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On the coefficients of integrated expansions and integrals of ultraspherical polynomials and their applications for solving differential equations

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Abstract

An analytical formula expressing the ultraspherical coefficients of an expansion for an infinitely differentiable function that has been integrated an arbitrary number of times in terms of the coefficients of the original expansion of the function is stated in a more compact form and proved in a simpler way than the formula suggested by Phillips and Karageorghis (27 (1990) 823). A new formula expressing explicitly the integrals of ultraspherical polynomials of any degree that has been integrated an arbitrary number of times of ultraspherical polynomials is given. The tensor product of ultraspherical polynomials is used to approximate a function of more than one variable. Formulae expressing the coefficients of differentiated expansions of double and triple ultraspherical polynomials in terms of the original expansion are stated and proved. Some applications of how to use ultraspherical polynomials for solving ordinary and partial differential equations are described.

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1. Introduction

Classical orthogonal polynomials are used extensively for the numerical solution of differential equations in spectral and pseudospectral methods, see for instance [2–7,9–11,13–15]. If these polynomials are used as basis functions, then the rate of decay of the expansion coefficients is determined by the smoothness properties of the function being expanded and not by any special boundary

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conditions satisfied by the function itself. If the function of interest is infinitely differentiable, then the nth expansion coefficient will decrease faster than any finite power of $(1/n)$ as n tends to infinity, cf. [13].

For spectral and pseudospectral methods; explicit formulae for the expansion coefficients of the derivatives (integrals) in terms of the original expansion coefficients of the function are needed. Also explicit expressions for the derivatives (integrals) of the basis functions themselves are required.

A formula expressing the Chebyshev coefficients of a general order derivative of an infinitely differentiable function in terms of its Chebyshev coefficients is given in [16] and a corresponding formula for the Legendre coefficients is obtained by Phillips [20].

A more general formula for the coefficients of an expansion of ultraspherical polynomials which has been differentiated an arbitrary number of times in terms of those in the original expansion is given in [17,18,5]. More general formulae express the ultraspherical coefficients of the moments of the general order derivative of an infinitely differentiable function in terms of its ultraspherical coefficients are given in [8].

As an alternative approach to differentiating solution expansions is to integrate the differential equation q times, where q is the order of the equation. An advantage of this approach is that the general equation in the algebraic system then contains a finite number of terms.

Phillips and Karageorghis [21] have followed this approach, see Fox and Parker [12], to obtain a formula for the coefficients of an expansion of ultraspherical polynomials that has been integrated an arbitrary number of times in terms of the coefficients of the original expansion. No formula expressing explicitly the integration of ultraspherical polynomials in terms of ultraspherical polynomials is known yet.

In the present paper we rederive the formula given in [21] in a simpler way and write it in a compact form, and we obtain a new explicit formula for the integration of ultraspherical polynomial of any degree that has been integrated an arbitrary number of times in terms of ultraspherical polynomials themselves. The tensor product of ultraspherical polynomials is used to approximate a function of more than one variable defined explicitly or implicitly by differential equation.

The paper is organized as follows: In Section 2, we give some properties of Jacobi and ultraspherical polynomials, and we state without proof two theorems from Doha [5]. The first relates the ultraspherical coefficients of the general order derivative of an infinitely differentiable function in terms of its original ultraspherical coefficients; the second expresses explicitly the derivatives of ultraspherical polynomials of any degree and for any order in terms of the ultraspherical polynomials themselves. In Section 3, we rederive a simpler and more compact formula corresponding to that given in [21]. A formula expressing directly the integration of ultraspherical polynomial of any degree that has been integrated an arbitrary number of times in terms of ultraspherical polynomials is given; results for Chebyshev polynomials of the first and second kinds and for Legendre polynomials are also obtained. We describe how the differentiated and integrated ultraspherical expansions can be applied to a model problem. Formulae expressing the coefficients of differentiated expansions of double and triple ultraspherical polynomials in terms of the coefficients of the original expansion are given in Section 4; the special case of double and triple Chebyshev polynomials of the first and second kind and of Legendre polynomials are also considered; an application of how to use double ultraspherical polynomials for solving Poisson's equation inside a square subject to the nonhomogeneous mixed boundary conditions is also considered. Use of ultraspherical polynomials to solve more general differential equations is also developed. Some concluding remarks are given in Section 5.

2. Some properties of Jacobi and ultraspherical polynomials

The Jacobi polynomials associated with the two real parameters ($\alpha > -1$, $\beta > -1$) are a sequence of polynomials $P_n^{(\alpha,\beta)}(x)$ ($n = 0, 1, \ldots$), each respectively of degree n. These polynomials are eigenfunctions of the following singular Sturm-Liouville equation:

$$(1-x^2)\phi''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]\phi'(x) + n(n + \alpha + \beta + 1)\phi(x) = 0.$$

A consequence of this property is that spectral accuracy can be achieved for an expansion in Jacobi polynomials. The ultraspherical polynomials are Jacobi polynomials with $\alpha = \beta$ and are thus a subclass of the Jacobi polynomials. It is convenient to weigh the ultraspherical polynomials so that

$$C_n^{(\alpha)}(x) = \frac{n!\Gamma(\alpha + \frac{1}{2})}{\Gamma(n + \alpha + \frac{1}{2})} P_n^{(\alpha-1/2,\alpha-1/2)}(x), \tag{1}$$

which gives $C_n^{(\alpha)}(1) = 1$, ($n = 0, 1, 2, \ldots$); this is not the usual standardization, but has the desirable properties that the $C_n^{(0)}(x)$ are identical with the Chebyshev polynomials of the first kind $T_n(x)$, the $C_n^{(1/2)}(x)$ are the Legendre polynomials $P_n(x)$, and $C_n^{(1)}(x)$ are equal to $(1/(n+1))U_n(x)$, where $U_n(x)$ are the Chebyshev polynomials of the second kind.

The following properties of Jacobi polynomials (see for instance, Luke [19, Vol. 1, Section 8.2, pp. 275–276]) are of fundamental importance in the sequel.

$$(2n + \alpha + \beta)P_n^{(\alpha-1,\beta)}(x) = (n + \alpha + \beta)P_n^{(\alpha,\beta)}(x) - (n + \beta)P_{n-1}^{(\alpha,\beta)}(x), \tag{2}$$

$$(2n + \alpha + \beta)P_n^{(\alpha,\beta-1)}(x) = (n + \alpha + \beta)P_n^{(\alpha,\beta)}(x) + (n + \alpha)P_{n-1}^{(\alpha,\beta)}(x), \tag{3}$$

$$D^q P_n^{(\alpha,\beta)}(x) = 2^{-q}(n + \alpha + \beta + 1)_q P_{n-q}^{(\alpha+q,\beta+q)}(x), \tag{4}$$

where

$$D \equiv \frac{\mathrm{d}}{\mathrm{d}x}, \quad (a)_m = \frac{\Gamma(a + m)}{\Gamma(a)}, \quad \lambda = \alpha + \beta + 1.$$

Let $f(x)$ be an infinitely differentiable function defined on $[-1, 1]$, then we can write

$$f(x) = \sum_{n=0}^{\infty} a_n C_n^{(\alpha)}(x), \tag{5}$$

and for the qth derivative of $f(x)$

$$f^{(q)}(x) = \sum_{n=0}^{\infty} a_n^{(q)} C_n^{(\alpha)}(x), \quad a_n^{(0)} = a_n \tag{6}$$

then

Theorem 1.

$$a_n^{(q)} = \frac{2^q(n+\alpha)\Gamma(n+2\alpha)}{(q-1)!n!}\sum_{j=1}^{\infty}\frac{(j+q-2)!(n+2j+q-2)!\Gamma(n+j+q+\alpha-1)}{(j-1)!\Gamma(n+j+\alpha)\Gamma(n+2j+q+2\alpha-2)}a_{n+2j+q-2}$$

$$= \frac{2^q(n+\alpha)\Gamma(n+2\alpha)}{(q-1)!n!}\sum_{\substack{i=n+q\\(i-n-q)\text{ even}}}^{\infty}\frac{i!\left(\frac{i-n+q-2}{2}\right)!\Gamma\left(\frac{i+n+q+2\alpha}{2}\right)}{\left(\frac{i-n-q}{2}\right)!\Gamma(i+2\alpha)\Gamma\left(\frac{i+n-q+2\alpha+2}{2}\right)}a_i,$$

$$n \geqslant 0,\ q \geqslant 1. \tag{7}$$

The formula expresses explicitly the derivatives of ultraspherical polynomials of any degree and for any order in terms of the ultraspherical polynomials themselves is given by the following theorem.

