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Constructing hierarchical Archimedean copulas with Lévy subordinators

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ABSTRACT

A probabilistic interpretation for hierarchical Archimedean copulas based on Lévy subordinators is given. Independent exponential random variables are divided by group-specific Lévy subordinators which are evaluated at a common random time. The resulting random vector has a hierarchical Archimedean survival copula. This approach suggests an efficient sampling algorithm and allows one to easily construct several new parametric families of hierarchical Archimedean copulas.

1. Introduction

A d-dimensional copula is *Archimedean* if it is given by

$$C_\psi(u_1,\dots,u_d) := \psi(\psi^{-1}(u_1)+\dots+\psi^{-1}(u_d)), \quad u_1,\dots,u_d \in [0,1]. \tag{1}$$

The function $\psi : [0,\infty) \to [0,1]$, called the *generator*, is necessarily continuous, non-increasing, with $\psi(0)=1$, and with $\lim_{x\to\infty}\psi(x)=0$. Eq. (1) defines a copula in every dimension $d \ge 2$ if and only if ψ is additionally completely monotone (c.m.); see [1]. The functional symmetry of C_ψ implies exchangeability of the underlying dependence structure, which is often not justified in reality. Hierarchical (or nested) Archimedean copulas are a popular concept for overcoming this drawback; see e.g. [2, p. 87]. This article focuses on hierarchical Archimedean copulas which are a grouped generalization of (1), given by

$$C_{\psi_0}(C_{\psi_1}(u_{1,1},\dots,u_{1,d_1}),\dots,C_{\psi_J}(u_{J,1},\dots,u_{J,d_J})). \tag{2}$$

Thus, the random vector is partitioned into $J \in \mathbb{N}$ groups of sizes $d_1,\dots,d_J \in \mathbb{N}$, respectively. The copula thus has dimension $d := d_1+\dots+d_J$. The following sufficient condition is given in [2, p. 88] for three- and four-dimensional copulas and may be found in [3] for more general structures: if for each $j \in \{0,\dots,J\}$ the function ψ_j is a c.m. generator, and if additionally

$$(\psi_0^{-1}\circ\psi_j)' \text{ is c.m.}, \quad j \in \{1,\dots,J\}, \tag{3}$$

then (2) defines a copula. The function ψ_0 is referred to as the *outer generator* and the functions ψ_j for $j \in \{1,\dots,J\}$ as *inner generators*. The generators involved are called *compatible* if condition (3) holds. A sampling routine for copulas of the

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form (2) is derived in [3] and some explicit examples of compatible generators $\psi_0, \ldots, \psi_J$ are presented. Further results and examples are provided in [4].

The construction of hierarchical dependence structures from well-known exchangeable ones is a topic of active research. For instance, there exist hierarchical versions of elliptical and Marshall–Olkin copulas; see e.g. [5,6]. Furthermore, the related concept of vines tackles this issue quite generally; see e.g. [7,8].

One application of hierarchical Archimedean copulas is portfolio credit risk. In this context, a prominent financial product is a collateralized debt obligation (CDO), i.e. a credit derivative whose payment streams depend on the credit status of a pool of firms. An important benchmark for CDOs is the iTraxx Europe basket, which consists of (credit default swaps on) $d = 125$ firms. These are partitioned into $J = 6$ groups according to industrial branches. With this segmentation in mind, [9,10] suggest using hierarchical Archimedean copulas to model the dependence structure of the firms' default times. Such approaches rely on Monte Carlo simulations and highlight the need for flexible and tractable parametric families.

The present article reveals that the aforementioned approaches constitute mixture models where the dependence within an industry sector is induced by a group-specific Lévy subordinator and the global dependence between different branches results from a common random time at which all Lévy subordinators are evaluated. More precisely, a random vector with hierarchical Archimedean survival copula is constructed using Lévy subordinators. This new approach is useful, since it provides an alternative view on hierarchical Archimedean copulas, and it is general enough to comprise *all* copulas of the form (2) whose generators are compatible according to (3). This hard-to-check nesting condition is therefore conveniently circumvented.

