $u \in \mathcal{C}_{0,T}$ solving the same equations in $]0,T[\times\mathbb{R}^3$. Here, T depends on u_0 and, in particular, $T=+\infty$ when u_0 is small enough.

(3) Furthermore, it has been proved by one of us that any solution $u \in \mathcal{C}_{0,T}$, $T \leqslant +\infty$, of $(NSI)_{u_0}$ with $u_0 \in BMO^{-1}$ is unique: see [8].

Because these results are optimal in a sense we will describe later, they are the highest point in a chain of works initiated by Fujita and Kato, Kato, Weissler, and continued by Taylor, Konozo, Yamazaki, Cannone, Planchon, Meyer, and many others: see the bibliography. Regarding them as perturbation results around the zero solution of $(NSI)_0$, we ask what happens when one tries to perturb any *a priori* given global solution to Navier–Stokes equations.

To be more precise, we define the set E of all the data $u_0 \in VMO^{-1}$ giving rise to a global solution of $(NSI)_{u_0}$ belonging to $\bigcap_{T>0}\mathcal{C}_{0,T}$, whatever its large time behaviour might be. Our main theorem essentially says that Koch and Tataru result, valid for $u_0=0$, extends to any such u_0.

Theorem 1.

(i) *If $u_0 \in E$ and if u is the solution attached to u_0, then $u \in \mathcal{C}_0$ and*

$$\lim_{t\to+\infty}\sqrt{t}\,\|u(t)\|_\infty = 0, \tag{6}$$

$$\lim_{t\to+\infty}\|u(t+\cdot)\|_{\mathcal{C}} = 0. \tag{7}$$

(ii) *There exists* $\varepsilon > 0$ *such that, for every* $v_0 \in BMO^{-1}(\mathbb{R}^3)$ *with*

$$\|u_0 - v_0\|_{BMO^{-1}} \leqslant \varepsilon,$$

the Navier–Stokes equations with data v_0 *admit a global solution* v *in* $\mathcal{C}$. *Moreover, the map* $v_0 \mapsto v$, *defined from the ball* $B(u_0,\varepsilon)$ *in* BMO^{-1} *to the space* $\mathcal{C}$, *is analytic at* u_0.

Here and in the sequel, we will say that a map Φ, defined from an open subset Ω of some Banach space F to another Banach space $\mathcal{F}$, is *analytic at* f_0, $f_0 \in \Omega$, when there exists $\varepsilon > 0$ and a sequence L_k, $k \geqslant 1$, of k-linear bounded operators, each defined from F^k to $\mathcal{F}$, such that $B(f_0,\varepsilon) \subset \Omega$ and

$$\Phi(g_0) = \Phi(f_0) + \sum_{k=1}^{+\infty} L_k(g_0 - f_0, \ldots, g_0 - f_0)$$

whenever $g_0 \in B(f_0,\varepsilon)$, the series being normally convergent in $\mathcal{F}$ for such a g_0.

3. Proof of the theorem

3.1. The main steps

Let $u_0 \in E$ and $u \in \bigcap_{T>0}\mathcal{C}_{0,T}$ a global solution of $(NSI)_{u_0}$. We begin with proving that $u \in \mathcal{C}_0$ and that (6) and (7) hold true. To this end, we use a strategy which has been defined by Gallagher, Iftimie and Planchon in [14], developing earlier ideas due to C. Calderón [3], and later rediscovered by Lemarié [23]. Fix $\varepsilon > 0$: since $u_0 \in VMO^{-1}$ there exists a decomposition:

$$u_0 = f_0 + g_0,$$

where $f_0 \in VMO^{-1} \cap L^2$, while g_0 is small enough in VMO^{-1} so that there exists g, solution of $(NSI)_{g_0}$ in $\mathcal{C}_0$, with $\|g\|_{\mathcal{C}_0} \leqslant \varepsilon$. Then the function $f = u - g$ satisfies the equation:

$$f(t) = \mathrm{e}^{t\Delta} f_0 - \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\big(u(s)\otimes f(s) + f(s)\otimes u(s)\big)\,\mathrm{d}s$$
$$- \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\big(f(s)\otimes f(s)\big)\,\mathrm{d}s. \tag{8}$$

The key point is now that, since $f_0 \in L^2$, f will be shown to fulfill a kind of energy inequality, which implies the following lemma.

Lemma 2. *With the notation above, we have*:

$$\lim_{T \to +\infty} \frac{1}{T} \int_1^T \left\| f(t) \right\|^4_{\dot{W}^{1/2,2}} \, dt = 0.$$

Admitting this statement for the moment, we may conclude. Observe that, because $\dot{W}^{1/2,2}$ is embedded in VMO^{-1}, there exists a time T' at which $\| f(T')\|_{VMO^{-1}} \leqslant \varepsilon$. By the foregoing Lemma 8, $g(t)$ persists in BMO^{-1} and $\|g(t)\|_{BMO^{-1}} \leqslant C\varepsilon$ for all $t > 0$. Thus, $\|u(T')\|_{BMO^{-1}} \leqslant C\varepsilon$: provided ε is small enough, the result of Koch and Tataru and the uniqueness in $\mathcal{C}_0$ of the solutions of (*NSI*) apply to $u(T' + \cdot)$, showing that $\|u(T' + \cdot)\|_{\mathcal{C}} \leqslant C\varepsilon$. In particular, we have $\sqrt{t}\,\|u(t)\|_\infty \leqslant C\varepsilon$ if $t \geqslant 2T'$. This proves the desired results on the asymptotic behaviour of u.

Consider now $v_0 \in BMO^{-1}$: we have to show that $(NSI)_{v_0}$ has a solution v in $\mathcal{C}$ whenever v_0 is close enough to u_0. Setting $w = u - v$ and $w_0 = u_0 - v_0$, this is equivalent to solving for small w_0 the equation:

$$\begin{aligned} w(t) = {} & e^{t\Delta} w_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}\bigl(u(s) \otimes w(s) + w(s) \otimes u(s)\bigr) \, ds \\ & - \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}\bigl(w(s) \otimes w(s)\bigr) \, ds. \end{aligned} \tag{9}$$

We formally define the bilinear operator B by the formula:

$$B(f, g) = \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}\bigl(f(s) \otimes g(s)\bigr) \, ds.$$

The continuity of this operator on the space $\mathcal{C}$ is the main estimate in Koch and Tataru paper:

Lemma 3 (Koch and Tataru estimate).

$$\exists A > 0 \; \forall f, g \in \mathcal{C} \quad \left\| B(f, g) \right\|_{\mathcal{C}} \leqslant A \| f \|_{\mathcal{C}} \|g\|_{\mathcal{C}}.$$

Therefore, we may define on $\mathcal{C}$ a continuous linear operator $L_{u,u}$ by:

$$L_{u,u}(w) = B(u, w) + B(w, u).$$

Also setting $Sw_0 = (\mathrm{e}^{t\Delta} w_0)_{t>0}$, we rewrite (9) as

$$w = Sw_0 - L_{u,u}(w) + B(w, w). \tag{10}$$

Now, the operator $L_{u,u}$ (with this particular u) has the following property:

Lemma 4. *The spectrum of $L_{u,u}$ in $\mathcal{C}$ is* $\{0\}$.

Again admitting momentarily this statement, we see that (10) is solvable in $\mathcal{C}$ for Sw_0 small enough, i.e., for w_0 small enough in BMO^{-1}, thanks to the abstract principle for solving quadratic nonlinear equations in Banach spaces which lies behind Koch and Tataru result—as well as all the aforementioned other results of the same type—that we now state explicitly.

Lemma 5. *Let $\mathcal{F}$ be a Banach space, L a continuous linear operator on $\mathcal{F}$ and B a bilinear operator, continuous on $\mathcal{F}$ in the sense that*

$$\|B\| \stackrel{def}{=} \sup_{\|f\|_{\mathcal{F}} = \|g\|_{\mathcal{F}} = 1} \big\|B(f, g)\big\|_{\mathcal{F}} < +\infty.$$

Then, if $I + L$ is invertible, and for all $z \in \mathcal{F}$ such that

$$\big\|(I + L)^{-1} z\big\|_{\mathcal{F}} \leqslant \frac{1}{4\|B\| \|(I + L)^{-1}\|}, \tag{11}$$

there is a solution $w \in \mathcal{F}$ to the equation:

$$w = z - L(w) + B(w, w). \tag{12}$$

Moreover, there exists for each $k \geqslant 1$ a k-linear operator T_k continuous on $\mathcal{F}^k$ with

$$w = \sum_{k=1}^{+\infty} T_k(z, \ldots, z),$$

where this series converges normally under the condition (11).