Theorem 2.

$$D^q C_k^{(\alpha)}(x) = \frac{2^q k!}{(q-1)!\Gamma(k+2\alpha)}$$

$$\sum_{\substack{m=0\\(k+m-q)\text{ even}}}^{k-q}\frac{(m+\alpha)\Gamma(m+2\alpha)\left(\frac{k-m+q-2}{2}\right)!\Gamma\left(\frac{k+m+q+2\alpha}{2}\right)}{m!\left(\frac{k-q-m}{2}\right)!\Gamma\left(\frac{k+m-q+2\alpha+2}{2}\right)}C_m^{(\alpha)}(x), \tag{8}$$

(for the proof of Theorems 1 and 2, see [5]).

3. The coefficients of integrated expansions

Following Phillips and Karageorghis [21], and let $b_n^{(q)}$, $q \geqslant 1$, denote the Jacobi expansion coefficients of $f(x)$, $x \in [-1,1]$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} b_n^{(q)} P_n^{(\alpha,\beta)}(x) \tag{9}$$

and let $f(x)$ be an infinitely differentiable function, then we may express the ℓth derivative of $f(x)$ in the form

$$f^{(\ell)}(x) = \sum_{n=0}^{\infty} b_n^{(q-l)} P_n^{(\alpha,\beta)}(x), \quad l \geqslant 0 \tag{10}$$

and in particular

$$f^{(q)}(x) = \sum_{n=0}^{\infty} b_n P_n^{(\alpha,\beta)}(x), \quad b_n = b_n^{(0)}, \tag{11}$$

we derive a recurrence relation involving the coefficients of successive derivatives of $f(x)$ when $\alpha = \beta$.

If we put $\alpha=\gamma$, $\beta=\gamma-1$ in (2) and $\alpha=\beta=\gamma$ in (3) and eliminate $P_n^{(\gamma,\gamma-1)}(x)$, we obtain the formula

$$P_n^{(\gamma-1,\gamma-1)}(x)=\frac{(n+2\gamma-1)(n+2\gamma)}{2(n+\gamma)(2n+2\gamma-1)}P_n^{(\gamma,\gamma)}(x)-\frac{(n+\gamma-1)}{2(2n+2\gamma-1)}P_{n-2}^{(\gamma,\gamma)}(x). \tag{12}$$

It is not difficult to show that

$$\sum_{n=0}^{\infty} b_n^{(q)}\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\gamma,\gamma)}(x)=\sum_{n=0}^{\infty} b_n^{(q-1)}P_n^{(\gamma,\gamma)}(x) \tag{13}$$

and if we replace the derivative in (13) using (4) with $\alpha=\beta=\gamma$, $q=1$, we obtain

$$\frac{1}{2}\sum_{n=0}^{\infty}(n+2\gamma+1)b_n^{(q)}P_{n-1}^{(\gamma+1,\gamma+1)}(x)=\sum_{n=0}^{\infty} b_n^{(q-1)}P_n^{(\gamma,\gamma)}(x). \tag{14}$$

If we replace γ by $\gamma+1$ in (12) and substitute for $P_n^{(\gamma,\gamma)}(x)$ in (14), we get the following formula relating $b_n^{(q)}$ to $b_n^{(q-1)}$

$$b_n^{(q)}=\frac{(n+2\gamma)}{(n+\gamma)(2n+2\gamma-1)}b_{n-1}^{(q-1)}-\frac{(n+\gamma+1)}{(n+2\gamma+1)(2n+2\gamma+3)}b_{n+1}^{(q-1)},\quad n=1,2,\ldots. \tag{15}$$

Phillips and Karageorghis [21] have stated and proved the following formula:

$$b_n^{(q)}=2^q(2n+2\gamma+1)\sum_{j=0}^{q}(-1)^j\binom{q}{j}\frac{\Gamma(n+2\gamma+1)\Gamma(n+\gamma-q+2j+1)}{\Gamma(n+\gamma+1)\Gamma(n+2\gamma-q+2j+1)}$$
$$\times\frac{\Gamma(2n+2\gamma-2q+2j+1)\Gamma(n+\gamma+j+1)}{\Gamma(2n+2\gamma+2j+2)\Gamma(n+\gamma-q+j+1)}b_{n-q+2j},\quad n\geqslant q\geqslant 1, \tag{16}$$

which relates the ultraspherical coefficients $b_n^{(q)}$ of $f(x)$ to the ultraspherical coefficients b_n of the qth derivative of $f(x)$. This formula is somewhat complicated and its derivation is too lengthy and involved.

Now, we rederive this formula in a simpler way and write it in a more compact form. Returning to (15), and if we put $\gamma=\alpha-\frac{1}{2}$, and use the transformation

$$b_n^{(q)}=\frac{2(n+\alpha)\Gamma(n+2\alpha)}{\Gamma(n+\alpha+\frac{1}{2})}A_n^{(q)} \tag{17}$$

we obtain the simple formula

$$2(n+\alpha)A_n^{(q)}=A_{n-1}^{(q-1)}-A_{n+1}^{(q-1)},\quad n\geqslant q\geqslant 1. \tag{18}$$

Note here that substitution of relations (1) and (17) into Eqs. (9)–(11), yields

$$f(x)=\sum_{n=0}^{\infty} b_n^{(q)}P_n^{(\alpha-1/2,\alpha-1/2)}(x)=\sum_{n=0}^{\infty} B_n^{(q)}C_n^{(\alpha)}(x), \tag{19}$$

$$f^{(l)}(x)=\sum_{n=0}^{\infty} B_n^{(q-l)}C_n^{(\alpha)}(x),\quad l=0,1,\ldots,q-1, \tag{20}$$

$$f^{(q)}(x)=\sum_{n=0}^{\infty} B_nC_n^{(\alpha)}(x), \tag{21}$$

where

$$B_n^{(l)} = \frac{2(n+\alpha)\Gamma(n+2\alpha)}{n!\,\Gamma(\alpha+\frac{1}{2})} A_n^{(l)}, \quad l = 0, 1, \ldots, q \tag{22}$$

and from (17)

$$B_n^{(l)} = \frac{\Gamma(n+\alpha+\frac{1}{2})}{n!\,\Gamma(\alpha+\frac{1}{2})} b_n^{(l)}, \quad l = 0, 1, \ldots, q. \tag{23}$$

Now, we are able to prove the following theorem.

Theorem 3.

$$A_n^{(q)} = 2^{-q} \sum_{j=0}^{q} (-1)^j \binom{q}{j} \frac{(n-q+2j+\alpha)\Gamma(n-q+j+\alpha)}{\Gamma(n+j+\alpha+1)} A_{n-q+2j}$$

$$n \geqslant q \textit{ for } \alpha \neq 0, \quad n > q \textit{ for } \alpha = 0. \tag{24}$$

Proof. For $q = 1$, the application of (18) with $q = 1$ yields the required result. Proceeding by induction, assuming that the theorem is valid for q, we want to show that

$$A_n^{(q+1)} = 2^{-(q+1)} \sum_{j=0}^{q+1} (-1)^j \binom{q+1}{j} \frac{(n-q-1+2j+\alpha)\Gamma(n-q-1+j+\alpha)}{\Gamma(n+j+\alpha+1)} A_{n-q-1+2j}. \tag{25}$$

Replacing q by $q+1$ in (18) leads to

$$2(n+\alpha)A_n^{(q+1)} = A_{n-1}^{(q)} - A_{n+1}^{(q)}. \tag{26}$$

Since the theorem is true for q, we may express both $A_{n-1}^{(q)}$ and $A_{n+1}^{(q)}$ in terms of the A_n. Thus the right-hand side of (26) becomes

$$\begin{aligned}
A_{n-1}^{(q)} - A_{n+1}^{(q)} &= 2^{-q} \sum_{j=0}^{q} (-1)^j \binom{q}{j} \frac{(n-q+2j+\alpha-1)\Gamma(n-q+j+\alpha-1)}{\Gamma(n+j+\alpha)} A_{n-q+2j-1} \\
&\quad - 2^{-q} \sum_{j=0}^{q} (-1)^j \binom{q}{j} \frac{(n-q+2j+\alpha+1)\Gamma(n-q+j+\alpha+1)}{\Gamma(n+j+\alpha+2)} A_{n-q+2j+1} \\
&= 2^{-q} \sum_{j=0}^{q+1} (-1)^j \binom{q}{j} \frac{(n-q+2j+\alpha-1)(n+j+\alpha)\Gamma(n-q+j+\alpha-1)}{\Gamma(n+j+\alpha+1)} A_{n-q+2j-1} \\
&\quad - 2^{-q} \sum_{j=0}^{q+1} (-1)^{j-1} \binom{q}{j-1} \frac{(n-q+2j+\alpha-1)\Gamma(n-q+j+\alpha)}{\Gamma(n+j+\alpha+1)} A_{n-q+2j-1}
\end{aligned}$$