The remainder of the article is organized as follows. In Section 2 the set of all compatible pairs of generators is determined in a convenient form. In Section 3 a probabilistic model based on Lévy subordinators is constructed which is equivalent to the aforementioned sufficient nesting condition. Moreover, the sampling strategy of [3] is reformulated from this alternative perspective. Section 4 illustrates the findings by means of an example and motivates the aforementioned application to portfolio credit risk modeling. Section 5 concludes.

2. Compatible generators

In contrast to [2–4], the present article does not provide specific examples for pairs (ψ_0, ψ_1) of an outer and an inner generator that satisfy the nesting condition (3). Rather a c.m. outer generator ψ_0 is fixed, and the set

$$M_{\psi_0} := \{\psi_1 \text{ c.m. generator} \mid (\psi_0^{-1} \circ \psi_1)' \text{ c.m.}\}$$

of inner generators which are compatible with ψ_0 is determined. This theoretical result is stated in Theorem 2.1. In order to establish it, the notion of a Lévy subordinator, i.e. a non-decreasing Lévy process, is useful. For further background on Lévy processes we refer the reader to the books [11,12]. For a given Lévy subordinator $\Lambda = \{\Lambda_t\}_{t \geq 0}$, the well-known *Lévy–Khinchin Theorem* (see e.g. [11, Theorem 8.1 combined with Theorem 21.5]) states that there is a non-negative number $\mu \geq 0$ and a measure ν on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \min\{t, 1\}\, \nu(\mathrm{d}t) < \infty \tag{4}$$

such that for each $t \geq 0$ the Laplace transform of Λ_t is given by

$$\mathbb{E}[\mathrm{e}^{-x\Lambda_t}] = \mathrm{e}^{-t\Psi(x)}, \qquad \Psi(x) := \mu x + \int_{(0,\infty)} (1 - \mathrm{e}^{-xt})\, \nu(\mathrm{d}t), \quad x \geq 0. \tag{5}$$

The function $\Psi : [0, \infty) \to [0, \infty)$ is called the *Laplace exponent* of Λ. Conversely, given $\mu \geq 0$ and a measure ν on $(0, \infty)$ satisfying (4), there exists a Lévy subordinator Λ which is determined by the Laplace transforms in (5). The measure ν is called the *Lévy measure* of Λ. Examples of popular Lévy subordinators are given in Table 2.

Returning to copulas, an application of the Lévy–Khinchin Theorem allows us to fully determine the set M_{ψ_0}. More precisely, for a given outer generator ψ_0, all compatible inner generators can be parameterized by ψ_0, a drift constant $\mu \geq 0$, and a Lévy measure ν on $(0, \infty)$.

Theorem 2.1 (*Compatible Generators*). *Let ψ_0 be a c.m. generator. Then*

$$M_{\psi_0} = \Big\{\psi_1 \mid \psi_1(x) = \psi_0\Big(\mu x + \int_{(0,\infty)} (1 - \mathrm{e}^{-xt})\, \nu(\mathrm{d}t)\Big), \textit{ where } \mu \geq 0 \textit{ and } \nu \textit{ is a measure on } (0,\infty) \text{ satisfying (4)},$$
$$\textit{and either } \mu > 0, \textit{ or } \nu\big((0, 1)\big) = \infty, \textit{ or both}\Big\}.$$

Proof. It follows from [13, p. 450] that a function $\Psi : [0, \infty) \to [0, \infty)$ is the Laplace exponent of a Lévy subordinator if and only if $\Psi(0) = 0$, $\lim_{x \downarrow 0} \Psi(x) = 0$, and Ψ has a c.m. derivative on $(0, \infty)$. Thus, for two c.m. generators ψ_0 and ψ_1 it follows that

$$(\psi_0^{-1} \circ \psi_1)' \text{ is c.m.} \Leftrightarrow \psi_0^{-1} \circ \psi_1 = \Psi \text{ is the Laplace exponent of a Lévy subordinator.}$$