The three lemmas above allow to solve (10) for w_0 small enough in BMO^{-1}, as desired. The analyticity result is a direct consequence of Lemma 5.

The strategy of proof of Theorem 1 is now explained: before going into the details and proving Lemmas 2 and 4, we give a proof of Lemma 5 for the convenience of the reader.

Let us first assume that $L = 0$ and recursively define the operators $\widetilde{T}_k$ by:

$$\begin{cases} \widetilde{T}_1(z) = z, \\ \widetilde{T}_k(z) = \sum_{j=1}^{k-1} B(\widetilde{T}_j(z), \widetilde{T}_{k-j}(z)), \quad k \geqslant 2. \end{cases}$$

By construction each $\widetilde{T}_k$ is the trace on the diagonal of $\mathcal{F}^k$ of some k-linear operator T_k (which is not uniquely defined). Also the constants C_k defined by the recurrence relation

$$\begin{cases} C_1 = 1, \\ C_k = \|B\| \sum_{j=1}^{k-1} C_j C_{k-j}, \quad k \geqslant 2, \end{cases}$$

are such that

$$\forall z \in \mathcal{F} \quad \big\| \widetilde{T}_k(z) \big\|_{\mathcal{F}} \leqslant C_k \|z\|_{\mathcal{F}}^k.$$

When $\|B\| = 1$, the C_k's are the so-called Catalan numbers, which can be computed via their generating function; by a simple reduction to this case one finds in general:

$$C_k = \frac{1}{4k-2} \frac{(2k)!}{(k!)^2} \|B\|^{k-1} \sim \frac{1}{\sqrt{\pi} k^{3/2}} \big(4\|B\|\big)^{k-1}.$$

Hence the series $\sum_{k=1}^{+\infty} \widetilde{T}_k(z)$ converges normally when $\|z\|_{\mathcal{F}} \leqslant 1/(4\|B\|)$, and is a solution to the equation $w = z + B(w, w)$, by construction.

In the general case, when $I + L$ is invertible, it suffices to notice that (12) is equivalent to

$$w = (I+L)^{-1} z + (I+L)^{-1} B(w, w)$$

and then to apply the result when $L = 0$. This ends the proof.

3.2. The linear operators $L_{a,b}$

Both Lemma 2 and Lemma 4 lie upon properties of the linear operators $L_{a,b}$ formally defined by:

$$L_{a,b}(f)(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} \mathbb{P} \operatorname{div}\big(a(s) \otimes f(s) + f(s) \otimes b(s)\big)\, \mathrm{d}s,$$

or more briefly,

$$L_{a,b}(f) = B(a, f) + B(f, b).$$

For $T \in]0, +\infty]$ we will need two new functional spaces. The first one, denoted by $\mathcal{L}_T$, is the space of the functions f, defined on $]0, T[\times \mathbb{R}^3$, such that

$$\|f\|_{\mathcal{L}_T} \overset{def}{=} \sup_{0<t<T} \big(\big\| f(t) \big\|_2 + \sqrt{t}\, \big\| \nabla f(t) \big\|_2 \big) < +\infty.$$

This space resembles the Leray–Hopf space of energy, which is more usually considered, with the advantage of being easier to handle and enough for our purpose. The second one, denoted by $\mathcal{C}_T^1$, is the subspace of $\mathcal{C}_T$ endowed with the norm,

$$\|f\|_{\mathcal{C}_T^1} \stackrel{def}{=} \|f\|_{\mathcal{C}_T} + N^1_{\infty,T}(f),$$

where by definition: $N^1_{\infty,T}(f) = \sup_{0<t<T} t\|\nabla f(t)\|_\infty$. We begin with stating and proving the following:

Lemma 6. *Let a, b in $\mathcal{C}_{0,T}$, $T \leqslant +\infty$, with the additional property that*

$$\lim_{t\to+\infty} \|a(t+\cdot)\|_{\mathcal{C}} + \|b(t+\cdot)\|_{\mathcal{C}} = 0$$

when $T = +\infty$. Then

(i) *$L_{a,b}$ is continuous on $\mathcal{C}_T$ and its spectrum is $\{0\}$;*
(ii) *if moreover $\operatorname{div} a = 0$ and $b \in \mathcal{C}_T^1$, $L_{a,b}$ is also continuous on $\mathcal{L}_T$ and on $\mathcal{C}_T^1$, and its spectrum on both spaces is $\{0\}$ as well.*

As we already mentioned, the continuity of $L_{a,b}$ on $\mathcal{C}_T$, uniformly with respect to $T \leqslant +\infty$, is nothing but Koch and Tataru estimate. Let us prove that, when $\operatorname{div} a = 0$ and $b \in \mathcal{C}_T^1$, $L_{a,b}$ is continuous on $\mathcal{L}_T$.

Take f in $\mathcal{L}_T$. The L^2 estimate for $L_{a,b}(f)$ is straightforward:

$$\begin{aligned}\|L_{a,b}(f)(t)\|_2 &\leqslant C\big(N_{\infty,T}(a) + N_{\infty,T}(b)\big)\int_0^t \frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}\|f(s)\|_2\,\mathrm{d}s\\ &\leqslant C\big(N_{\infty,T}(a) + N_{\infty,T}(b)\big)\|f\|_{\mathcal{L}_T}.\end{aligned}$$

For the $\dot W^{1,2}$ estimate, we use that a is divergence-free and write (∂ denoting any first-order partial derivative):

$$\begin{aligned}\partial L_{a,b}(f)(t) = &\int_0^{t/2} \partial \mathrm{e}^{(t-s)\Delta}\mathbb{P}\operatorname{div}\big(a(s)\otimes f(s) + f(s)\otimes b(s)\big)\,\mathrm{d}s\\ &+ \int_{t/2}^t \partial \mathrm{e}^{(t-s)\Delta}\mathbb{P}\big((a(s)\cdot\nabla)f(s) + (f(s)\cdot\nabla)b(s)\big)\,\mathrm{d}s\\ &+ \int_{t/2}^t \partial \mathrm{e}^{(t-s)\Delta}\mathbb{P}\big((\operatorname{div} f(s))b(s)\big)\,\mathrm{d}s. \end{aligned} \tag{13}$$

We may now estimate:

$$\begin{aligned}\big\|\partial L_{a,b}(f)(t)\big\|_2 \leqslant{} & C\big(N_{\infty,T}(a)+N_{\infty,T}(b)\big)\int_0^{t/2}\frac{1}{t-s}\frac{1}{\sqrt{s}}\big\|f(s)\big\|_2\,\mathrm{d}s\\ & + C\big(N_{\infty,T}(a)+N^1_{\infty,T}(b)\big)\\ & \times\int_{t/2}^{t}\frac{1}{\sqrt{t-s}}\left(\frac{1}{\sqrt{s}}\big\|\nabla f(s)\big\|_2+\frac{1}{s}\big\|f(s)\big\|_2\right)\mathrm{d}s\\ & + CN_{\infty,T}(b)\int_{t/2}^{t}\frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}\big\|\operatorname{div} f(s)\big\|_2\,\mathrm{d}s\\ \leqslant{} & C\frac{1}{\sqrt{t}}\big(N_{\infty,T}(a)+N_{\infty,T}(b)+N^1_{\infty,T}(b)\big)\|f\|_{\mathcal{L}_T}.\end{aligned}$$

We thus have obtained that $L_{a,b}(f)\in\mathcal{L}_T$, with

$$\|L_{a,b}(f)\|_{\mathcal{L}_T}\leqslant C\big(N_{\infty,T}(a)+N_{\infty,T}(b)+N^1_{\infty,T}(b)\big)\|f\|_{\mathcal{L}_T}.$$

