Functional Callan–Symanzik equation for QED

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Abstract

An exact evolution equation, the functional generalization of the Callan–Symanzik method, is given for the effective action of QED where the electron mass is used to turn the quantum fluctuations on gradually. The usual renormalization group equations are recovered in the leading order but no Landau pole appears.

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1. Introduction

The idea of dimensional transmutation [1] is that a dimensionless parameter is traded for a dimensionful one. This replacement can be used to generate a renormalization flow where the field amplitude which is a quantity related to the size of the quantum fluctuations is evolved. These flows (“fluctuation flows”) prove to be equivalent with the usual momentum flows at one-loop order. This scheme is developed in this Letter for QED by constructing the evolution of the one-particle irreducible (1PI) generator functional where a mass parameter controls the quantum fluctuations. We derive an exact equation describing this evolution and will recover the usual one-loop momentum flows. We conclude with some observations on the connection between the two renormalization schemes.

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external field [7] where the evolution of the generator functional of the one-particle irreducible (1PI) graphs with the amplitude of the external field led to the dependence of the 1PI graphs on the external gauge.

2. Evolution equation

We start with the following bare Lagrangian in dimension \(d = 4 - \varepsilon\)
\[
\mathcal{L} = \frac{1}{2 e^2 \mu^2} A_{\mu} \Box (T^{\mu\nu} + \alpha L^{\mu\nu}) A_{\nu} + \overline{\psi} (i \gamma - A - z m_0) \Psi,  
\]
(1)
where \(T^{\mu\nu}\) and \(L^{\mu\nu}\) are, respectively, the transverse and longitudinal projectors in the inverse photon propagator, and the gauge parameter \(\alpha\) characterizes the gauge fixing. The parameter \(z\) is introduced to control the amplitude of the fluctuations. For \(z \gg 1\) the theory is dominated by a free mass term contribution and is perturbative. As \(z\) decreases the interaction with the gauge field becomes more important and quantum corrections increase in amplitude. Our aim is to study the evolution in \(z\) of the generator functional \(G_z\) of the 1PI diagrams, the effective action. The functional \(W_z\) of the connected diagrams is given by
\[
ex W_z[\eta, \psi, j^\mu] = \int D[\overline{\psi}, \psi, A_{\mu}] \\
\times \exp\left\{ i \int_x \mathcal{L} + i \int_x \left( j^{\mu} A_{\mu} + \eta \psi + \overline{\psi} \eta \right) \right\}, \tag{2}
\]
and has the following functional derivatives
\[
\frac{\delta}{\delta \eta} W = i \langle \psi \rangle = i \psi,  \\
\frac{\delta}{\delta \psi} W = i \langle \overline{\psi} \rangle = i \overline{\psi},  \\
\frac{\delta}{\delta j^{\mu}} W = i A^{\mu} = i A^{\mu}.  
\tag{3}
\]
The effective action \(G_z[\overline{\psi}, \psi, A_{\mu}]\) is defined as the Legendre transform of \(W_z\),
\[
W_z = i G_z + i \int_x \left( j^{\mu} A_{\mu} + \eta \psi + \overline{\psi} \eta \right), \tag{4}
\]
and has the following functional derivatives
\[
\frac{\delta}{\delta \psi} G = -\eta,  \\
\frac{\delta}{\delta \overline{\psi}} G = -\eta,  \\
\frac{\delta}{\delta A_{\mu}} G = -j^{\mu}.  
\tag{5}
\]
The functional manipulations which lead to the evolution of \(G_z\) are the same as those discussed in [7]: we first remark that the derivatives (3) and (5) of the functionals \(W_z\) and \(G_z\) with respect to their variables imply that
\[
\frac{\partial_z}{\partial \psi} W = -m_0 \int_x \overline{\psi} \psi - m_0 \int_x \frac{\delta}{\delta \eta} W \frac{\delta}{\delta \eta}.  
\tag{6}
\]
Then the relation
\[
\frac{\delta}{\delta \psi} G \frac{\delta}{\delta \overline{\psi}} G = -i \left( \frac{\delta}{\delta \eta} W \frac{\delta}{\delta \eta} \right)^{-1}  
\tag{7}
\]
leads us to the final equation for the generator functional of the 1PI graphs,
\[
\frac{\partial_z}{\partial \psi} G_z + m_0 \int_x \overline{\psi} \psi = i m_0 \text{Tr} \left\{ \left( \frac{\delta}{\delta \eta} W \frac{\delta}{\delta \eta} \right)^{-1} \right\},  
\tag{8}
\]
where ‘Tr’ denotes the trace over spacetime and Dirac indices. The inverse matrix \(\delta^2 G^{-1}\) has to be taken with respect to the field variables, spacetime and Dirac indices. We stress that Eq. (8) is exact and does not make reference to any truncation of any sort. Note the similarity between Eq. (8) and other RG equations [5]. This similarity reflects the qualitative agreement between the usual momentum flows and our method, as we will discuss at the end of this Letter.