Keeping the outer generator ψ_0 fixed, this implies that all possible inner generators ψ_1 are given by $\psi_1 = \psi_0 \circ \Psi$ for the Laplace exponent Ψ of a Lévy subordinator Λ. More clearly, ψ_1 is the Laplace transform of Λ_V, where Λ is some subordinator and V is an independent random variable with Laplace transform ψ_0. Note that such a random variable V exists due to Bernstein's Theorem; see [14]. Since $\lim_{x\to\infty} \psi_1(x) = 0$, one must choose Laplace exponents Ψ satisfying $\lim_{x\to\infty} \Psi(x) = \infty$. This excludes Lévy subordinators Λ with the property that $\mathbb{P}(\Lambda_t = 0) > 0$ for some $t > 0$. These are precisely the compound Poisson subordinators with zero drift $\mu = 0$ and Lévy measure ν satisfying $\nu\big((0, 1)\big) < \infty$. The claim is hence established by applying the Lévy–Khinchin representation. □

3. Probabilistic construction and sampling

Simulation studies, which are common e.g. in financial applications, require fast algorithms for sampling copulas. In particular, large dimensions such as $d > 100$ are of interest for the pricing of portfolio credit derivatives. It is therefore important to provide efficient sampling routines for hierarchical Archimedean copulas. However, due to the non-trivial compatibility condition (3) only a limited repertoire of compatible generators is known; see for instance [3,4] for specific examples, alternative methods of construction, and sampling strategies.

It is shown in the previous section that if the nesting condition (3) holds, the functions $\psi_0^{-1} \circ \psi_j, j \in \{1, \dots, J\}$, are Laplace exponents of Lévy subordinators. Conversely, suitably combining Lévy subordinators in a probabilistic construction leads to well-defined hierarchical Archimedean copulas. From this perspective the sampling methodology of [3] for copulas of the form (2) can be reformulated using Lévy subordinators. More precisely, let $\{E_{j,i}\}_{j=1,\dots,J,\ i=1,\dots,d_j}$ be independent and identically distributed (i.i.d.) exponential random variables with mean 1. Furthermore, let $V > 0$ be an independent random variable, interpreted as random time, with Laplace transform $\psi_0(x) = \mathbb{E}[e^{-xV}]$. Independently of all previously defined random variables, let $\Lambda^{(1)}, \dots, \Lambda^{(J)}$ be J independent Lévy subordinators with corresponding Laplace exponents $\Psi_1, \dots, \Psi_J$. Further, assume that $\lim_{x\to\infty} \Psi_j(x) = \infty$ for all $j = 1, \dots, J$, which is equivalent to the fact that $\Lambda_t^{(j)} > 0$ a.s. for all $t > 0$ and $j = 1, \dots, J$. Define the random vector

$$\left(\frac{E_{1,1}}{\Lambda_V^{(1)}}, \dots, \frac{E_{1,d_1}}{\Lambda_V^{(1)}}, \frac{E_{2,1}}{\Lambda_V^{(2)}}, \dots, \frac{E_{2,d_2}}{\Lambda_V^{(2)}}, \dots\dots, \frac{E_{d,1}}{\Lambda_V^{(J)}}, \dots, \frac{E_{J,d_J}}{\Lambda_V^{(J)}} \right). \tag{6}$$

It is shown in Theorem 3.1 that a hierarchical Archimedean copula with compatible c.m. generators can be constructed as the survival copula of a random vector of the form (6). Recall that the survival copula of a random vector is the copula that couples the univariate survival functions of its components to obtain the multivariate survival function. This survival analog to Sklar's Theorem can be found e.g. in [15, pp. 195].