There is a useful variant of this estimate, based on the relation

$$\int_0^t=\int_0^{\alpha t}+\int_{\alpha t}^t$$

to be used in (13), with $\alpha\in[1/2,1[$ to be chosen. It gives:

$$\begin{aligned}\big\|\partial L_{a,b}(f)(t)\big\|_2 \leqslant{} & C\frac{1}{\sqrt{t}}\left(\ln\frac{1}{1-\alpha}\right)\big(N_{\infty,T}(a)+N_{\infty,T}(b)\big)\|f\|_{\mathcal{L}_T}\\ & + C\sqrt{\frac{1-\alpha}{t}}N^1_{\infty,T}(b)\|f\|_{\mathcal{L}_T},\end{aligned}$$

and therefore

$$\big\|L_{a,b}(f)\big\|_{\mathcal{L}_T}\leqslant C\left\{\left(\ln\frac{1}{1-\alpha}\right)\big(N_{\infty,T}(a)+N_{\infty,T}(b)\big)+\sqrt{1-\alpha}\,N^1_{\infty,T}(b)\right\}\|f\|_{\mathcal{L}_T}. \tag{14}$$

The last continuity property we have to prove, namely that of $L_{a,b}$ on $\mathcal{C}^1_T$, is obtained in a similar way. Let $f\in\mathcal{C}^1_T$: we already know that $L_{a,b}(f)\in\mathcal{C}_T$. To estimate $\|\partial L_{a,b}(f)\|_\infty$, we start from (13); the second and third integrals in the right-hand member are estimated in L^∞ as in L^2 by:

$$C\big(N_{\infty,T}(a)+N^1_{\infty,T}(b)\big)\int_{t/2}^{t}\frac{1}{\sqrt{t-s}}\left(\frac{1}{\sqrt{s}}\big\|\nabla f(s)\big\|_\infty+\frac{1}{s}\big\|f(s)\big\|_\infty\right)\mathrm{d}s$$

$$+\,CN_{\infty,T}(b)\int_{t/2}^{t}\frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}\big\|\operatorname{div} f(s)\big\|_\infty\,\mathrm{d}s$$

$$\leqslant C\frac{1}{t}\big(N_{\infty,T}(a)+N_{\infty,T}(b)+N^1_{\infty,T}(b)\big)\|f\|_{\mathcal{C}^1_T}.$$

We used here the classical fact that any operator of the form $P(D)\mathrm{e}^{t\Delta}$, where $P(D)$ is a pseudo-differential operator of convolution type and of degree $d>0$, is bounded on L^∞ with norm proportional to $t^{-d/2}$. For the remaining integral in (13) we need the more precise fact that such an operator is a convolution with a function $\frac{1}{t^{(3+d)/2}}\psi(\frac{\cdot}{\sqrt{t}})$, where ψ is smooth and decays like $|x|^{-3-d}$ at infinity. Thus, for all $x\in\mathbb{R}^3$, we have:

$$\left|\int_0^{t/2}\partial\mathrm{e}^{(t-s)\Delta}\mathbb{P}\operatorname{div}\big(a(s)\otimes f(s)+f(s)\otimes b(s)\big)(x)\,\mathrm{d}s\right|$$

$$\leqslant C\sum_{k\in\mathbb{Z}^3}\big(1+|k|\big)^{-5}\frac{1}{t^{5/2}}\int_0^{t/2}\int_{y\in B(x+\sqrt{t}k,C\sqrt{t})}\big(\big|a(s,y)\big|+\big|b(s,y)\big|\big)\big|f(s,y)\big|\,\mathrm{d}y\,\mathrm{d}s$$

$$\leqslant C\big(N_{c,T}(a)+N_{c,T}(b)\big)N_{c,T}(f)\frac{1}{t}.$$

Finally, we have obtained that $L_{a,b}(f)\in\mathcal{C}^1_T$ with the estimate,

$$\big\|L_{a,b}(f)\big\|_{\mathcal{C}^1_T}\leqslant C\big(\|a\|_{\mathcal{C}_T}+\|b\|_{\mathcal{C}^1_T}\big)\|f\|_{\mathcal{C}^1_T}. \tag{15}$$

As before, we could have replaced $t/2$ by αt, $\alpha\in[1/2,1[$, in the above calculations. We let the reader verify that this would have given the following:

$$\big\|L_{a,b}(f)\big\|_{\mathcal{C}^1_T}\leqslant C\left\{\left(\frac{1}{1-\alpha}\ln\frac{1}{1-\alpha}\right)\big(\|a\|_{\mathcal{C}_T}+\|b\|_{\mathcal{C}_T}\big)+\sqrt{1-\alpha}\,N^1_{\infty,T}(b)\right\}\|f\|_{\mathcal{C}^1_T}. \tag{16}$$

Let us now show that the spectrum of $L_{a,b}$ on $\mathcal{C}_T$, on $\mathcal{L}_T$ and on $\mathcal{C}^1_T$ is $\{0\}$. We may only consider the case $T=+\infty$, extending a and b by 0 on $[T,+\infty[\times\mathbb{R}^3$ if T is finite. Recall that we assume:

$$\lim_{t\to+\infty}\big\|a(t+\cdot)\big\|_{\mathcal{C}}+\big\|b(t+\cdot)\big\|_{\mathcal{C}}=0. \tag{17}$$

Let $\lambda \neq 0$, $g \in \mathcal{C}$ (respectively $\mathcal{L}_\infty, \mathcal{C}^1_\infty$), and consider the equation:

$$\lambda f - L_{a,b}(f) = g. \tag{18}$$

We have to show it has a unique solution in $\mathcal{C}$ (respectively $\mathcal{L}_\infty, \mathcal{C}^1_\infty$): since $L_{a,b}$ depends linearly on a and b, it is enough to prove it when $\lambda = 1$.

We are going to construct a global solution to (18) from finitely many local solutions obtained on appropriate time intervals. Let us begin with the following observation, whose proof is left to the reader.

Lemma 7. *Let $0 < t < t' < +\infty$ and $\tilde{a} = a(t + \cdot, \cdot)$, defined on $]0, t' - t[\times \mathbb{R}^3$. Then*

$$N_{c,t'-t}(\tilde{a}) \leqslant \sqrt{\ln \frac{t'}{t}}\, N_\infty(a), \qquad N_{\infty,t'-t}(\tilde{a}) \leqslant \sqrt{1 - \frac{t}{t'}}\, N_\infty(a).$$

Let $\delta > 0$ be a small parameter, to be fixed in a short while. We deduce from the preceding lemma, (17) and the fact that $a, b \in \mathcal{C}_0$, the existence of an integer N and $N+1$ overlapping intervals $I_j =]t_j, t'_j[$, with $t_0 = 0$, $t'_N = +\infty$ and $t_j < t'_{j-1}$ if $1 \leqslant j \leqslant N$, such that

$$\forall j \in \{0, \ldots, N\} \quad \|a_j\|_{\mathcal{C}_{t'_j - t_j}} + \|b_j\|_{\mathcal{C}_{t'_j - t_j}} \leqslant \delta, \tag{19}$$

where by definition $a_j = a(t_j + \cdot, \cdot),\ 0 < t < t'_j - t_j$, and similarly for b_j. Then, Koch and Tataru estimate implies that, for δ chosen small enough, we have:

$$\forall j \in \{0, \ldots, N\}\ \forall \tau \leqslant \delta \quad \|L_{a_j,b_j}\|_{\mathcal{C}_\tau} \leqslant 1/2.$$

Similarly, we may and do choose at first α in (14) and (16) close enough to 1 so that

$$C\sqrt{1-\alpha}\, N^1_{\infty,+\infty}(b) \leqslant 1/4,$$

then δ so as to obtain:

$$\forall j \in \{0, \ldots, N\}\ \forall \tau \leqslant \delta \quad \|L_{a_j,b_j}\|_{\mathcal{L}_\tau} \leqslant 1/2, \tag{20}$$

$$\forall j \in \{0, \ldots, N\}\ \forall \tau \leqslant \delta \quad \|L_{a_j,b_j}\|_{\mathcal{C}^1_\tau} \leqslant 1/2. \tag{21}$$

We are now in position to solve Eq. (18) (recall that $\lambda = 1$). We begin with the case where $g \in \mathcal{L}_\infty$.