3. Gradient expansion

In order to find an approximate solution of the evolution equation, we make an expansion in the amplitude and the gradient of the fields and use the ansatz
\[
G_z[\overline{\psi}, \psi, A_{\mu}] = \int_x \left[ A_{\mu}(x) \Box^{-1} \mu^{\nu} A_{\nu}(x) + \overline{\psi}(x) G^{-1} \psi(x) - Z(z) \overline{\psi}(x) A(x) \psi(x) \right],  
\tag{9}
\]
where the photon and fermion propagators are
\[ \mathcal{D}^{-1\mu\nu}(p) = \frac{\beta_T(z)}{\epsilon^2 p^2} T^{\mu\nu}(p) + a \frac{\beta_L(z)}{\epsilon^2 p^2} L^{\mu\nu}(p), \]
\[ G^{-1}(p) = Z(z)p - m(z) - zm_0, \]
with
\[ T^{\mu\nu}(p) = g^{\mu\nu} - p^\mu p^\nu / p^2 \]
and
\[ L^{\mu\nu}(p) = p^\mu p^\nu / p^2. \]

The evolution of \( \beta_T(z), \beta_L(z), Z(z) \) and \( m(z) \) are obtained by expanding both sides of Eq. (8) in powers of the gauge field and by identifying the operators on the left- and right-hand sides. For this end we need the inverse of the matrix \( \Gamma^{(2)} \)
\[ \Gamma^{(2)} = \left( \begin{array}{ccc} \frac{\delta^2 \Gamma}{\delta \psi \delta \psi} & \frac{\delta^2 \Gamma}{\delta \psi \delta \bar{\psi}} & \frac{\delta^2 \Gamma}{\delta \bar{\psi} \delta \bar{\psi}} \\ \frac{\delta^2 \Gamma}{\delta \bar{\psi} \delta \psi} & \frac{\delta^2 \Gamma}{\delta \bar{\psi} \delta \bar{\psi}} & \frac{\delta^2 \Gamma}{\delta \bar{\psi} \delta \bar{\psi}} \\ \frac{\delta^2 \Gamma}{\delta \bar{\psi} \delta \bar{\psi}} & \frac{\delta^2 \Gamma}{\delta \bar{\psi} \delta \bar{\psi}} & \frac{\delta^2 \Gamma}{\delta \bar{\psi} \delta \bar{\psi}} \end{array} \right), \]
which will be obtained by expanding in the diagonal part \( \Gamma^{(2)}_\Delta \) in momentum space,
\[ \left( \Gamma^{(2)}_\Delta \right)^{-1} = \left( \Gamma^{(2)} \right)^{-1} \left( \Gamma^{(2)}_\Delta \right)^{-1} - \left( \Gamma^{(2)}_\Delta \right)^{-1} \left( \Gamma^{(2)} \right)^{-1} \left( \Gamma^{(2)}_\Delta \right)^{-1} - \left( \Gamma^{(2)}_\Delta \right)^{-1} \left( \Gamma^{(2)} \right)^{-1} \left( \Gamma^{(2)}_\Delta \right)^{-1} - \cdots. \]