$$= 2^{-q}\sum_{j=0}^{q+1}(-1)^j\binom{q}{j}\frac{(n-q+2j+\alpha-1)(n+j+\alpha)\Gamma(n-q+j+\alpha-1)}{\Gamma(n+j+\alpha+1)}A_{n-q+2j-1}$$

$$+2^{-q}\sum_{j=0}^{q+1}(-1)^j\binom{q}{j-1}\frac{(n-q+2j+\alpha-1)(n-q+j+\alpha-1)\Gamma(n-q+j+\alpha-1)}{\Gamma(n+j+\alpha+1)}$$

$$\times A_{n-q+2j-1}$$

$$= 2^{-q}\sum_{j=0}^{q+1}(-1)^j\frac{(n-q+2j+\alpha-1)\Gamma(n-q+j+\alpha-1)}{\Gamma(n+j+\alpha+1)}$$

$$\times\left[\binom{q}{j}(n+j+\alpha)+\binom{q}{j-1}(n-q+j+\alpha-1)\right]A_{n-q+2j-1}$$

$$= 2^{-q}(n+\alpha)\sum_{j=0}^{q+1}(-1)^j\binom{q+1}{j}\frac{(n-q+2j+\alpha-1)\Gamma(n-q+j+\alpha-1)}{\Gamma(n+j+\alpha+1)}A_{n-q+2j-1}$$

which yields immediately that

$$A_n^{(q+1)} = 2^{-(q+1)}\sum_{j=0}^{q+1}(-1)^j\binom{q+1}{j}\frac{(n-q+2j+\alpha-1)\Gamma(n-q+j+\alpha-1)}{\Gamma(n+j+\alpha+1)}A_{n-q+2j-1}$$

this completes the inductive step and proves the theorem.

Formula (24) is simpler and more compact than formula (16) given by Phillips and Karageorghis [21].

In particular, the special cases for Chebyshev polynomials of the first and second kinds may be obtained directly by taking $\alpha = 0$, 1 respectively, and for the Legendre polynomials by taking $\alpha = \frac{1}{2}$. These are given as corollaries to Theorem 3. $\square$

Corollary 1. *Let $f(x)$ be an infinitely differentiable function on $[-1,1]$. The Chebyshev coefficients $B_n^{(q)}$ of $f(x)$ in (19) are related to the Chebyshev coefficients B_n of the qth derivative of $f(x)$ in (21), for the case $\alpha = 0$, by*

$$B_n^{(q)} = \frac{2}{\sqrt{\pi}}A_n^{(q)}, \quad q = 0, 1, \ldots,$$

where

$$A_n^{(q)} = 2^{-q}\sum_{j=0}^{q}(-1)^j\binom{q}{j}\frac{(n+j-q-1)!}{(n+j)!}(n+2j-q)A_{n+2j-q}, \quad n \geqslant q \geqslant 0. \tag{27}$$

Corollary 2. *For the case $\alpha = 1$, the coefficients $B_n^{(q)}$ of $f(x)$ in (19) are related to the coefficients B_n of the qth derivative of $f(x)$ in (21) by*

$$B_n^{(q)} = \frac{4}{\sqrt{\pi}}(n+1)^2 A_n^{(q)}, \quad q = 0, 1, \ldots,$$

where

$$A_n^{(q)} = 2^{-q}\sum_{j=0}^{q}(-1)^j \binom{q}{j} \frac{(n-q+j)!}{(n+j+1)!}(n-q+2j+1)A_{n-q+2j}, \quad n \geqslant q \geqslant 0. \tag{28}$$

Corollary 3. *The corresponding Legendre coefficients ($\alpha = \frac{1}{2}$), are related by*

$$B_n^{(q)} = (2n+1)A_n^{(q)}, \quad q = 0, 1, \ldots,$$

where

$$A_n^{(q)} = 2^{q}\sum_{j=0}^{q}(-1)^j \binom{q}{j} \frac{(2n-2q+2j)!(n+j)!}{(2n+2j+1)!(n-q+j)!}(2n-2q+4j+1)A_{n+2j-q},$$
$$n \geqslant q \geqslant 0. \tag{29}$$

3.1. Computation of q times repeated integration of $C_n^{(\alpha)}(x)$

Theorem 4. *If we define the q times repeated integration of $C_n^{(\alpha)}(x)$ by*

$$I_n^{(q,\alpha)}(x) = \int \int^{q \text{ times}} \cdots \int C_n^{(\alpha)}(x)\,\mathrm{d}x\,\mathrm{d}x \ldots \mathrm{d}x \tag{30}$$

then

$$I_n^{(q,\alpha)}(x) = \frac{2^{-q} n!}{\Gamma(n+2\alpha)}\sum_{j=0}^{q}(-1)^j \binom{q}{j} \frac{\Gamma(n-j+\alpha)\Gamma(n+q-2j+2\alpha)}{(n+q-2j)!\Gamma(n+q-j+\alpha+1)}(n+q-2j+\alpha)$$
$$\times C_{n+q-2j}^{(\alpha)}(x), \quad q \geqslant 0,\ n \geqslant q+1 \ \mathit{for}\ \alpha = 0;\ q \geqslant 0,\ n \geqslant q \ \mathit{for}\ \alpha \neq 0. \tag{31}$$

Proof. If we integrate Eq. (21), q times, with respect to x, we get

$$f(x) = \sum_{n=0}^{\infty} B_n I_n^{(q,\alpha)}(x) = \sum_{n=0}^{\infty} \frac{2(n+\alpha)\Gamma(n+2\alpha)}{n!\Gamma(\alpha+\frac{1}{2})} A_n I_n^{(q,\alpha)}(x). \tag{32}$$

Making use of Eqs. (22), (24) and substitution into (19) give

$$f(x) = \sum_{n=0}^{\infty} \frac{2(n+\alpha)\Gamma(n+2\alpha)}{n!\Gamma(\alpha+\frac{1}{2})} 2^{-q}\sum_{j=0}^{q} \binom{q}{j} \frac{(n-q+2j+\alpha)\Gamma(n-q+j+\alpha)}{\Gamma(n+j+\alpha+1)}$$
$$\times A_{n-q+2j} C_n^{(\alpha)}(x). \tag{33}$$

Expanding (33) and collecting similar terms and comparing the result with (32), we get the following formula which expresses explicitly the integration of the ultraspherical polynomials of any degree,

q times, in terms of the ultraspherical polynomials themselves in the form

$$I_n^{(q,\alpha)}(x)=\frac{2^{-q}n!}{\Gamma(n+2\alpha)}\sum_{j=0}^{q}(-1)^j\binom{q}{j}\frac{\Gamma(n-j+\alpha)\Gamma(n+q-2j+2\alpha)}{(n+q-2j)!\Gamma(n+q-j+\alpha+1)}$$

$$\times(n+q-2j+\alpha)C_{n+q-2j}^{(\alpha)}(x).$$

This completes the proof of Theorem 4. □

The particular expressions for Chebyshev polynomials of the first and second kinds and for Legendre polynomials, may be obtained as special cases of formula (31). We give these as corollaries as follows:

Corollary 4. *For* $\alpha=0$

$$I_n^{(q,0)}(x)=\frac{n}{2^q}\sum_{j=0}^{q}(-1)^j\binom{q}{j}\frac{(n-j-1)!}{(n-j+q)!}T_{n+q-2j}(x),\quad n>q\geqslant 1. \tag{34}$$

Corollary 5. *For* $\alpha=1/2$

$$I^{(q,1/2)}(x)=2^q\sum_{j=0}^{q}(-1)^j\binom{q}{j}\frac{(2n-2j)!(n+q-j)!}{(n-j)!(2n+2q-2j+1)!}(2n+2q-4j+1)P_{n+q-2j}(x),$$
$$n\geqslant q\geqslant 1. \tag{35}$$

Corollary 6. *For* $\alpha=1$

$$I_n^{(q,1)}(x)=\frac{2^{-q}}{(n+1)}\sum_{j=0}^{q}(-1)^j\binom{q}{j}\frac{(n-j)!(n+q-2j+1)}{(n+q-j+1)!}U_{n+q-2j}(x),\quad n\geqslant q\geqslant 1. \tag{36}$$

3.2. The differentiated and integrated systems for a model problem

We demonstrate in this section how the differentiated and integrated ultraspherical expansions can be applied to the following model problem.