Restricting (18) to $I_0 =]0, t'_0[$ and using (20), we obtain a unique $f_0 \in \mathcal{L}_{t'_0}$ such that

$$\forall t \in I_0 \quad f_0(t) - L_{a_0,b_0}(f_0)(t) = g(t).$$

Thus, by construction, $L_{a_0,b_0}(f_0)(t_1) \in L^2$. Again using (20), there exists a unique $\tilde{f}_1 \in \mathcal{L}_{t_1'-t_1}$ such that for all $\tau \in]0, t_1' - t_1[$,

$$\tilde{f}_1(\tau) - L_{a_1,b_1}(\tilde{f}_1)(\tau) = g(t_1 + \tau) + \mathrm{e}^{\tau\Delta} L_{a_0,b_0}(f_0)(t_1). \tag{22}$$

We define the function f_1 on $I_0 \cup I_1$ by $f_1 = f_0$ on I, $f_1(t) = \tilde{f}_1(t - t_1)$ on I_1: that this definition is consistent follows from the fact that $f_0(t_1 + \tau)$ is a solution of (22) on $]0, t_0' - t_1[$, while this solution is unique by (20). It is not difficult to check that $f_1 \in \mathcal{L}_{t_1'}$.

We iterate this construction $N - 1$ times, defining $\tilde{f}_2, \ldots, \tilde{f}_N$, with

$$f_2 \in \mathcal{L}_{t_2'}, \quad \ldots, \quad f_{N-1} \in \mathcal{L}_{t_{N-1}'}, \quad f_N \in \mathcal{L}_\infty.$$

Then, the function $f = f_N$ is a solution of (18). Its uniqueness follows from its construction.

The case where $g \in \mathcal{C}$ is solved in a similar way, the only point needing to be precised being the following persistence result.

Lemma 8. *If $a, b, f \in \mathcal{C}_T$, then $L_{a,b}(f)(t) \in BMO^{-1}$ for each $t \in]0, T[$, uniformly with respect to t.*

Proving it reduces to show that $B(a, f)(t) \in BMO^{-1}$, and this is a consequence—since $\mathbb{P}$ maps L^∞ to BMO—of the following:

$$\exists C > 0\ \forall t \in]0, T[\quad \left\| \int_0^t \mathrm{e}^{(t-s)\Delta} a(s) \otimes f(s)\, \mathrm{d}s \right\|_\infty \leqslant C. \tag{23}$$

Indeed, on the one hand we have:

$$\left\| \int_{t/2}^t \mathrm{e}^{(t-s)\Delta} a(s) \otimes f(s)\, \mathrm{d}s \right\|_\infty \leqslant C \int_{t/2}^t \|a(s)\|_\infty \|f(s)\|_\infty \,\mathrm{d}s$$
$$\leqslant C N_{\infty,T}(a) N_{\infty,T}(f).$$

On the other hand, if $x \in \mathbb{R}^3$ is fixed, we have:

$$\int_0^{t/2} \mathrm{e}^{(t-s)\Delta} \left|a(s) \otimes f(s)\right|(x)\, \mathrm{d}s$$
$$\leqslant \frac{C}{t^{3/2}} \int_0^{t/2} \int \mathrm{e}^{-|x-y|^2/(4t)} \left|a(s, y)\right| \left|f(s, y)\right| \mathrm{d}y\, \mathrm{d}s$$

$$\begin{aligned}&\leqslant C\sum_{k\in\mathbb{Z}^3}\mathrm{e}^{-|k|^2/10}\frac{1}{t^{3/2}}\int_0^{t/2}\int_{y\in B(x+\sqrt{t}k,C\sqrt{t})}\big|a(s,y)\big|\big|f(s,y)\big|\,\mathrm{d}y\,\mathrm{d}s\\&\leqslant CN_{c,T}(a)N_{c,T}(f).\end{aligned}$$

We therefore have proved (23) and Lemma 8. This allows to solve (18) in $\mathcal{C}$ exactly as we did in $\mathcal{L}_\infty$.

Finally, solving (18) in $\mathcal{C}^1_\infty$ is just a repetition of the same arguments. We skip the details, and finish there the proof of Lemma 6.

A first application of this lemma is the proof of Lemma 4. Recall that, by hypothesis, u is a solution of $(NSI)_{u_0}$ in $\mathcal{C}_0$, which satisfies (7). But $(NSI)_{u_0}$ rewrites as

$$u = Su_0 - L_{u,0}(u),$$

with $Su_0 = (\mathrm{e}^{t\Delta}u_0)_{t>0}$ belonging to $\mathcal{C}^1_\infty$ and u being divergence-free: thanks to Lemma 6, we may apply the following simple observation (left to the reader).

Lemma 9. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Banach spaces, L a linear operator continuous on both, with both spectral radii in* $[0,1[$. *Then $I+L$ is invertible on $\mathcal{E}_1\cap\mathcal{E}_2$.*

Thus u belongs to $\mathcal{C}^1_\infty$, too; we invoke again Lemma 6, this time with $a=b=u$, to get the desired result.

The proof of Lemma 2 is a little more involved, as we shall now see.

3.3. The energy estimate

We consider the solution f of (8): we know that it is in $\bigcap_{T>0}\mathcal{C}_{0,T}$, and also that $f_0\in VMO^{-1}\cap L^2$. We first prove the following:

Lemma 10. $f\in\bigcap_{T>0}\mathcal{L}_T$.

Proof. The proof starts with rewriting (8) as

$$f + L_{u,g}(f) = Sf_0$$

with $Sf_0 = (\mathrm{e}^{t\Delta}f_0)_{t>0}$. Since u, f, g, and Sf_0 are all elements of $\bigcap_{T>0}\mathcal{C}_{0,T}$, the equation above also holds into this space. Because $g\in\mathcal{C}^1_\infty$, as we will show, Lemma 6 implies that $L_{u,g}$ has a null spectral radius on every space $\mathcal{C}_T$ and $\mathcal{L}_T$. We get that $f\in\mathcal{L}_T$ for all $T>0$ on applying Lemma 9.

The reason why g belongs to $\mathcal{C}^1_\infty$ is now to be explained. Recall that $\|g\|_{\mathcal{C}_{0,\infty}}\leqslant\varepsilon$ and that g solves $(NSI)_{g_0}$, i.e., that

$$g + L_{g,0}(g) = Sg_0.$$

Provided ε is small enough, Koch and Tataru estimate and (15) ensure that

$$\|L_{g,0}\|_{\mathcal{C}_\infty} < 1 \quad \text{and} \quad \|L_{g,0}\|_{\mathcal{C}^1_\infty} < 1.$$

Thus Lemma 9 works again: since Sg_0 belongs to $\mathcal{C}^1_\infty$, the same is true for g. This ends the proof.

Following [7] (see also [9]), we infer from (8) that f is a weak solution of the differential equation,

$$\frac{\partial f}{\partial t} - \Delta f + \mathbb{P}\operatorname{div}(u \otimes f + f \otimes g) = 0.$$

In particular, thanks to the preceding lemma we have for every $0 < T < T' < +\infty$

$$\begin{aligned}\int \bigl|f(T')\bigr|^2 + 2\int_T^{T'}\int \bigl|\nabla f(t)\bigr|^2 \,\mathrm{d}t &= -2\int_T^{T'}\int \bigl(u(t)\otimes f(t)\bigr)\cdot \nabla f(t)\,\mathrm{d}t \\ &\quad - 2\int_T^{T'}\int \bigl(f(t)\otimes g(t)\bigr)\cdot\nabla f(t)\,\mathrm{d}t + \int \bigl|f(T)\bigr|^2,\end{aligned}$$

where all the integrals above are defined in the sense of Lebesgue. Recall that $u(t) \in L^\infty$ and $\operatorname{div} u(t) = 0$: the cancellation property specific to Navier–Stokes equations gives us

$$\forall t > 0 \quad \int \bigl(u(t)\otimes f(t)\bigr)\cdot\nabla f(t) = 0. \tag{24}$$