where \( \Gamma^{(2)}_\Delta = \Gamma^{(2)} - \Gamma^{(2)}_\Delta \). We obtain in this manner the result
\[ \left( \frac{\delta \Gamma}{\delta \bar{\psi} \delta \psi} \right)^{-1} = \Gamma + G(-c + cGc - cGcGc + a^\mu D_{\mu\nu} \bar{b}^\nu - a^\mu D_{\mu\nu} \bar{b}^\nu Gc - cGd^\mu D_{\mu\nu} \bar{b}^\nu G + \cdots), \]
where
\[ a^\mu(p, q) = -Z(z)\gamma^\mu \psi(-p - q), \]
\[ b^\nu(p, q) = \bar{\psi}(-p - q)Z(z)\gamma^\nu, \]
\[ c(p, q) = -Z(z)\bar{\psi}(-p - q), \]
and the tilde stands for the transposed in momentum and spinor spaces. The evolution equations read finally as
\[ a_\xi \mathcal{D}^{-1\mu\nu}(p) = -2im_0Z^2(z) \int_q \{ G^2(q) \gamma^\mu G(p + q) \gamma^\nu \} + \mathcal{O}(h^2), \]
\[ a_\xi G^{-1}(p) + m_0 = -i \int_q D_{\mu\nu}(q - p)\gamma^\mu G(q) \gamma^\nu + \mathcal{O}(h^2), \]
\[ a_\xi Z \gamma^\mu = -i \int_q D_{\mu\nu}(q)\gamma^\nu G(q) \gamma^\mu G(q) \gamma^\nu + \mathcal{O}(h^2). \]

The following remarks are important:

- The photon does not acquire a mass in the evolution in \( z \) and the IR divergences are the usual ones.
- The longitudinal part of the photon propagator does not evolve with \( z \). Accordingly, the fluctuations do not generate longitudinal contributions to photons.

Furthermore a technical point, it is the evolution equation of the fermion mass only which needs regulator, the other equations are divergence-free with the given effective action ansatz.

To compare this result with the traditional method based on the loop-expansion, note that the change \( zm_0 \to (z + \delta z)m_0 \) of the bare electron mass changes the fermion propagator as \( G \to G + G\delta \bar{z}m_0G \) in the internal lines of the Feynman graphs. This observation, the starting point of the Callan–Symanzik method is sufficient to realize that the evolution equations (15) can obviously be written as
whose integrals give immediately
\[
D^{-1 \mu \nu}(p) = D_{\text{tree}}^{-1 \mu \nu}(p) + i \int_q \text{tr} \{ \mathcal{G}(q) \gamma^\mu \mathcal{G}(p + q) \gamma^\nu \} + \mathcal{O}(\hbar^2),
\]
\[
\mathcal{G}^{-1}(p) = \mathcal{G}_{\text{tree}}^{-1}(p) - i \int_q D_{\mu \nu}(q - p) \gamma^\mu \mathcal{G}(q) \gamma^\nu + \mathcal{O}(\hbar^2).
\]
\[
Z_{\gamma \mu} = Z_{\text{tree}} \gamma^\mu - \int_q D_{\gamma \nu}(q) \gamma^\nu \mathcal{G}(q) \gamma^\mu \mathcal{G}(q) \gamma^\nu + \mathcal{O}(\hbar^2),
\]
the usual one-loop corrections to the 1PI functions appearing in the ansatz (9).