3.2.1. The differentiated system for a model problem

Consider the solution of the differential equation

$$f''(x)+\lambda f(x)=g(x),\quad x\in[-1,1], \tag{37}$$

subject to

$$f(\pm 1)=0, \tag{38}$$

and λ is a scalar. Suppose that we approximate $f(x)$ by a truncated expansion of ultraspherical polynomials

$$f_N(x)=\sum_{n=1}^{N/2} a_{2n}(C_{2n}^{(\alpha)}(x)-C_0^{(\alpha)}(x))+\sum_{n=1}^{N/2-1} a_{2n+1}(C_{2n+1}^{(\alpha)}(x)-C_1^{(\alpha)}(x)), \tag{39}$$

where we assume that N is even. We seek to determine a_n, $n=2,3,\ldots,N$, using Galerkin method. Note here that the form of (39) ensures that the boundary conditions (38) are satisfied.

Let us first consider the differentiated system. Since $f_N''(x)$ is a polynomial of degree at most $N-2$, we may write

$$f_N''(x)=\sum_{n=0}^{N-2} a_n^{(2)} C_n^{(\alpha)}(x), \tag{40}$$

where

$$a_n^{(2)}=\frac{(n+\alpha)\Gamma(n+2\alpha)}{n!}\sum_{\substack{i=n+2\\(i+n)\ \text{even}}}^{N}\frac{i!(i-n)(i+n+2\alpha)}{\Gamma(i+2\alpha)}a_i \tag{41}$$

is obtained from (7) by taking $q=2$. The coefficients a_n are chosen so that $f_N(x)$ satisfies

$$f_N''(x)+\lambda f_N(x)=g_N(x), \tag{42}$$

where

$$g_N(x)=\sum_{n=0}^{N} b_n C_n^{(\alpha)}(x). \tag{43}$$

Substituting (39) and (40) into (42), yields

$$a_0^{(2)}-\lambda\sum_{n=1}^{N/2} a_{2n}=b_0, \tag{44}$$

$$a_1^{(2)}-\lambda\sum_{n=1}^{N/2-1} a_{2n+1}=b_1, \tag{45}$$

$$a_m^{(2)}-\lambda a_m=b_m, \quad m=2,\ldots,N-2. \tag{46}$$

If we substitute from relation (41) into Eqs. (44)–(46), we get the recurrence relations which should be solved to find the ultraspherical expansion coefficients, a_n, $n=2,3,\ldots,N$. This in turn may be put in the form

$$A\mathbf{a}=\mathbf{b}, \tag{47}$$

where the structure of the coefficient matrix A is

$$\begin{pmatrix} x & 0 & x & 0 & x & \cdots & 0 & x & 0 & x \\ 0 & x & 0 & x & 0 & \cdots & x & 0 & x & 0 \\ x & 0 & x & 0 & x & \cdots & 0 & x & 0 & x \\ 0 & x & 0 & x & 0 & \cdots & x & 0 & x & 0 \\ 0 & 0 & x & 0 & x & \cdots & 0 & x & 0 & x \\ 0 & & & & & \cdots & x & 0 & x & 0 \\ 0 & & & & & \cdots & 0 & x & 0 & x \end{pmatrix}.$$

This matrix is not banded but its entries can be determined from (44) to (46) and (41).

3.2.2. The integrated system for the model problem

Let us now derive the integrated system associated with the solution of (37). We integrate (37) twice with respect to x to obtain

$$f_N(x) + \lambda \int\int f_N(x)(\mathrm{d}x)^2 = \int\int g_N(x)(\mathrm{d}x)^2 + d_0 + d_1 C_1^{(\alpha)}(x) \tag{48}$$

and using (39) and (43), enables one to put (48) in the form

$$\sum_{n=2}^{N} a_n[C_n^{(\alpha)}(x) + \lambda I_n^{(2,\alpha)}(x)] - \sum_{n=1}^{N/2} a_{2n}[C_0^{(\alpha)}(x) + \lambda I_0^{(2,\alpha)}(x)]$$
$$- \sum_{n=1}^{N/2-1} a_{2n+1}[C_1^{(\alpha)}(x) + \lambda I_1^{(2,\alpha)}(x)] = \sum_{n=0}^{N} b_n I_n^{(2,\alpha)}(x) + d_0 + d_1 C_1^{(\alpha)}(x), \tag{49}$$

where $I_n^{(2,\alpha)}$ is obtained from formula (31) of Theorem 4, as

$$I_n^{(2,\alpha)}(x) = \frac{1}{4}\left[\frac{(n+2\alpha)(n+2\alpha+1)}{(n+1)(n+2)(n+\alpha)(n+\alpha+1)} C_{n+2}^{(\alpha)}(x) - \frac{2}{(n+\alpha-1)(n+\alpha+1)} C_n^{(\alpha)}(x) \right.$$
$$\left. + \frac{n(n-1)}{(n+\alpha-1)(n+\alpha)(n+2\alpha-2)(n+2\alpha-1)} C_{n-2}^{(\alpha)}(x)\right], \tag{50}$$

$n \geqslant 3$ for $\alpha = 0$; $n \geqslant 2$ for $\alpha \neq 0$,

$$I_0^{(2,\alpha)}(x) = \begin{cases} \dfrac{1}{4(1+\alpha)}[(1+2\alpha)C_2^{(\alpha)}(x) + C_0^{(\alpha)}(x)], & \alpha \neq 0, \\ \dfrac{1}{4}[T_2(x) + T_0(x)], & \alpha = 0, \end{cases} \tag{51}$$

$$I_1^{(2,\alpha)}(x) = \begin{cases} \dfrac{1}{12(2+\alpha)}[(1+2\alpha)C_3^{(\alpha)}(x) + 3C_1^{(\alpha)}(x)], & \alpha \neq 0, \\ \dfrac{1}{24}[T_3(x) + 3T_1(x)], & \alpha = 0, \end{cases} \tag{52}$$

$$I_2^{(2,0)}(x) = \frac{1}{48}T_4(x) - \frac{1}{6}T_2(x) - \frac{3}{16}T_0(x) \tag{53}$$

and d_0, d_1 are constants of integration. Using (49)–(53), we arrive at the following results:
(i) For $\alpha=0$

$$\sum_{n=2}^{N} a_n T_n(x) + \lambda\left[\left\{-\frac{a_2}{6}+\frac{a_4}{24}-\frac{1}{4}\sum_{n=1}^{N/2} a_{2n}\right\} T_2(x) + \left\{-\frac{a_3}{16}+\frac{a_5}{48}-\frac{1}{24}\sum_{n=1}^{N/2-1} a_{2n+1}\right\} T_3(x)\right.$$
$$\left. + \frac{1}{4}\sum_{n=4}^{N}\left\{\frac{a_{n-2}}{n(n-1)} - \frac{2a_n}{(n-1)(n+1)} + \frac{a_{n+2}}{n(n+1)}\right\} T_n(x)\right] + e_{N+1}T_{N+1}(x) + e_{N+2}T_{N+2}(x)$$
$$= \sum_{n=2}^{N} \tilde{d}_n T_n(x) + AT_1(x) + B, \tag{54}$$

where $\tilde{d}_n$ are the coefficients of twice-integrated $g_N(x)$. Eq. (54) leads to the following recurrence relations:

$$\left.\begin{aligned}
&\left(1-\frac{5\lambda}{12}\right)a_2 - \frac{5\lambda}{24}a_4 - \frac{\lambda}{4}\sum_{n=3}^{N/2} a_{2n} = \tilde{d}_2,\\
&\left(1-\frac{5\lambda}{48}\right)a_3 - \frac{\lambda}{48}a_5 - \frac{\lambda}{24}\sum_{n=3}^{N/2-1} a_{2n+1} = \tilde{d}_3,\\
&\frac{\lambda}{4n(n-1)}a_{n-2} + \left(1-\frac{\lambda}{2(n-1)(n+1)}\right)a_n + \frac{\lambda}{4n(n+1)}a_{n+2} = \tilde{d}_n, \quad n=4,5,\ldots,N.
\end{aligned}\right\} \tag{55}$$