We thus have:

$$\begin{aligned}\int \bigl|f(T')\bigr|^2 + 2\int_T^{T'}\int \bigl|\nabla f(t)\bigr|^2\,\mathrm{d}t &\leqslant \int \bigl|f(T)\bigr|^2 + 2\int_T^{T'}\int \bigl|f(t)\otimes g(t)\bigr|\bigl|\nabla f(t)\bigr|\,\mathrm{d}t \\ &\leqslant \int \bigl|f(T)\bigr|^2 + 6N_\infty(g)\int_T^{T'} \bigl\|f(t)\bigr\|_2 \bigl\|\nabla f(t)\bigr\|_2 \frac{\mathrm{d}t}{\sqrt{t}}.\end{aligned}$$

Since $\|g\|_{\mathcal{C}} \leqslant \varepsilon$, we obtain:

$$\begin{aligned}&\int \bigl|f(T')\bigr|^2 + 2\int_T^{T'}\int \bigl|\nabla f(t)\bigr|^2\,\mathrm{d}t \\ &\quad\leqslant \int \bigl|f(T)\bigr|^2 + 6\varepsilon\sqrt{\ln\frac{T'}{T}}\sup_{T\leqslant t\leqslant T'}\bigl\|f(t)\bigr\|_2\Biggl(\int_T^{T'}\int \bigl|\nabla f(t)\bigr|^2\,\mathrm{d}t\Biggr)^{1/2}.\end{aligned} \tag{25}$$

We now set $t_k = \mathrm{e}^k$, $k \in \mathbb{N}$. From (25) we first deduce that, for all $T' \in [t_k, t_{k+1}]$, we have:

$$\int \left|f(T')\right|^2 \leqslant \int \left|f(t_k)\right|^2 + \frac{9\varepsilon^2}{2} \sup_{t_k \leqslant t \leqslant t_{k+1}} \left\| f(t) \right\|_2^2,$$

whence if $\varepsilon \leqslant 1/3$

$$\sup_{t_k \leqslant t \leqslant t_{k+1}} \left\| f(t) \right\|_2^2 \leqslant 2 \int \left|f(t_k)\right|^2$$

and then

$$\int \left|f(t_{k+1})\right|^2 \leqslant \left(1 + 9\varepsilon^2\right) \int \left|f(t_k)\right|^2.$$

This implies

$$\forall t \geqslant 1 \quad \int \left|f(t)\right|^2 \leqslant C t^\alpha, \tag{26}$$

where $\alpha = \ln(1 + 9\varepsilon^2)$.

Returning to (25), we now have:

$$\int_{t_k}^{t_{k+1}} \int \left|\nabla f(t)\right|^2 \mathrm{d}t \leqslant \int \left|f(t_k)\right|^2 + 9\varepsilon^2 \sup_{t_k \leqslant t \leqslant t_{k+1}} \left\| f(t) \right\|_2^2 \leqslant C\left(1 + 9\varepsilon^2\right)^k,$$

which gives, for all $T > 1$,

$$\int_1^T \int \left|\nabla f(t)\right|^2 \mathrm{d}t \leqslant C T^\alpha. \tag{27}$$

Since $\dot{W}^{1/2,2}$ is the interpolation space midway between L^2 and $\dot{W}^{1,2}$, we obtain from the inequalities (26) and (27),

$$\forall T > 1 \quad \int_1^T \left\| f(t) \right\|_{\dot{W}^{1/2,2}}^4 \mathrm{d}t \leqslant C T^{2\alpha}.$$

Thus, Lemma 2 holds as soon as ε is small enough, and Theorem 1 is completely proved.

Remark. The same energy estimate, with essentially the same proof, appears in [14].

4. Stability of more regular solutions

4.1. The optimality of Koch and Tataru result

It is time to go back to Koch and Tataru result and explain in what sense it is optimal. Other results to which it is to be compared were all obtained, through the abstract principle we described in Lemma 5, in the following situation. One is given a Banach space F, which will contain the data u_0, and another one into which the solution u will be constructed, denoted by $\mathcal{F}$. Both have the property of being critical, i.e., invariant under the transformations canonically associated to the Navier–Stokes equations:

$$\forall\lambda > 0\ \forall x_0 \in \mathbb{R}^3 \quad \bigl\|\lambda u_0(\lambda\cdot - x_0)\bigr\|_F = \|u_0\|_F, \tag{$H1$}$$

$$\forall\lambda > 0\ \forall x_0 \in \mathbb{R}^3 \quad \bigl\|\lambda u(\lambda^2\cdot, \lambda\cdot - x_0)\bigr\|_{\mathcal{F}} = \|u\|_{\mathcal{F}}. \tag{$H2$}$$

Moreover, the heat semigroup maps continuously F into $\mathcal{F}$: there exists a constant C such that

$$\|Su_0\|_{\mathcal{F}} \leqslant C\|u_0\|_F \tag{$H3$}$$

for every $u_0 \in F$.

Then, in order to give a meaning to the bilinear form B, it is always assumed that the space $\mathcal{F}$ is continuously embedded into the space $L^2_c([0,+\infty[\times\mathbb{R}^3)$—by definition the latter is the space of all the functions u defined on $[0,+\infty[\times\mathbb{R}^3$ which are square integrable on any compact of $[0,+\infty[\times\mathbb{R}^3$. As Koch and Tataru have pointed out, this leads to the existence of a constant C such that

$$\forall f \in \mathcal{F} \quad \left(\int_0^1\int_{Q_0} \bigl|f(t,x)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \leqslant C\|f\|_{\mathcal{F}},$$

where Q_0 is the unit cube $[0,1[^3$. By $(H2)$, the above inequality implies in turn,

$$\sup_{Q\in\mathcal{Q}}\left(\frac{1}{|Q|}\int_0^{l_Q^2}\int_Q \bigl|f(t,x)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \leqslant C\|f\|_{\mathcal{F}},$$

or in other words

$$N_c(f) \leqslant C\|f\|_{\mathcal{F}} \tag{$H4$}$$

for every $f \in \mathcal{F}$. We let the reader check that this inequality allows to define $B(f,g)$, for $f, g \in \mathcal{F}$, as a tempered distribution.

The last hypothesis to be assumed, and the most relevant one, is the continuity of B on $\mathcal{F}$:

$$\sup_{\|f\|_{\mathcal{F}}=\|g\|_{\mathcal{F}}=1} \|B(f,g)\|_{\mathcal{F}} < +\infty. \tag{$H5$}$$

When the hypothesis $(H1)$–$(H5)$ are fulfilled, Lemma 5 applies straightforwardly and gives the existence of a global solution in $\mathcal{F}$ to Navier–Stokes equations $(NSI)_{u_0}$, for data u_0 small enough in F.

If, in addition, the space F is such that

$$\lim_{T\to 0} \|Su_0\|_{\mathcal{F}_T} = 0 \tag{$H6$}$$

for every $u_0 \in F$, where $\mathcal{F}_T$ is the space of the restrictions to $[0, T[\times\mathbb{R}^3$ of elements of $\mathcal{F}$, then a local existence result for any data in F is available, too.

For instance this scheme is applicable to $F = \dot{W}^{1/2,2}$, L^3, Lorentz spaces above L^3 (the closure of $\mathcal{S}$ in) $\dot{B}^{s_p}_{p,q}$, $s_p = -1 + 3/p$, $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, among others: see [11,17, 16,28,20,4,1,22,23].

Now the optimality of Koch and Tataru result lies in the fact that, whenever $(H3)$ and $(H4)$ hold, we must have:

$$N_c(Su_0) \leqslant C\|u_0\|_F$$

for every $u_0 \in F$, and that the finiteness of $N_c(Su_0)$ is equivalent to u_0 being in BMO^{-1}. Hence any space F to which the above-described scheme is applicable must embed in BMO^{-1}, while Koch and Tataru showed how to apply the scheme to BMO^{-1} itself.

Regarding the local results, let us mention that the condition $u_0 \in VMO^{-1}$, which we have assumed, is slightly more demanding than $(H6)$ alone. However, this is a natural hypothesis, since it says that u_0 belongs to the closure of the Schwartz class in BMO^{-1}, and as a matter of fact, our proof of the stability result does not work under the hypothesis $(H6)$.

4.2. A general principle ensuring regularity and stability

It is therefore a natural question to ask what happens when the data u_0 belongs to a Banach space F embedded in VMO^{-1}, for example to $\dot{W}^{1/2,2}$, to L^3, or to the closure of $\mathcal{S}$ in $\dot{B}^{s_p}_{p,q}$, $s_p = -1 + 3/p$, $p < \infty$. There are two questions to consider: first the regularity of the solution, then its stability.