Theorem 3.1 (*Probabilistic Construction via Lévy Subordinators*). *The survival copula of the random vector defined in (6) has the form (2) with $\psi_j = \psi_0 \circ \Psi_j$ for $j = 1, \dots, J$. Moreover, the univariate survival functions of the components are given by*

$$\mathbb{P}\big(E_{j,i}/\Lambda_V^{(j)} > x\big) = (\psi_0 \circ \Psi_j)(x), \quad x > 0,\ j = 1, \dots, J,\ i = 1, \dots, d_j.$$

Proof. The joint survival function of the random vector in (6) is computed as follows:

$$\mathbb{P}\left(\frac{E_{j,i}}{\Lambda_V^{(j)}} > x_{j,i},\ \text{for all } j, i \right) = \mathbb{E}\left[e^{-\sum_{j=1}^{J} \Lambda_V^{(j)} \sum_{i=1}^{n_j} x_{j,i}} \right] = \mathbb{E}\left[\prod_{j=1}^{J} e^{-V\Psi_j\left(\sum_{i=1}^{n_j} x_{j,i}\right)} \right]$$

$$= \mathbb{E}\left[e^{-V\sum_{j=1}^{J} \Psi_j\left(\sum_{i=1}^{n_j} x_{j,i}\right)} \right] = \psi_0\left(\sum_{j=1}^{J} \psi_0^{-1} \circ (\psi_0 \circ \Psi_j)\left(\sum_{i=1}^{n_j} x_{j,i} \right) \right).$$

The component $E_{j,i}/\Lambda_V^{(j)}$ has the following survival function:

$$\mathbb{P}\big(E_{j,i}/\Lambda_V^{(j)} > x\big) = \mathbb{E}\left[e^{-x\Lambda_V^{(j)}} \right] = \mathbb{E}\left[e^{-V\Psi_j(x)} \right] = (\psi_0 \circ \Psi_j)(x).$$

Hence, the survival copula has the claimed form. □

Reinterpreting hierarchical Archimedean copulas in terms of Lévy subordinators implies that the input of the corresponding sampling strategy does not need to fulfill complicated compatibility conditions. Instead, the copula is specified by an arbitrary positive random variable V and J quite arbitrary Lévy subordinators. Theorem 3.1 ensures that the resulting generators are compatible and Eq. (6) suggests a convenient sampling strategy, which is formulated as a generic algorithm below.

Algorithm 1 (*Sampling Hierarchical Archimedean Copulas*).

(1) Sample the time point V with distribution given by the Laplace transform ψ_0, i.e. the outer generator.

(2) For each group $j = 1, \dots, J$, sample the subordinator $\Lambda^{(j)}$ at time V, i.e. sample the random variable $\Lambda_V^{(j)}$.

Table 1
List of c.m. Archimedean generators: A (Ali–Mikhail–Haq), F (Frank), J (Joe), C (Clayton), G (Gumbel), and IG (Inverse Gaussian). The numbers (12), (14), (19), and (20) correspond to [18, p. 94]. In G, (12), and (14), S is $1/\vartheta$-stable distributed, S and W are independent. In (19) and (20), the Gamma distributions of V are influenced by a random parameter W drawn beforehand.