Relations (55) may be put in the form of algebraic system for the coefficients $a_n, n=2,\ldots,N$, of the form

$$B\mathbf{a} = \mathbf{f}. \tag{56}$$

(ii) For $\alpha \neq 0$

$$\sum_{n=2}^{N} a_n C_n^{(\alpha)}(x)$$
$$+\lambda\left[\left\{-\frac{a_2}{2(\alpha+1)(\alpha+2)} + \frac{3a_4}{(\alpha+3)(\alpha+4)(2\alpha+2)(2\alpha+3)} - \frac{(2\alpha+1)}{4(\alpha+1)}\sum_{n=1}^{N/2} a_{2n}\right\} C_2^{(\alpha)}(x)\right.$$
$$+\left\{-\frac{a_3}{2(\alpha+2)(\alpha+4)} + \frac{5a_5}{(\alpha+4)(\alpha+5)(2\alpha+3)(2\alpha+4)} - \frac{(2\alpha+1)}{12(\alpha+2)}\sum_{n=1}^{N/2-1} a_{2n+1}\right\} C_3^{(\alpha)}(x)$$

$$+\frac{1}{4}\sum_{n=4}^{N}\left\{\begin{array}{l}\dfrac{(n+2\alpha-1)(n+2\alpha-2)}{n(n-1)(n+\alpha-2)(n+\alpha-1)}a_{n-2}-\dfrac{2}{(n+\alpha-1)(n+\alpha+1)}a_n\\ \quad+\dfrac{(n+1)(n+2)}{(n+\alpha+1)(n+\alpha+2)(n+2\alpha)(n+2\alpha+1)}a_{n+2}\end{array}\right\}C_n^{(\alpha)}(x)\Bigg]$$

$$+\tilde{e}_{N+1}C_{n+1}^{(\alpha)}(x)+\tilde{e}_{N+2}C_{n+2}^{(\alpha)}(x)=\sum_{n=2}^{N}\hat{d}_nC_n^{(\alpha)}(x)+\tilde{A}C_1^{(\alpha)}(x)+\tilde{B}, \tag{57}$$

which leads to similar recurrence relations like those in (55), namely,

$$\left[1-\frac{\lambda}{2(\alpha+1)(\alpha+2)}-\frac{\lambda(2\alpha+1)}{4(\alpha+1)}\right]a_2$$

$$+\lambda\left[\frac{3}{(\alpha+3)(\alpha+4)(2\alpha+2)(2\alpha+3)}-\frac{(2\alpha+1)}{4(\alpha+1)}\right]a_4-\frac{\lambda(2\alpha+1)}{4(\alpha+1)}\sum_{n=3}^{N/2}a_{2n}=\hat{d}_2,$$

$$\left[1-\frac{\lambda}{2(\alpha+2)(\alpha+4)}-\frac{\lambda(2\alpha+1)}{12(\alpha+2)}\right]a_3$$

$$+\lambda\left[\frac{5}{(\alpha+4)(\alpha+5)(2\alpha+3)(2\alpha+4)}-\frac{(2\alpha+1)}{12(\alpha+2)}\right]a_5-\frac{\lambda(2\alpha+1)}{12(\alpha+2)}\sum_{n=3}^{N/2-1}a_{2n+1}=\hat{d}_3,$$

$$\frac{\lambda(n+2\alpha-1)(n+2\alpha-2)}{4n(n-1)(n+\alpha-2)(n+\alpha-1)}a_{n-2}+\left(1-\frac{\lambda}{2(n+\alpha-1)(n+\alpha+1)}\right)a_n$$

$$+\frac{\lambda(n+1)(n+2)}{4(n+\alpha+1)(n+\alpha+2)(n+2\alpha)(n+2\alpha+1)}a_{n+2}=\hat{d}_n,\quad n=4,5,\dots,N \tag{58}$$

which may be put in the form

$$C\mathbf{a}=\mathbf{g}. \tag{59}$$

It is of fundamental importance to notice that the structure of the matrices B and C is different from that of A. Apart from the first two rows of B and C, the matrices have a band width of five. The three systems (47), (56) and (59) can be decoupled into separate systems for the even and odd coefficients a_n. In this way one needs to solve two systems of order n instead of one of order $2n$, which leads to a substantial savings. Further, we believe that in the cases of systems (56) and (59), these can be decoupled into two systems which, with the exception of the first row, are tridiagonal and therefore can be solved very efficiently.

4. The coefficients of differentiated expansions of double and triple ultraspherical polynomials

We define the double ultraspherical polynomials as

$$C_{mn}^{(\alpha)}(x,y)=C_m^{(\alpha)}(x)C_m^{(\alpha)}(y), \tag{60}$$

where $C_m^{(\alpha)}(x)$, $C_n^{(\alpha)}(y)$ are ultraspherical polynomials of degrees m and n in the variables x and y, respectively. These polynomials are satisfying the biorthogonality relation

$$\int_{-1}^{1}\int_{-1}^{1}[(1-x)^2(1-y^2)]^{\alpha-1/2}C_{ij}^{(\alpha)}(x,y)C_{kl}^{(\alpha)}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

$$=\begin{cases} \dfrac{\pi i!j!}{(i+\alpha)(j+\alpha)\Gamma(i+2\alpha)\Gamma(j+2\alpha)}\left[\dfrac{\Gamma(2\alpha)\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\right]^2 & i=k,\ j=l,\ \alpha\neq 0,\\ \pi^2 & i=j=k=l=0,\ \alpha=0,\\ \dfrac{\pi^2}{4}, & i=k\neq 0,\ j=l\neq 0,\ \alpha=0,\\ \dfrac{\pi^2}{2}, & i=k\neq 0,\ j=l=0,\ \text{or}\\ & \quad i=k=0,\ j=l\neq 0,\ \alpha=0,\\ 0 & \text{for all other values of } i,j,k,l. \end{cases}$$

It is worthy to note here that typical orthogonal polynomials, the double Chebyshev polynomials of the first kind $T_{mn}(x,y)$ and of the second kind $U_{mn}(x,y)$ and the double Legendre polynomials $P_{mn}(x,y)$, are particular forms of the double ultraspherical polynomials. Namely, we have

$$T_{mn}(x,y)=C_{mn}^{(0)}(x,y)=T_m(x)T_n(y),$$

$$U_{mn}(x,y)=C_{mn}^{(1)}(x,y)=\frac{1}{(m+1)(n+1)}U_m(x)U_n(y),$$

$$P_{mn}(x,y)=C_{mn}^{(1/2)}(x,y)=P_m(x)P_n(y).$$

Let $u(x,y)$ be a continuous function defined on the square $(-1\leqslant x,y\leqslant 1)$, and let it has continuous and bounded partial derivatives of any order with respect to its variables x and y. Then it is possible to express

$$u(x,y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y), \tag{61}$$

$$u^{(p,q)}(x,y)=\frac{\partial^{p+q}u(x,y)}{\partial x^p\partial y^q}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}^{(p,q)}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y), \tag{62}$$

where $a_{mn}^{(p,q)}$ denote the ultraspherical expansion coefficients of $u^{(p,q)}(x,y)$ and $a_{mn}^{(0,0)}=a_{mn}$. Using the expressions (see, [5])

$$2(m+\alpha)C_m^{(\alpha)}(x)=\frac{m+2\alpha}{m+1}D_xC_{m+1}^{(\alpha)}(x)-\frac{m}{m+2\alpha-1}D_xC_{m-1}^{(\alpha)}(x), \tag{63}$$

$$2(n+\alpha)C_n^{(\alpha)}(y)=\frac{n+2\alpha}{n+1}D_yC_{n+1}^{(\alpha)}(y)-\frac{n}{n+2\alpha-1}D_yC_{n-1}^{(\alpha)}(y) \tag{64}$$

with the assumptions that

$$D_x\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}^{(p-1,q)}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}^{(p,q)}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y),$$

$$D_y\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}^{(p,q-1)}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{mn}^{(p,q)}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y)$$

it is not difficult to derive the relations

$$\frac{(m+2\alpha-1)}{2m(m+\alpha-1)}a_{m-1,n}^{(p,q)}-\frac{(m+1)}{2(m+\alpha+1)(m+2\alpha)}a_{m+1,n}^{(p,q)}=a_{mn}^{(p-1,q)},\quad m,p\geqslant 1, \tag{65}$$

$$\frac{(n+2\alpha-1)}{2n(n+\alpha-1)}a_{m,n-1}^{(p,q)}-\frac{(n+1)}{2(n+\alpha+1)(n+2\alpha)}a_{m+1,n}^{(p,q)}=a_{mn}^{(p,q-1)},\quad n,q\geqslant 1. \tag{66}$$

To simplify the computing, we define a related set of coefficients $b_{mn}^{(p,q)}$ by writing

$$a_{mn}^{(p,q)}=\frac{(m+\alpha)(n+\alpha)\Gamma(m+2\alpha)\Gamma(n+2\alpha)}{m!}b_{mn}^{(p,q)},\quad m,n\geqslant 0,\ \ p,q=0,1,2,\ldots\ . \tag{67}$$

Eqs. (65) and (66) take the simpler forms

$$b_{m-1,n}^{(p,q)}-b_{m+1,n}^{(p,q)}=2(m+\alpha)b_{m,n}^{(p-1,q)},\quad m,p\geqslant 1, \tag{68}$$

$$b_{m,n-1}^{(p,q)}-b_{m,n+1}^{(p,q)}=2(n+\alpha)b_{m,n}^{(p,q-1)},\quad n,q\geqslant 1. \tag{69}$$

Repeated application of (68) keeping n and q fixed, see [22], yields

$$b_{mn}^{(p,q)}=2\sum_{i=1}^{\infty}(m+2i+\alpha-1)b_{m+2i-1,n}^{(p-1,q)},\quad p\geqslant 1 \tag{70}$$

and the same with (69) keeping m and p fixed yields

$$b_{mn}^{(p,q)}=2\sum_{j=1}^{\infty}(n+2j+\alpha-1)b_{m,n+2j-1}^{(p,q-1)},\quad q\geqslant 1. \tag{71}$$

4.1. Relations between the coefficients

The main result of this section is to prove the following theorem.