The regularity of a given solution u in $\mathcal{C}_0$ associated to a data u_0 belonging to a strict subspace of VMO^{-1} has already been studied, at least for small enough u_0: see [13]. Closely related works are [25] and [5]. Apart from leaving the restriction on the size of u_0 and replacing it by the hypothesis $u_0 \in \mathcal{C}_0$, what we are going to prove is not original.

The stability problem has been considered by Gallagher, Iftimie and Planchon in [14], and we will slightly improve their result, measuring the size of the allowed perturbation in the topology of BMO^{-1} instead of other stronger topologies.

It turns out that such regularity and stability results may be derived from a general principle that we now state. We recall that, by Theorem 1, E denotes the set of all the Cauchy data in VMO^{-1} giving rise to a global solution of (NSI) in $\mathcal{C}_0$.

Lemma 11. *Let F be a Banach space, continuously embedded in VMO^{-1}. Let $\mathcal{F}$ be another Banach space such that $(H3)$ holds. Assume that, for any two global solutions u, v of (NSI) in $\mathcal{C}_0$ associated to data $u_0, v_0 \in F \cap E$, the operator $L_{u,v}$ is continuous on $\mathcal{F}$ and has spectrum $\{0\}$. Let $u_0 \in F \cap E$, and u the solution of $(NSI)_{u_0}$ in $\mathcal{C}_0$. Then*

- (i) $u \in \mathcal{F}$;
- (ii) *there exists $\varepsilon > 0$ such that any $v_0 \in F$ satisfying $\|u_0 - v_0\|_{BMO^{-1}} < \varepsilon$ gives rise to a global solution $v \in \mathcal{F}$ of $(NSI)_{v_0}$. Moreover, the map $v_0 \mapsto v$, from $F \cap B_{BMO^{-1}}(u_0, \varepsilon)$ to $\mathcal{F}$, is analytic at u_0.*

The most natural example of space $\mathcal{F}$ is that of all the functions $f(t,x)$ continuously valued in F for $t \geqslant 0$ and such that $\lim_{t\to+\infty} \|f(t)\|_F = 0$, which we denote by $C_0([0,+\infty[; F)$. In this case and for particular choices of F, the point (i) has already been proved by Gallagher, Iftimie and Planchon, as well as a weaker version of the point (ii) (the admitted perturbations $u_0 - v_0$ being measured in the norm of F). However other choices of $\mathcal{F}$ will be useful.

Remark. The space $\mathcal{F}$ into which we embed our solutions does not necessarily fulfill the hypothesis $(H4)$, and therefore may not be appropriate for *defining* the solutions and getting *uniqueness*. This is why it might be necessary to introduce another space: Gallagher, Iftimie and Planchon choose one of the spaces $\tilde{L} = \bigcap_{T>0} \tilde{L}^r(]0,T[, \dot{B}^s_{p,q})$, $s = -1 + 3/p + 2/r$, $2 < r < 2/(1 - 3/p)$; we take $\bigcap_{T>0} \mathcal{C}_{0,T}$, which is a canonical choice in our context. The two classes of solutions thereby defined are in fact the same: a sketch of the argument is that any solution in $\tilde{L}$ is in $\bigcap_{T>0} \mathcal{C}_{0,T}$ by the discussion in the previous subsection and by uniqueness, and conversely any solution in $\bigcap_{T>0} \mathcal{C}_{0,T}$ is in $\tilde{L}$ by Theorem 1, Lemma 11 above, applied to the case of Besov spaces (see Theorem 12), and Theorem 2.1 in [14]. The reader will find the detailed comparison in [8], as well as the comparison with still another seemingly different class of solutions proposed in [14]. We end here this discussion and turn to the proof of the lemma.

The point (i) is based on an idea we already used several times. We write $(NSI)_{u_0}$ as

$$u + L_{u,0}(u) = Su_0,$$

and apply Lemma 9: this gives $u \in \mathcal{F}$. If now $\|u_0 - v_0\|_{BMO^{-1}} < \varepsilon$, where ε is the same as in point (ii) of Theorem 1, we obtain a global solution v of $(NSI)_{v_0}$ in $\mathcal{C}_0$ on applying Theorem 1, which belongs to $\mathcal{F}$ by the preceding argument. Finally $w = u - v$ is such that

$$w + L_{u,v}(w) = Sw_0,$$

which implies

$$w = \sum_{k \geqslant 1} (-1)^k L_{u,v}^k (Sw_0),$$

this series converging normally in $\mathcal{F}$ by hypothesis. This shows the analyticity result, and ends the proof.

4.3. Application

We will not exhaustively describe the various spaces which were considered in the literature and to which Lemma 11 applies, but rather restrict ourselves to the following three cases: $F = \dot{W}^{1/2,2}$, $F = L^3$ and $F =$ (the closure of $\mathcal{S}$ in) $\dot{B}^{s_p}_{p,q}$, $3/2 < p < \infty$, $s_p = -1 + 3/p$, $1 \leqslant q \leqslant \infty$.[1] Remark that the first case is included in the third one.

Theorem 12. *Let $u_0 \in F \cap E$, where F is one of the spaces listed above, and u be the solution of $(NSI)_{u_0}$ in $\mathcal{C}_0$. Then*

$$u \in C_0\big([0, +\infty[; F\big)$$

and there exists $\varepsilon > 0$ depending on u_0 such that every $v_0 \in F$ with $\|u_0 - v_0\|_{BMO^{-1}} < \varepsilon$ belongs to E, the associated solution being in $C_0([0, +\infty[; F)$. Moreover the map $v_0 \mapsto v$, from $F \cap B_{BMO^{-1}}(u_0, \varepsilon)$ to $C_0([0, +\infty[; F)$, is analytic at u_0.

Several arguments in the proof of this theorem are merely repetitions of what we have already done. We will therefore be allusive sometimes. In particular we will neglect the time-localised estimates, only writing their global versions.

The case $F = L^3$ is the simplest. We apply Lemma 11 with $\mathcal{F} = C_0([0, +\infty[; L^3)$.

Let u, v be two global solutions of (NSI) in $\mathcal{C}_0$, associated to data $u_0, v_0 \in L^3 \cap E$, and consider the operator $L_{u,v}$ acting on L^3-valued functions. The proof is based on the simple inequality (where $0 < s < t$):

$$\begin{aligned} &\big\| e^{(t-s)\Delta} \mathbb{P} \operatorname{div}\big(u(s) \otimes f(s) + f(s) \otimes v(s)\big)\big\|_3 \\ &\quad \leqslant C\big(N_{\infty,t}(u) + N_{\infty,t}(v)\big) \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} \big\| f(s)\big\|_3. \end{aligned} \tag{28}$$

This implies that $L_{u,v}(f)$ is continuously valued in L^3, with

$$\big\|L_{u,v}(f)(t)\big\|_3 \leqslant C\big(N_{\infty,t}(u) + N_{\infty,t}(v)\big) \sup_{0 \leqslant s \leqslant t} \big\| f(s)\big\|_3. \tag{29}$$

Note that $L_{u,v}(f)$ is continuous even at $t = 0$, with $L_{u,v}(f)(0) = 0$, since u and v belong to $\mathcal{C}_0$ by hypothesis. If moreover $\lim_{t \to +\infty} \|f(t)\|_3 = 0$, then for $0 < T < t$ we have thanks to the estimate (28),

[1] Here, the restriction on the lower value of p is not essential, and the case $1 \leqslant p \leqslant 3/2$ could be treated as well.

$$\left\|L_{u,v}(f)(t)\right\|_3 \leqslant C\big(N_\infty(u)+N_\infty(v)\big)\left(\sqrt{\frac{T}{t-T}}\sup_{0<s<T}\left\|f(s)\right\|_3+\sup_{s\geqslant T}\left\|f(s)\right\|_3\right),$$

and therefore

$$\lim_{t\to+\infty}\left\|L_{u,v}(f)(t)\right\|_3=0.$$

This shows that $L_{u,v}(f)\in C_0([0,+\infty[;L^3)$.