Family	ϑ	$\psi_\vartheta(x)$	λ_l	λ_u	Distribution
A	$[0,1)$	$\frac{1-\vartheta}{e^x-\vartheta}$	0	0	$\mathbb{P}(V=k)=(1-\vartheta)\vartheta^{k-1}, k\in\mathbb{N}$
F	$(0,\infty)$	$-\frac{1}{\vartheta}\log(e^{-x}(e^{-\vartheta}-1)+1)$	0	0	$\mathbb{P}(V=k)=\frac{(1-e^{-\vartheta})^k}{k\vartheta}, k\in\mathbb{N}$
J	$[1,\infty)$	$1-(1-e^{-x})^{1/\vartheta}$	0	$2-2^{\frac{1}{\vartheta}}$	$\mathbb{P}(V=k)=(-1)^{k+1}\binom{1/\vartheta}{k}, k\in\mathbb{N}$
C	$(0,\infty)$	$(1+x)^{-1/\vartheta}$	$2^{-\frac{1}{\vartheta}}$	0	$V\sim\Gamma(1/\vartheta,1)$
G	$[1,\infty)$	$e^{-x^{1/\vartheta}}$	0	$2-2^{\frac{1}{\vartheta}}$	$V\stackrel{d}{=}S$
IG	$(0,\infty)$	$e^{(1-\sqrt{1+2\vartheta^2x})/\vartheta}$	0	0	$V\sim\mathrm{IG}(\vartheta,1)$
(12)	$[1,\infty)$	$(1+x^{1/\vartheta})^{-1}$	$2^{-\frac{1}{\vartheta}}$	$2-2^{\frac{1}{\vartheta}}$	$V\stackrel{d}{=}SW^\vartheta,\ W\sim\mathrm{Exp}(1)$
(14)	$[1,\infty)$	$(1+x^{1/\vartheta})^{-\vartheta}$	$\frac{1}{2}$	$2-2^{\frac{1}{\vartheta}}$	$V\stackrel{d}{=}SW^\vartheta,\ W\sim\Gamma(\frac{1}{\vartheta},1)$
(19)	$(0,\infty)$	$\vartheta/\log(x+e^\vartheta)$	1	0	$V\sim\Gamma(\frac{W}{\vartheta},e^\vartheta)\vert_{W\sim\mathrm{Exp}(1)}$
(20)	$(0,\infty)$	$(\log(x+e))^{-1/\vartheta}$	1	0	$V\sim\Gamma(W,e)\vert_{W\sim\Gamma(1/\vartheta,1)}$

Table 2
List of popular Lévy subordinators: (i) Compound Poisson processes with drift $\mu>0$, jump intensity $\beta>0$, and jump size distribution determined by its Laplace transform ψ_ϑ. The remaining subordinators are infinitely active (i.a.), i.e. have infinitely many jumps on a bounded interval. These include the (ii) Gamma process, (iii) inverse Gaussian process, and (iv) exponentially tilted stable process.

	$\Psi(x)$	Parameters	Distribution of Λ_t
(i)	$\mu x+\beta\Big(1-\psi_\vartheta(x)\Big)$	$\mu>0, \beta>0, J\sim\psi_\vartheta$	Compound Poisson
(ii)	$\beta\log\Big(1+\frac{x}{\eta}\Big)$	$\beta>0,\ \eta>0$	$\Gamma(\beta t,\eta)$,i.a.
(iii)	$\beta(\sqrt{2x+\eta^2}-\eta)$	$\beta>0,\ \eta>0$	$\mathrm{IG}(\beta t/\eta,(\beta t)^2)$, i.a.
(iv)	$(x+h)^\alpha-h^\alpha$	$\alpha\in(0,1),\ h\geq 0$	$S(\alpha,1,(\cos(\pi\alpha/2)t)^{1/\alpha},0,h;1)$, i.a.

(3) Sample i.i.d. $E_{j,i}\sim\mathrm{Exp}(1), j=1,\dots,J, i=1,\dots d_j$.
(4) Return $(U_{1,1},\dots,U_{J,d_J})$, where $U_{j,i}=\psi_0\circ\Psi_j\big(E_{j,i}/\Lambda_V^{(j)}\big), j=1,\dots,J, i=1,\dots d_j$.

Efficient sampling strategies for several subordinators are well-known; see for instance [12, pp. 171] and the references therein. Note that there exist approximate sampling strategies for general Lévy subordinators with given drift and Lévy measure; see e.g. [16,17]. Examples of popular choices for V and the subordinators are found in Tables 1 and 2.