Theorem 5. *The coefficients $b_{mn}^{(p,q)}$ are related to the coefficients $b_{mn}^{(0,q)}$, $b_{mn}^{(p,0)}$ and the original coefficients b_{mn} by*

$$b_{mn}^{(p,q)}=\frac{2^p}{(p-1)!}\sum_{i=1}^{\infty}\frac{(i+p-2)!\Gamma(m+i+p+\alpha-1)}{(i-1)!\Gamma(m+i+\alpha)}(m+2i+p+\alpha-2)b_{m+2i+p-2,n}^{(0,q)},$$

$$p\geqslant 1, \tag{72}$$

$$b_{mn}^{(p,q)} = \frac{2^q}{(q-1)!}\sum_{j=1}^{\infty}\frac{(j+q-2)!\Gamma(n+j+q+\alpha-1)}{(j-1)!\Gamma(n+j+\alpha)}(n+2j+q+\alpha-2)b_{m,n+2j+q-2}^{(p,0)},$$

$$q \geqslant 1, \tag{73}$$

$$b_{mn}^{(p,q)} = \frac{2^{p+q}}{(p-1)!(q-1)!}$$
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(i+p-2)!(j+q-2)!\Gamma(m+i+p+\alpha-1)\Gamma(n+j+q+\alpha-1)}{(i-1)!(j-1)!\Gamma(m+i+\alpha)\Gamma(n+j+\alpha)}$$
$$\times(m+2i+p+\alpha-2)(n+2j+q+\alpha-2)b_{m+2i+p-2,n+2j+q-2}, \quad p,q \geqslant 1 \tag{74}$$

for all $m,n \geqslant 0$.

In order to prove the theorem, the following two lemmas are required:

Lemma 1.

$$\sum_{i=1}^{M}(m+2i+\alpha-1)\frac{(M-i+p-1)!\Gamma(m+i+M+p+\alpha-1)}{(M-i)!\Gamma(m+i+M+\alpha)}$$
$$= \frac{1}{p}\frac{(M+p-1)!\Gamma(m+M+p+\alpha)}{(M-1)!\Gamma(m+M+\alpha)}, \quad m,p \geqslant 1. \tag{75}$$

Lemma 2.

$$\sum_{j=1}^{N}(n+2j+\alpha-1)\frac{(N-j+q-1)!\Gamma(n+j+N+q+\alpha-1)}{(N-j)!\Gamma(n+j+N+\alpha)}$$
$$= \frac{1}{q}\frac{(N+q-1)!\Gamma(n+N+q+\alpha)}{(N-1)!\Gamma(n+N+\alpha)}, \quad n,q \geqslant 1. \tag{76}$$

The interested reader is referred to [5] for the proof of any of these two lemmas.

Proof of Theorem 5. Firstly, we prove formula (72). For $p=1$, application of (70) with $p=1$ yields the required formula. Proceeding by induction, assuming that the relation is valid for p (keeping n and q fixed), we want to show that

$$b_{mn}^{(p+1,q)} = \frac{2^{p+1}}{p!}\sum_{i=1}^{\infty}\frac{(i+p-1)!\Gamma(m+i+p+\alpha)}{(i-1)!\Gamma(m+i+\alpha)}(m+2i+p+\alpha-1)b_{m+2i+p-1,n}^{(0,q)}. \tag{77}$$

From (70), replacing p by $p+1$, and assuming the validity of (72) for p,

$$b_{mn}^{(p+1,q)} = \frac{2^{p+1}}{(p-1)!}\sum_{i=1}^{\infty}(m+2i+\alpha-1)\left\{\sum_{k=1}^{\infty}\frac{(k+p-2)!\Gamma(m+2i+k+p+\alpha-2)}{(k-1)!\Gamma(m+2i+k+\alpha-1)}\right.$$
$$\left.\times(m+2i+2k+p+\alpha-3)b_{m+2i+2k+p-3,n}^{(0,q)}\right\} \tag{78}$$

let $i+k-1=M$, then (78) takes the form

$$b_{mn}^{(p+1,q)} = \frac{2^{p+1}}{(p-1)!} \sum_{M=1}^{\infty} \left[\sum_{\substack{i,k=1\\ i+k=M+1}}^{M} (m+2i+\alpha-1) \frac{(k+p-2)!\,\Gamma(m+2i+k+p+\alpha-2)}{(k-1)!\,\Gamma(m+2i+k+\alpha-1)} \times (m+2M+p+\alpha-1) b_{m+2M+p-1,n}^{(0,q)} \right]$$

which may also be written as

$$b_{mn}^{(p+1,q)} = \frac{2^{p+1}}{(p-1)!} \sum_{M=1}^{\infty} \left\{ \sum_{i=1}^{M} (m+2i+\alpha-1) \frac{(M-i+p-1)!\,\Gamma(m+M+i+p+\alpha-1)}{(M-i)!\,\Gamma(m+M+i+\alpha)} \times (m+2M+p+\alpha-1) \right\} b_{m+2M+p-1,n}^{(0,q)}.$$

Application of Lemma (1) to the second series yields Eq. (77) and the proof of formula (72) is completed.

It can also be shown that formula (73) is true by following the same procedure with (71) keeping m and p fixed. Formula (74) is obtained immediately by substituting (72) into (73). This completes the proof of Theorem 5. $\square$

Now, substitution of (72), (73) and (74) into (67) give the relations between the coefficients $a_{mn}^{(p,q)}$, $a_{mn}^{(0,q)}$, $a_{mn}^{(p,0)}$ and a_{mn} as

$$a_{mn}^{(p,q)} = \frac{2^p(m+\alpha)\Gamma(m+2\alpha)}{(p-1)!\,m!} \sum_{i=1}^{\infty} \frac{(i+p-2)!\,\Gamma(m+i+p+\alpha-1)(m+2i+p-2)!}{(i-1)!\,\Gamma(m+i+\alpha)\Gamma(m+2i+p+2\alpha-2)} \times a_{m+2i+p-2,n}^{(0,q)}, \quad p \geqslant 1, \tag{79}$$

$$a_{mn}^{(p,q)} = \frac{2^q(n+\alpha)\Gamma(n+2\alpha)}{(q-1)!\,n!} \sum_{j=1}^{\infty} \frac{(j+q-2)!\,\Gamma(n+j+q+\alpha-1)(n+2j+q-2)!}{(j-1)!\,\Gamma(n+j+\alpha)\Gamma(n+2j+q+2\alpha-2)} \times a_{m,n+2j+q-2}^{(p,0)}, \quad q \geqslant 1, \tag{80}$$

$$a_{mn}^{(p,q)} = \frac{2^{p+q}(m+\alpha)(n+\alpha)\Gamma(m+2\alpha)\Gamma(n+2\alpha)}{(p-1)!(q-1)!\,m!\,n!} \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \frac{(i+p-2)!(j+q-2)!}{(i-1)!(j-1)!} \times \frac{\Gamma(m+i+p+\alpha-1)\Gamma(n+j+q+\alpha-1)(m+2i+p-2)!(n+2j+q-2)!}{\Gamma(m+i+\alpha)\Gamma(n+j+\alpha)\Gamma(m+2i+p+2\alpha-2)\Gamma(n+2j+q+2\alpha-2)} \times a_{m+2i+p-2,n+2j+q-2}, \quad p,q \geqslant 1. \tag{81}$$

In particular, the special cases for the "bivariate" Chebyshev polynomials of the first and second kinds may be obtained directly by taking $\alpha = 0, 1$, respectively, and for the "bivariate" Legendre polynomials by taking $\alpha = \frac{1}{2}$. These are given as corollaries to the previous theorem.