Finally, that the spectrum of $L_{u,v}$ on $C_0([0,+\infty[;L^3)$ is $\{0\}$ can be obtained through (29), along the same lines as in the proof of Lemma 6: we let the details to the reader.

Let us now consider that F is (the closure of $\mathcal{S}$ in) $\dot{B}^{s_p}_{p,q}$, $3/2<p<\infty$, $s_p=-1+3/p$, $1\leqslant q\leqslant\infty$. We choose in this case

$$\mathcal{F}=C_0\big([0,+\infty[;F\big)\cap\mathcal{E}_{2p},$$

where $\mathcal{E}_{2p}$ is by definition the space of all functions f such that $f(t)\in L^{2p}$ for almost every $t>0$, and

$$\sup_{t>0}t^{(1-3/(2p))/2}\left\|f(t)\right\|_{2p}<+\infty,$$

with

$$\lim_{\substack{t\to0\\ t\to+\infty}}t^{(1-3/(2p))/2}\left\|f(t)\right\|_{2p}=0.$$

Again, let u, v be two global solutions of (*NSI*) in $\mathcal{C}_0$, associated to data $u_0,v_0\in F\cap E$, and consider the operator $L_{u,v}$. We first concentrate on its behaviour on the space $\mathcal{E}_{2p}$ alone, and start by proving that, if $f\in\mathcal{E}_{2p}$, then $L_{u,v}f\in\mathcal{E}_{2p}$ with the estimate

$$\|L_{u,v}f\|_{\mathcal{E}_{2p}}\leqslant C\big(N_\infty(u)+N_\infty(v)\big)\|f\|_{\mathcal{E}_{2p}}. \tag{30}$$

Indeed we have as in (28),

$$\begin{aligned}&\left\|\mathrm{e}^{(t-s)\Delta}\mathbb{P}\operatorname{div}\big(u(s)\otimes f(s)+f(s)\otimes v(s)\big)\right\|_{2p}\\ &\quad\leqslant C\big(N_\infty(u)+N_\infty(v)\big)(t-s)^{-1/2}s^{-1+3/(4p)}\|f\|_{\mathcal{E}_{2p}},\end{aligned}$$

which leads to

$$\left\|L_{u,v}f(t)\right\|_{2p}\leqslant C\big(N_\infty(u)+N_\infty(v)\big)\|f\|_{\mathcal{E}_{2p}}t^{-1/2+3/(4p)}.$$

This easily implies the continuity of $L_{u,v}$ on $\mathcal{E}_{2p}$. That its spectrum is reduced to $\{0\}$ is deduced from (30) as in Lemma 6 once more.

From this first step, Lemma 11 and the well-known fact that $Su_0\in\mathcal{E}_{2p}$ whenever $u_0\in F$, we deduce the following:

Lemma 13. *Let $u_0 \in F \cap E$, where F is (the closure of $\mathcal{S}$ in) $\dot{B}^{s_p}_{p,q}$, and u be the solution of $(NSI)_{u_0}$ in $\mathcal{C}_0$. Then $u \in \mathcal{E}_{2p}$, and there exists $\varepsilon > 0$ depending on u_0 such that every $v_0 \in F$ with $\|u_0 - v_0\|_{BMO^{-1}} < \varepsilon$ belongs to E, the associated solution v being in $\mathcal{E}_{2p}$.*

We may therefore assume that $u \in \mathcal{E}_{2p}$, and choose $\varepsilon > 0$ such that $v \in \mathcal{E}_{2p}$ as well, whenever $\|u_0 - v_0\|_{BMO^{-1}} < \varepsilon$.

The next step is the following continuity result on the bilinear operator B.

Lemma 14. *Let $3/2 < p < \infty$ and $f, g \in \mathcal{E}_{2p}$. Then $B(f,g)(t) \in \dot{B}^{s_p}_{p,1}$ for every $t > 0$, and*

$$\big\| B(f,g)(t)\big\|_{\dot{B}^{s_p}_{p,1}} \leqslant C\|f\|_{\mathcal{E}_{2p}}\|g\|_{\mathcal{E}_{2p}}.$$

We take a sequence $(\Delta_j)_{j\in\mathbb{Z}}$ of Littlewood–Paley operators; by this we mean that $\Delta_j = \psi(-4^j\Delta)$, where ψ is an infinitely differentiable function defined on $]0,+\infty[$ and supported on $]1/4, 4[$, satisfying the identity

$$\sum_{j\in\mathbb{Z}} \psi\big(4^j\xi\big) = 1$$

for every $\xi > 0$. We recall that we may—and do—define the norm on the Besov space $\dot{B}^{\sigma}_{p,q}$ by the following (see [29]):

$$\|f\|_{\dot{B}^{\sigma}_{p,q}} = \big\|2^{j\sigma}\big\|\Delta_j(f)\big\|_p\big\|_{l^q}.$$

We have by standard arguments

$$\big\|\Delta_j \mathrm{e}^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}\big\|_{p,p} \leqslant C2^j\big(1+4^j(t-s)\big)^{-1}.$$

Thus if $f, g \in \mathcal{E}_{2p}$ we get:

$$\big\|\Delta_j B(f,g)(t)\big\|_p \leqslant C\|f\|_{\mathcal{E}_{2p}}\|g\|_{\mathcal{E}_{2p}} \int_0^t 2^j\big(1+4^j(t-s)\big)^{-1} s^{-1+3/(2p)}\,\mathrm{d}s.$$

Summing over $j \in \mathbb{Z}$, this inequality gives:

$$\big\|B(f,g)(t)\big\|_{\dot{B}^{s_p}_{p,1}} \leqslant C\|f\|_{\mathcal{E}_{2p}}\|g\|_{\mathcal{E}_{2p}} \int_0^t (t-s)^{-3/(2p)} s^{-1+3/(2p)}\,\mathrm{d}s \leqslant C\|f\|_{\mathcal{E}_{2p}}\|g\|_{\mathcal{E}_{2p}},$$

which ends the proof of the lemma.

Returning to the operator $L_{u,v}$, and since $\dot{B}^{s_p}_{p,1}$ is included into $\dot{B}^{s_p}_{p,q}$ for every q, we obtain:

$$\big\|L_{u,v}(f)(t)\big\|_{\dot{B}^{s_p}_{p,q}} \leqslant C\big(\|u\|_{\mathcal{E}_{2p}} + \|v\|_{\mathcal{E}_{2p}}\big)\|f\|_{\mathcal{E}_{2p}}.$$

We leave to the reader the fact that $L_{u,v} f$ is continuously valued in $\dot{B}^{s_p}_{p,q}$ for $t \in [0, +\infty[$, vanishing at 0 and at infinity.

We thus have obtained the continuity of $L_{u,v}$ on $\mathcal{F}$, with the estimate:

$$\|L_{u,v} f\|_{\mathcal{F}} \leqslant C\big(\|u\|_{\mathcal{E}_{2p}} + N_\infty(u) + \|v\|_{\mathcal{E}_{2p}} + N_\infty(v)\big)\|f\|_{\mathcal{E}_{2p}}.$$

Together with (30), this implies:

$$\begin{aligned}\big\|L^2_{u,v} f\big\|_{\mathcal{F}} \leqslant{}& C\big(\|u\|_{\mathcal{E}_{2p}} + N_\infty(u) + \|v\|_{\mathcal{E}_{2p}} + N_\infty(v)\big)\\ &\times \big(N_\infty(u) + N_\infty(v)\big)\|f\|_{\mathcal{F}}.\end{aligned}$$

The important term in this inequality is the factor $(N_\infty(u) + N_\infty(v))$, which allows to argue as in Lemma 6 to prove that the spectrum of $L^2_{u,v}$ is $\{0\}$, hence the same for $L_{u,v}$. This finishes the proof of Theorem 12.

Remark. We owe to the referee the idea of introducing the space $\mathcal{E}_{2p}$ in the case of Besov spaces, which leads to quite a simple proof. However, it is possible to work in the space $C_0([0, +\infty[; F)$ alone, as we did in a first version of this paper, but at the expense of more elaborated arguments. We just quote here without proving it the result we had obtained, which might be of independent interest: *if $u, v \in \mathcal{C}_0$, the operator $L_{u,v}$ is continuous and has spectrum $\{0\}$ on $C_0([0, +\infty[; F)$, where F is as above.*

5. A further comment by way of conclusion

Our proof of Theorem 1 relies on the cancellation property of the trilinear form associated to the Navier–Stokes equations, since it is this property which allowed us to obtain the energy estimate (see (24)). This is in contrast with the many constructions of solutions (global or local) due to Koch, Tataru and their predecessors: these are not based on the cancellation property, and remain valid for a more general class of equations and systems. Therefore, one wonders whether the stability of global solutions is essentially linked to the cancellation property or not.