4. Examples and applications

The probabilistic model based on Lévy subordinators allows one to obtain a better understanding of the distribution given by the copula (2). On the basis of the implemented parametric models for the random variable V and the Lévy subordinators, it is possible to draw conclusions about implied dependence measures. As an example, pairwise upper-tail dependence coefficients are treated. Denote by (X,Y) a random vector with marginal distributions F_X, F_Y and copula C. Assuming the existence of all limits, the *upper-tail dependence coefficient* $\lambda_{u,C}$ is defined as

$$\lambda_{u,C}:=\lim_{u\uparrow 1}\mathbb{P}(X>F_X^-(u)\mid Y>F_Y^-(u))=\lim_{u\uparrow 1}\frac{1-2u+C(u,u)}{1-u},$$

where $F_X^-(y):=\inf\{x\in\mathbb{R}\mid F_X(x)\geq y\}$ and similarly F_Y^-. If $C=C_\psi$ is Archimedean and $\lambda_{u,\psi}:=\lambda_{u,C_\psi}$, then the formula simplifies to $\lambda_{u,\psi}=2-2\lim_{x\downarrow 0}\psi'(2x)/\psi'(x)$; see [2, p. 103].

Applied to the random vector defined in (6) with copula (2), a pair of random variables from two distinct groups has the Archimedean survival copula C_{ψ_0}, and a pair from the same group $j\in\{1,\dots,J\}$ has survival copula $C_{\psi_0\circ\Psi_j}$. That means inter-sector pairs are affected by the mixing variable V, whereas intra-sector pairs are affected by the mixing variable $\Lambda_V^{(j)}$. Corollary 4.2 in [2, p. 90] implies that $C_{\psi_0\circ\Psi_j}\geq C_{\psi_0}$, and, hence, also $\lambda_{u,\psi_0\circ\Psi_j}\geq\lambda_{u,\psi_0}$. This means that intra-sector tail dependence is always greater than or equal to inter-sector tail dependence. Going one step further, it is interesting to investigate how the parametric models for V and the Lévy subordinator translate into properties of the implied upper-tail dependence coefficients. If either $\mathbb{E}[V]$ or $\mathbb{E}[\Lambda_1^{(j)}]$ is finite, it is possible to draw conclusions about the implied upper-tail dependence coefficients. In particular, it follows from [19, Proposition 4.4] that finite expectation of V implies zero upper-tail dependence of C_{ψ_0}. Moreover, it is necessary to have $\mathbb{E}[\Lambda_1]=\infty$ to obtain an intra-sector upper-tail dependence which is strictly larger than the inter-sector one. The relations in Table 3 are verified by taking the respective limits and constructing suitable examples and counterexamples.

As a motivating example for the above investigations, consider the random vector of default times of $d=125$ companies in a credit portfolio, which is subdivided into $J=6$ groups corresponding to industrial branches; the choices for d and J being motivated by the standard iTraxx Europe conventions. As a model for the default times one might use the random vector