Corollary 7. *For* $\alpha = 0$, *and if*

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty}{}'' a_{mn} T_m(x) T_n(y), \tag{82}$$

$$u^{(p,q)}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty}{}'' a_{mn}^{(p,q)} T_m(x) T_n(y) \tag{83}$$

then the coefficients $a_{mn}^{(p,q)}$ *which are related to* $a_{mn}^{(0,q)}$, $a_{mn}^{(p,0)}$ *and* a_{mn} *are given in* [6, *formulae* (16)–(18), *pp.* 86–87]. *Note here that the double primes in* (82) *and* (83) *indicate that the first term is* $\frac{1}{4}a_{00}$, a_{m0} *and* a_{0n} *are to be taken as* $\frac{1}{2}a_{m0}$ *and* $\frac{1}{2}a_{0n}$ *for* $m, n > 0$, *respectively.*

Corollary 8. *For* $\alpha = 1$, *and if*

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn} U_m(x) U_n(y),$$

$$u^{(p,q)}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn}^{(p,q)} U_m(x) U_n(y)$$

then the coefficients $A_{mn}^{(p,q)}$ *are related to the coefficients* $A_{mn}^{(0,q)}, A_{mn}^{(p,0)}$ *and* A_{mn} *by*

$$A_{mn}^{(p,q)} = \frac{2^p(m+1)}{(p-1)!} \sum_{i=1}^{\infty} \frac{(i+p-2)!(m+i+p-1)!}{(i-1)!(m+i)!} A_{m+2i+p-2,n}^{(0,q)}, \quad p \geqslant 1, \tag{84}$$

$$A_{mn}^{(p,q)} = \frac{2^q(n+1)}{(q-1)!} \sum_{j=1}^{\infty} \frac{(j+q-2)!(n+j+q-1)!}{(j-1)!(n+j)!} A_{m,n+2j+q-2}^{(p,0)}, \quad q \geqslant 1, \tag{85}$$

$$A_{mn}^{(p,q)} = \frac{2^{p+q}(m+1)(n+1)}{(p-1)!(q-1)!} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(i+p-2)!(j+q-2)!(m+i+p-1)!(n+j+q-1)!}{(i-1)!(j-1)!(m+i)!(n+j)!} \times A_{m+2i+p-2,n+2j+q-2}, \quad p, q \geqslant 1 \tag{86}$$

for all $m, n \geqslant 0$.

Corollary 9. *For* $\alpha = \frac{1}{2}$, *and if*

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} P_m(x) P_n(y),$$

$$u^{(p,q)}(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}^{(p,q)} P_m(x) P_n(y)$$

then the coefficients $a_{mn}^{(p,q)}$ *which are related to the coefficients* $a_{mn}^{(0,q)}$, $a_{mn}^{(p,0)}$ *and* a_{mn} *are given in* [7, *formulae* (32)–(34), *pp.* 31–32].

4.2. Extension to triple ultraspherical series expansions

Let $u(x,y,z)$ be a continuous function defined on the cube $C(-1 \leqslant x,y,z \leqslant 1)$, and let it has continuous and bounded partial derivatives of any order with respect to its variables x, y and z. Then it is possible to express

$$u(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} a_{lmn} C_c^{(\alpha)}(x) C_m^{(\alpha)}(y) C_n^{(\alpha)}(z),$$

$$u^{(p,q,r)}(x,y,z) = \frac{\partial^{p+q+r} u(x,y,z)}{\partial x^p \partial y^q \partial z^r} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} a_{lmn}^{(p,q,r)} C_c^{(\alpha)}(x) C_m^{(\alpha)}(y) C_n^{(\alpha)}(z).$$

Further, let

$$a_{lmn}^{(p,q,r)} = \frac{(l+\alpha)(m+\alpha)(n+\alpha)\Gamma(l+2\alpha)\Gamma(m+2\alpha)\Gamma(n+2\alpha)}{l!m!n!} b_{lmn}^{(p,q,r)},$$
$$l,m,n \geqslant 0, \quad p,q,r = 0,1,2,\ldots \tag{87}$$

then we state the following theorem, which is to be considered as an extension of Theorem 5 of Section 4.1.

Theorem 6. *The coefficients* $b_{lmn}^{(p,q,r)}$ *are related to the coefficients with superscripts* $(0,q,r)$, $(p,0,r)$, $(p,q,0)$, $(0,0,r)$, $(0,q,0)$, $(0,0,p)$ *and* b_{lmn} *by*

$$b_{lmn}^{(p,q,r)} = \frac{2^p}{(p-1)!}\sum_{i=1}^{\infty} \frac{(i+p-2)!\Gamma(l+i+p+\alpha-1)}{(i-1)!\Gamma(l+i+\alpha)} (l+2i+p+\alpha-2) b_{l+2i+p-2,m,n}^{(0,q,r)},$$
$$p \geqslant 1, \tag{88}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^q}{(q-1)!}\sum_{j=1}^{\infty} \frac{(j+q-2)!\Gamma(m+j+q+\alpha-1)}{(j-1)!\Gamma(m+j+\alpha)} (m+2j+q+\alpha-2) b_{l,m+2j+q-2,n}^{(p,0,r)}$$
$$q \geqslant 1, \tag{89}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^r}{(r-1)!}\sum_{k=1}^{\infty} \frac{(k+r-2)!\Gamma(n+k+r+\alpha-1)}{(k-1)!\Gamma(n+k+\alpha)} (n+2k+r+\alpha-2) b_{l,m,n+2k+r-2}^{(p,q,0)},$$
$$r \geqslant 1, \tag{90}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^{p+q}}{(p-1)!(q-1)!}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{(i+p-2)!(j+q-2)!\Gamma(l+i+p+\alpha-1)}{(i-1)!(j-1)!\Gamma(l+i+\alpha)} \times \frac{\Gamma(m+j+q+\alpha-1)}{\Gamma(m+j+\alpha)} (l+2i+p+\alpha-2)(m+2j+q+\alpha-2) b_{l+2i+p-2,m+2j+q-2,n}^{(0,0,r)},$$
$$p,q \geqslant 1, \tag{91}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^{p+r}}{(p-1)!(r-1)!} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{(i+p-2)!(k+r-2)!\Gamma(l+i+p+\alpha-1)}{(i-1)!(k-1)!\Gamma(l+i+\alpha)}$$
$$\frac{\Gamma(n+k+r+\alpha-1)}{\Gamma(n+k+\alpha)}(l+2i+p+\alpha-2)(n+2k+r+\alpha-2) b_{l+2i+p-2,m,n+2k+r-2}^{(0,q,0)},$$
$$p,r \geqslant 1, \tag{92}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^{q+r}}{(q-1)!(r-1)!} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{(j+q-2)!(k+r-2)!\Gamma(m+j+q+\alpha-1)}{(j-1)!(k-1)!\Gamma(m+j+\alpha)}$$
$$\times \frac{\Gamma(n+k+r+\alpha-1)}{\Gamma(n+k+\alpha)}(m+2j+q+\alpha-2)(n+2k+r+\alpha-2) b_{l,m+2j+q-2,n+2k+r-2}^{(p,0,0)},$$
$$q,r \geqslant 1, \tag{93}$$

$$b_{lmn}^{(p,q,r)} = \frac{2^{p+q+r}}{(p-1)!(q-1)!(r-1)!} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{(i+p-2)!(j+q-2)!(k+r-2)!}{(i-1)!(j-1)!(k-1)!}$$
$$\times \frac{\Gamma(l+i+p+\alpha-1)\Gamma(m+j+q+\alpha-1)\Gamma(n+k+r+\alpha-1)}{\Gamma(l+i+\alpha)\Gamma(m+j+\alpha)\Gamma(n+k+\alpha)}$$
$$\times (l+2i+p+\alpha-2)(m+2j+q+\alpha-2)$$
$$\times (n+2k+r+\alpha-2) b_{l+2i+p-2,m+2j+q-2,n+2k+r-2}, \quad p,q,r \geqslant 1. \tag{94}$$

The formulae corresponding to expansions in triple Chebyshev polynomials of the first and second kinds and of triple Legendre polynomials may be obtained as special cases by taking $\alpha = 0, 1$ and $\frac{1}{2}$, respectively in formulae (88)–(94).