An answer will be provided by considering the following system in dimension 1 with unknown $u = (u_1, u_2)$:

$$\begin{cases} \frac{\partial u_1}{\partial t} - u_1'' = -\big(u_2\sqrt{u_1^2 + u_2^2}\big)', \\ \frac{\partial u_2}{\partial t} - u_2'' = \big(u_1\sqrt{u_1^2 + u_2^2}\big)'. \end{cases}$$

This example is inspired from another one, of a similar form but designed for another purpose, cited in [21] and attributed to E. Heinze. Though the non-linearity is not given by a bilinear term, the solutions of this system obey the same invariance laws (see $(H1)$–$(H2)$) as the solutions of Navier–Stokes equations. Indeed, a suitable adaptation of Lemma 5 gives the same existence results. However, a simple computation shows that the initial data (valued in $\mathbb{R}^2$)

$$u_0(x) = a(\sin x, \cos x)$$

with $0 < a \leqslant 1$ gives rise to the global solution

$$u(t, x) = a(t)(\sin x, \cos x),$$

where

$$a(t) = \left(1 + \left(\frac{1}{a} - 1\right)e^t\right)^{-1}.$$

Thus the solution obtained for $a = 1$ is not stable, and does not tend to 0 at infinity.

In the light of this example, we think that the cancellation property is an essential feature for the stability results in Navier–Stokes equations to hold, because it is the key to the asymptotic behaviour of the solutions. What remains valid for a larger class of equations, however, is that any global solution tending to 0 as $t \to +\infty$ in an appropriate topology is stable.

Acknowledgements

It is our pleasure to thank Isabelle Gallagher and Fabrice Planchon for having shown their interest in the stability problem in earlier discussions, and later having shared their ideas on the asymptotic properties of the solutions.

We also warmly thank the referee for his suggestions and bibliographical informations, which helped us to improve this paper.

P. Auscher and S. Dubois partially supported by the European Commission (IHP Network “Harmonic Analysis and Related Problems” 2002–2006, contract HPRN-CT-2001-00273-HARP).

References

[1] O. Barraza, Self-similar solutions in weak L^p spaces for Navier–Stokes equations, Rev. Mat. Iberoamericana 12 (1996) 411–439.

[2] H. Beirão da Veiga, P. Secchi, L^p-stability for the strong solutions of the Navier–Stokes equations in the whole space, Arch. Rational Mech. Anal. 98 (1987) 65–70.

[3] C.P. Calderón, Existence of weak solutions for the Navier–Stokes equations with initial data in L^p, Trans. Amer. Math. Soc. 318 (1990) 179–200.

[4] M. Cannone, Ondelettes, paraproduits et Navier–Stokes, Diderot, 1995.

[5] M. Cannone, F. Planchon, On the regularity of the bilinear term for solutions to the incompressible Navier–Stokes equations, Rev. Mat. Iberoamericana 16 (1) (2000) 1–16.

[6] J.-Y. Chemin, Théorèmes d’unicité pour le système de Navier–Stokes tridimensionnel, J. Anal. Math. 77 (1999) 27–50.

[7] S. Dubois, Mild solutions to the Navier–Stokes equations and energy equalities, Adv. Differential Equations, submitted for publication.
[8] S. Dubois, Equations de Navier–Stokes dans l'espace : espaces critiques et solutions d'énergie finie, PhD thesis, Université de Picardie Jules-Verne, 2002.
[9] S. Dubois, What is a solution to the Navier–Stokes equations?, C. R. Acad. Sci. Paris Sér. I Math. 335 (2002) 27–32.
[10] H. Fujita, T. Kato, On the nonstationary Navier–Stokes system, Rend. Sem. Mat. Univ. Padova 32 (1962) 243–260.
[11] H. Fujita, T. Kato, On the Navier–Stokes initial value problem I, Arch. Rational Mech. Anal. 16 (1964) 269–315.
[12] G. Furioli, P.-G. Lemarié-Rieusset, E. Terraneo, Unicité dans $L^3(\mathbb{R}^3)$ et d'autres espaces fonctionnels limites pour Navier–Stokes, Rev. Mat. Iberoamericana 16 (3) (2000) 605–667.
[13] G. Furioli, P.-G. Lemarié-Rieusset, E. Zahrouni, A. Zhioua, Un théorème de persistance de la régularité en norme d'espaces de Besov pour les solutions de Koch et Tataru des équations de Navier–Stokes dans $\mathbb{R}^3$, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000) 339–342.
[14] I. Gallagher, D. Iftimie, F. Planchon, Asymptotics and stability for global solutions to the Navier–Stokes equations, Ann. Inst. Fourier (Grenoble), submitted for publication.
[15] I. Gallagher, D. Iftimie, F. Planchon, Non-explosion en temps grand et stabilité de solutions globales des équations de Navier–Stokes, C. R. Acad. Sci. Paris Sér. I Math. 334 (2002) 289–292.
[16] Y. Giga, T. Miyakawa, Navier-Stokes flows in $\mathbb{R}^3$ and Morrey spaces, Comm. Partial Differential Equations 14 (1989) 577–618.
[17] T. Kato, Strong L^p solutions of the Navier–Stokes equations in $\mathbb{R}^m$ with applications to weak solutions, Math. Z. 187 (1984) 471–480.
[18] T. Kawanago, Stability estimate for strong solutions of the Navier–Stokes system and its applications, Electron. J. Differential Equations 15 (1998) 1–23.
[19] H. Koch, D. Tataru, Well-posedness for the Navier–Stokes equations, Adv. Math. 157 (2001) 22–35.
[20] H. Kozono, M. Yamazaki, Semilinear heat equations and the Navier–Stokes equations with distributions in new function spaces as initial data, Comm. Partial Differential Equations 19 (1994) 959–1014.
[21] O.A. Ladyženskaya, V.A. Solonnikov, N.N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, in: Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI, 1968.
[22] Y. Le Jan, A.-S. Sznitman, Stochastic cascades and 3-dimensional Navier–Stokes equations, C. R. Acad. Sci. Paris Sér. I Math. 324 (1997) 823–826.
[23] P.-G. Lemarié-Rieusset, Recent Progress in the Navier–Stokes Problem, in: Res. Notes Math., vol. 431, Chapman & Hall/CRC Press, London–New York/Boca Raton, FL, 2002.
[24] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. 63 (1934) 193–248.
[25] F. Planchon, Asymptotic behavior of global solutions to the Navier–Stokes equations in $\mathbb{R}^3$, Rev. Mat. Iberoamericana 14(1) (1998) 71–93.
[26] G. Ponce, R. Racke, T.C. Sideris, E.S. Titi, Global stability of large solutions to the 3d Navier–Stokes equations, Comm. Math. Phys. 159 (1994) 329–341.
[27] J. Serrin, The initial value problem for the Navier–Stokes equations, in: R.E. Langer (Ed.), Nonlinear Problems, Univ. of Wisconsin Press, 1963, pp. 69–98.
[28] M.E. Taylor, Analysis on Morrey spaces and applications to Navier–Stokes and other evolution equations, Comm. Partial Differential Equations 17 (1992) 1407–1456.
[29] H. Triebel, Theory of Function Spaces II, Birkhäuser, Basel, 1992.
[30] F. Weissler, The Navier–Stokes initial value problem in L^p, Arch. Rational Mech. Anal. 74 (1980) 219–230.
[31] M. Wiegner, Decay and stability in L_p for strong solutions of the Cauchy problem for Navier–Stokes equations, in: J.G. Heywood, et al. (Eds.), The Navier–Stokes Equations. Theory and Numerical Methods. Proceedings Oberwolfach (1988), in: Lecture Notes in Math., vol. 1431, Springer-Verlag, Berlin/New York, 1990, pp. 95–99.