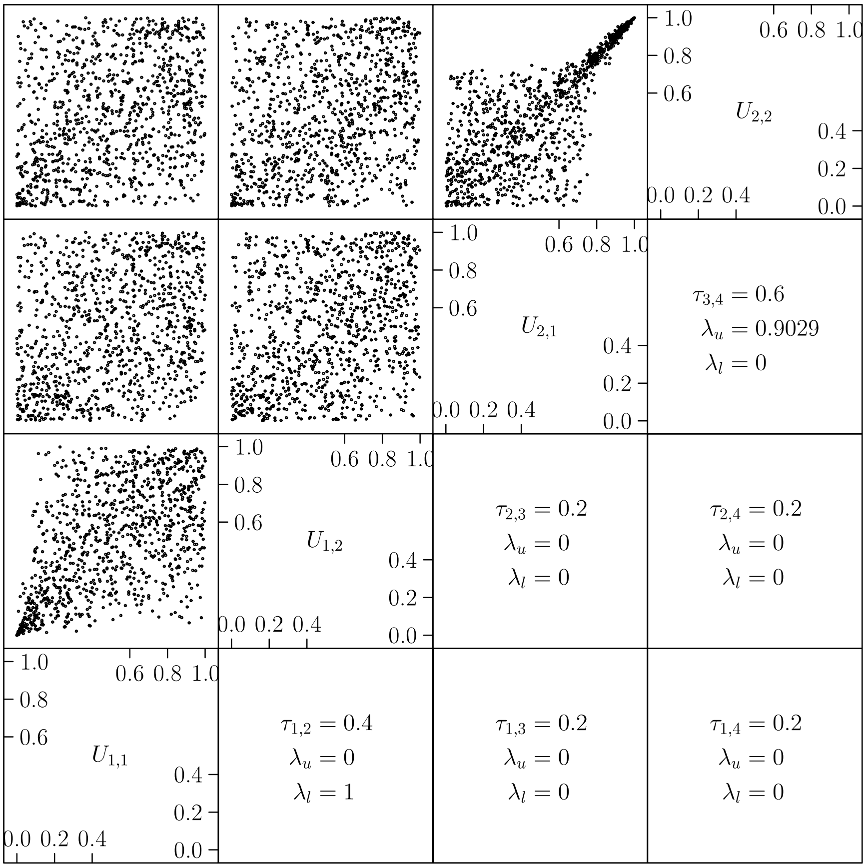

Fig. 1. 1000 vectors of random variates from the nested Archimedean copula of Example 4.1, constructed from an inverse Gaussian random time V and the Laplace exponents of a Gamma subordinator and a compound Poisson subordinator, respectively.

defined by (6), where the univariate margins are transformed to idiosyncratic survival functions. Such an approach is used for instance by [9,10], demonstrating the popularity of this concept for applications. The implied survival copula of the default times is hence the hierarchical Archimedean copula (2). If the bivariate survival copula of a pair of two companies exhibits large upper-tail dependence, this means that an early default of one firm is likely to coincide with an early default of the other firm. Typically firms belonging to the same industry sector are similarly affected by consumer trends, macro-economic effects, or political decisions. One example is the accumulation of defaults in the banking sector during the recent financial crisis. It is thus intuitive to impose a stronger upper-tail dependence within groups than between groups. Moreover, within a certain sector large jumps of the corresponding Lévy subordinator favor early defaults. In this regard, heavy-tailed choices such as stable subordinators or compound Poisson subordinators with heavy-tailed jump size distribution imply stronger dependence within a branch than light-tailed Lévy subordinators. Nevertheless, there might also be global economic shocks affecting all industrial branches at the same time. This effect is incorporated in the model via the common random time point V. The larger the realization of V, the earlier the default times. Therefore, a heavy-tailed distribution of V implies a strong overall dependence between the branches.

Finally, an explicit example is provided to demonstrate the construction developed.

Example 4.1 (*An IG ∘ (Γ, compound Poisson) Archimedean Copula*). A nested Archimedean copula based on $V \sim \mathrm{IG}(\vartheta, 1)$, i.e. an inverse Gaussian distribution, is constructed. The Laplace transform of V is assumed to be given by $\psi_0(x) = \exp((1 - \sqrt{1 + 2\vartheta^2 x})/\vartheta)$. For the first (of two) sectors, the Laplace exponent of a Gamma subordinator with zero drift is chosen, with Laplace exponent $\Psi_1(x) = \beta \log(1 + x)$ for an intensity parameter $\beta > 0$. For the second sector, the Laplace exponent is determined as $\Psi_2(x) = x + 1 - \exp(-x^\alpha)$, $\alpha \in (0, 1)$, corresponding to a compound Poisson subordinator with drift 1, jump intensity 1, and jumps following an α-stable distribution. For the resulting hierarchical Archimedean copula, the inner generators are given by $\psi_0 \circ \Psi_1$ and $\psi_0 \circ \Psi_2$. Note that the upper-tail dependence parameters involved are given by $\lambda_{u,\psi_0} = 0$ for the outer copula, and by $\lambda_{u,\psi_0 \circ \Psi_1} = 0$ and $\lambda_{u,\psi_0 \circ \Psi_2} = 2 - 2^\alpha$ for the inner copulas. Fig. 1 shows 1000 random variates drawn from this nested Archimedean copula, where the parameters involved are chosen in such a way that pairwise Kendall's taus are given by 0.2 between groups, 0.4 within the first group, and 0.6 within the second group. The lower-tail dependence parameters are also illustrated for the sake of completeness.