4.3. *The tau method for Poisson's equation inside a square*

Consider Poisson's equation in the square S $(-1 \leqslant x, y \leqslant 1)$

$$D_x^2 u(x,y) + D_y^2 u(x,y) = f(x,y), \quad (-1 \leqslant x, y \leqslant 1) \tag{95}$$

subject to the nonhomogeneous mixed boundary conditions

$$\left.\begin{aligned} u + \alpha_1 D_x u &= \gamma_1(y),\ x = -1 \\ u + \alpha_2 D_x u &= \gamma_2(y),\ x = 1 \end{aligned}\right\} \quad -1 \leqslant y \leqslant 1, \tag{96}$$

$$\left.\begin{aligned} u + \beta_1 D_y u &= \delta_1(x),\ y = -1 \\ u + \beta_2 D_y u &= \delta_2(x),\ y = 1 \end{aligned}\right\} \quad -1 \leqslant x \leqslant 1 \tag{97}$$

and assume that both $u(x, y)$ and $f(x, y)$ are approximated by a truncated double ultraspherical series

$$u(x, y)=\sum_{n=0}^{N}\sum_{m=0}^{M} a_{mn}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y), \tag{98}$$

$$f(x, y)=\sum_{n=0}^{N}\sum_{m=0}^{M} f_{mn}C_m^{(\alpha)}(x)C_n^{(\alpha)}(y). \tag{99}$$

Assume also that the functions $\gamma_1(y)$, $\gamma_2(y)$, $\delta_1(x)$ and $\delta_2(x)$ have the following truncated ultraspherical expansions:

$$\gamma_1(y)=\sum_{n=0}^{N}\gamma_n^{(1)}C_n^{(\alpha)}(y), \tag{100}$$

$$\gamma_2(y)=\sum_{n=0}^{N}\gamma_n^{(2)}C_n^{(\alpha)}(y), \tag{101}$$

$$\delta_1(x)=\sum_{m=0}^{M}\delta_m^{(1)}C_m^{(\alpha)}(x), \tag{102}$$

$$\delta_2(x)=\sum_{m=0}^{M}\delta_m^{(2)}C_m^{(\alpha)}(x) \tag{103}$$

then the ultraspherical tau equations for Poisson's equation (95) are given by

$$a_{mn}^{(2,0)}+a_{mn}^{(0,2)}=f_{mn}, \quad 0\leqslant m\leqslant M-2,\ 0\leqslant n\leqslant N-2 \tag{104}$$

while the boundary conditions (96) and (97) with (100)–(103) yield

$$\left.\begin{aligned}&\sum_{m=0}^{M}(-1)^m[a_{mn}+\alpha_1 a_{mn}^{(1,0)}]=\gamma_n^{(1)},\\&\sum_{m=0}^{M}[a_{mn}+\alpha_2 a_{mn}^{(1,0)}]=\gamma_n^{(2)},\end{aligned}\right\}\quad n=0,1,2,\ldots,N, \tag{105}$$

$$\left.\begin{aligned}&\sum_{n=0}^{N}(-1)^n[a_{mn}+\beta_1 a_{mn}^{(0,1)}]=\delta_m^{(1)},\\&\sum_{n=0}^{N}[a_{mn}+\beta_2 a_{mn}^{(1,0)}]=\delta_m^{(2)},\end{aligned}\right\}\quad n=0,1,2,\ldots,M. \tag{106}$$

The $2M+2N+4$ boundary conditions given by (105) and (106) are not all linearly independent; there exist four linear relations among them. Thus, Eqs. (104)–(106) give $(M+1)(N+1)$ equations for the $(M+1)(N+1)$ unknowns a_{mn} $(0\leqslant m\leqslant M,\ 0\leqslant n\leqslant N)$.

The coefficients, $a_{mn}^{(1,0)}$, $a_{mn}^{(0,1)}, a_{mn}^{(2,0)}$ and $a_{mn}^{(0,2)}$ of the first and second partial derivatives of the approximation $u(x, y)$ are related to the coefficients a_{mn}, of $u(x, y)$ by invoking (67) and (70) with $p=1$ and $p=2$, and (67) and (71) with $q=1$ and $q=2$, respectively. This allows us to replace $a_{mn}^{(1,0)}$, $a_{mn}^{(0,1)}$, $a_{mn}^{(2,0)}$, and $a_{mn}^{(0,2)}$ in (104), (105) and (106) by an explicit expressions in terms of the a_{mn}. In this way we can set up a linear system for a_{mn} $(0\leqslant m\leqslant M,\ 0\leqslant n\leqslant N)$ which may be solved using standard techniques.

4.4. Use of ultraspherical polynomials to solve differential equations

Consider the linear ordinary differential equation of order n of the form

$$\sum_{i=0}^{N} f_i(x) D_x^i y(x) = g(x), \tag{107}$$

where $f_i(x)$ and $g(x)$ are functions of x only. Suppose the equation to be solved in the interval $[-1,1]$ subject to n linear boundary conditions, and assume we approximate $y(x)$ by a truncated expansion of ultraspherical polynomials

$$y(x) = \sum_{j=0}^{N} a_j C_j^{(\alpha)}(x), \tag{108}$$

where N is the degree of approximation, $a_0, a_1, \ldots, a_N$ are unknown coefficients to be determined. Substituting (108) into (107) yields

$$\sum_{i=0}^{N} \left\{ f_i(x) \sum_{j=0}^{N} a_j D_x^i C_j^{(\alpha)}(x) \right\} = g(x) \tag{109}$$

which may be written in the form

$$\sum_{j=0}^{N} \left\{ a_j \sum_{i=0}^{N} f_i(x) D_x^i C_j^{(\alpha)}(x) \right\} = g(x). \tag{110}$$

The boundary conditions associated with (107) give rise to n equations connecting the coefficients a_j, and the remaining equations may be obtained in two ways:

(i) we may equate the coefficients of the various $c_j^{(\alpha)}(x)$ after expanding the two sides of (109) in ultraspherical series.
(ii) we may collocate at $m = N - n$ selected points in $(-1, 1)$.

The system of equations obtained from the collocation is of the form

$$\sum_{j=0}^{N} \left\{ a_j \sum_{i=0}^{N} f_i(x_k) D_x^i C_j^{(\alpha)}(x_k) \right\} = g(x_k), \quad k = 1, 2, \ldots, m, \tag{111}$$

where x_k are the collocation points, which are usually chosen at the zeros of $C_m^{(\alpha)}(x)$ (see for instance [1]). Since the derivatives $D_x^i C_j^{(\alpha)}(x)$ are now expressible explicitly in terms of $C_j^{(\alpha)}(x)$, then the problem of computing them is solved by using formula (8). Therefore, the resulting linear system obtained from (111) and the n linear boundary conditions can easily be solved using standard direct solvers.

The method just described is easily extended to higher dimensions. Consider, for example, the second order partial differential equation

$$A_1(x,y)u_{xx} + A_2(x,y)u_{x,y} + A_3(x,y)u_{yy} + A_4(x,y)u_x + A_5(x,y)u_y + A_6(x,y)u = f(x,y), \tag{112}$$

where the coefficients $A_1, A_2, \ldots, A_6$ and f are functions of x and y only. Suppose the solution of the equation is required in the square S $(-1 \leqslant x, y \leqslant 1)$, subject to general linear boundary conditions of the form

$$B_1(x, y)u_x + B_2(x, y)u_y + B_3(x, y)u = g(x, y) \tag{113}$$

on the sides of the square S.

Suppose the function $u(x, y)$ can be approximated by the double finite ultraspherical series

$$u(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} a_{mn} C_m^{(\alpha)}(x) C_n^{(\alpha)}(y) \tag{114}$$

for sufficiently large values of the integers M and N. Since $u(x, y)$ satisfies (112) we have approximately

$$\begin{aligned}\sum_{m=0}^{M} \Bigg\{ \sum_{n=0}^{N} a_{mn} [A_1 D_x^2 C_m^{(\alpha)}(x) C_n^{(\alpha)}(y) + A_2 D_x C_m^{(\alpha)}(x) D_y C_n^{(\alpha)}(y) \\ + A_3 C_m^{(\alpha)}(x) D_y^2 C_n^{(\alpha)}(y) + A_4 D_x C_m^{(\alpha)}(x) C_n^{(\alpha)}(y) + A_5 C_m^{(\alpha)}(x) D_y C_n^{(\alpha)}(y) \\ + A_6 C_m^{(\alpha)}(x) C_n^{(\alpha)}(y)] \Bigg\} = f(x, y).\end{aligned} \tag{115}$$

On collocating Eq. (115) at $(M-1)(N-1)$ distinct points (x_i, y_j), $i = 1, 2, \ldots, M-1$, $j = 1, 2, \ldots, N-1$, in S, there results a set of $(M-1)(N-1)$ linear equations for the coefficients a_{mn}. If we now collocate Eq. (113) at $2(M+N)$ points on the sides of the square S, we find the remaining equations for the unique determination of the coefficients a_{mn}.

As in ordinary differential equations, the derivatives of the ultraspherical polynomials occurring in (115) are computed by use of formula (8), and numbers x_i, y_j are chosen at the zeros of the appropriate ultraspherical polynomials.

5. Concluding remarks

This paper deals with formulae relating the coefficients in the differentiated (integrated) expansions of ultraspherical polynomials to those of the original expansion that has been differentiated (integrated) any number of times. It also investigates formulae associated with the q times differentiation (integration) of ultraspherical polynomials and describes how they can be used to solve two-point boundary value problems. Such formulae may be used to solve boundary value problems of any order and also elliptic partial differential equations.

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