Table 3
Upper-tail dependence parameters within a group and between groups, depending on the first moments of V and $\Lambda_1^{(j)}$, respectively.

	$\mathbb{E}[\Lambda_1^{(j)}] < \infty$	$\mathbb{E}[\Lambda_1^{(j)}] = \infty$
$\mathbb{E}[V] < \infty$	$0 = \lambda_{u,\psi_0} = \lambda_{u,\psi_0 \circ \Psi_j}$	$0 = \lambda_{u,\psi_0} \leq \lambda_{u,\psi_0 \circ \Psi_j}$
$\mathbb{E}[V] = \infty$	$\lambda_{u,\psi_0} = \lambda_{u,\psi_0 \circ \Psi_j}$	$\lambda_{u,\psi_0} \leq \lambda_{u,\psi_0 \circ \Psi_j}$

5. Conclusion

A new interpretation of hierarchical Archimedean copulas involving Lévy subordinators was presented. This construction guaranteed that the induced generators were compatible. Therefore, an appealing methodology for finding compatible generators, and, hence, new parametric families of hierarchical Archimedean copulas, was derived. Furthermore, a general sampling routine based on the underlying probabilistic construction was stated and illustrated. Finally, an example of a new family and a motivation from credit risk modeling was provided.

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References

[1] C.H. Kimberling, A probabilistic interpretation of complete monotonicity, Aequationes Mathematics 10 (1974) 52–164.
[2] H. Joe, Multivariate models and dependence concepts, Chapman and Hall/CRC (1997).
[3] A.J. McNeil, Sampling nested Archimedean copulas, Journal of Statistical Computation and Simulation 78 (6) (2008) 567–581.
[4] M. Hofert, Sampling Archimedean copulas, Computational Statistics and Data Analysis 52 (2008) 5163–5174.
[5] S. Daul, E. De Giorgi, F. Lindskog, A.J. McNeil, The grouped t-copula with an application to credit risk, RISK (November) (2003) 73–76.
[6] J.-F. Mai, M. Scherer, Reparameterizing Marshall–Olkin copulas with applications to sampling, Journal of Statistical Computation and Simulation (2009) (in press).
[7] T. Bedford, R.M. Cooke, Vines — A new graphical model for dependent random variables, Annals of Statistics 30 (4) (2002) 1031–1068.
[8] K. Aas, C. Czado, A. Frigessi, H. Bakken, Pair-copula constructions of multiple dependence, Insurance: Mathematics and Economics 44 (2) (2006) 182–198.
[9] N. Whelan, Sampling from Archimedean copulas, Quantitative Finance 4 (3) (2004) 339–352.
[10] M. Hofert, M. Scherer, CDO pricing with nested Archimedean copulas, Working Paper, 2008.
[11] K.-I. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge University Press, 1999.
[12] R. Cont, P. Tankov, Financial Modelling with Jump Processes, Chapman and Hall/CRC, 2004.
[13] W. Feller, An Introduction to Probability Theory and Its Applications, Volume II, John Wiley and Sons, Inc., 1966.
[14] S. Bernstein, Sur les fonctions absolument monotones, Acta Mathematica 52 (1) (1929) 1–66.
[15] A.J. McNeil, R. Frey, P. Embrechts, Quantitative Risk Management, Princeton University Press, 2005.
[16] L. Bondesson, On simulation from infinitely divisible distributions, Advances in Applied Probability 14 (1982) 855–869.
[17] P. Damien, P.M. Laud, A.F.M. Smith, Approximate random variate generation from infinitely divisible distributions with applications to Bayesian inference, Journal of the Royal Statistical Society 57 (3) (1995) 547–563.
[18] R.B. Nelsen, An Introduction to Copulas, Springer-Verlag, 1999.
[19] H. Li, Orthant tail dependence of multivariate extreme value distributions, Journal of Multivariate Analysis 100 (1) (2009) 243–256.