4. Beta functions

The computation of the integrals is straightforward and leads to the following result,
\[
\beta_L(z) = 0,
\]
\[
\beta_T(z) = \frac{e^2 \mu^\varepsilon}{2 \pi^2} \frac{m_0}{m_0 + m(z)},
\]
\[
Z'(z) = \frac{e^2 \mu^\varepsilon}{8 \pi^2 \alpha} \frac{Z(z)m_0}{z_0 + m(z)},
\]
\[
m'(z) = -m_0 + \frac{e^2 \mu^\varepsilon}{8 \pi^2 \alpha} \left( \frac{3}{\beta_T(z)} + \frac{1}{\alpha} \right) + \text{finite},
\]
where the prime denotes a derivative with respect to \(z\).

In what follows, we will keep the bare theory and \(\varepsilon \neq 0\). We define \(m(z) = m_0 \phi(z)\) and write \(\beta(z) = \beta_T(z)\), such that Eq. (18) read
\[
\phi'(z) = \frac{e^2 \mu^\varepsilon}{8 \pi^2 \varepsilon} \left( \frac{3}{\beta(z)} + \frac{1}{\alpha} \right),
\]
\[
\beta'(z) = \frac{e^2 \mu^\varepsilon}{6 \pi^2} \frac{1}{z + \phi(z)},
\]
\[
Z'(z) = \frac{e^2 \mu^\varepsilon}{8 \pi^2 \alpha} \frac{Z(z)}{z + \phi(z)}.
\]

In order to compare these scaling laws with the one-loop result, we look at the situation of weak fluctuations where the fermion mass term is dominant in the bare action, i.e., where \(z\) is close to some initial value \(z_0 \gg 1\) and the parameter \(m(z)\) is close to zero. In this regime we have \(z + \phi(z) \equiv z\) and \(Z(z) \simeq \beta(z) \simeq 1\). This approximation is equivalent to keeping in Eqs. (19) the terms of the order \(e^2\) only:
\[
\phi'(z) \simeq -\frac{e^2 \mu^\varepsilon}{2 \pi^2 \varepsilon},
\]
\[
z \beta'(z) \simeq \frac{e^2 \mu^\varepsilon}{6 \pi^2 z},
\]
\[
z Z'(z) \simeq \frac{e^2 \mu^\varepsilon}{8 \pi^2 z}.
\]

5. Landau pole

It is interesting to note that the integration of Eq. (20),
\[
\beta(z) \simeq 1 + \frac{e^2}{6 \pi^2 \varepsilon} \ln(z/z_0)
\]
reproduces the well-known one-loop value of the Landau-pole,
\[
\frac{z}{z_0} = \frac{\mu_0}{\mu} = \exp\left( -\frac{6 \pi^2}{e^2} \right).
\]
But the essential difference is that while the traditional Landau-pole is obstructing the further increase of the UV cut-off, the pole in the present scheme prevents us to further decrease the bare mass and indicates some problem with large amplitude quantum fluctuations. Since the initial condition of our evolution equation is already an effective potential corresponding to the regulated theory with a fixed cut-off, the UV Landau pole cannot be seen in our scheme. In both cases the pole reflects the problem that the loop corrections dominate the tree-level part of the effective action when ‘too many’ modes are treated in the one-loop approximations.

But this similarity is an artifact of the autonomous solution of the truncated equation (20). Instead one should allow the other parameters, at least the fermion mass, to run as well. For this end we write the evolution equation of $\beta$ as

$$\frac{1}{\beta'(z)} = \frac{6\pi^2}{\varepsilon^2 \mu^2} (z + \phi(z)),$$

which gives

$$\frac{\beta''(z)}{\beta'(z)} = \left(3 \frac{6\pi^2}{4\alpha \varepsilon} - \frac{6\pi^2}{\varepsilon^2 \mu^2}\right) \beta'(z) + \frac{9}{4\varepsilon} \beta'(z),$$

after using the first equation of (19). It is easy to integrate this equation and find by means of (19) again:

$$\beta(z) = \left(\frac{z_0 + \phi(z_0)}{z + \phi(z)}\right)^{4\varepsilon/9} \times \exp\left\{\frac{1 - \beta(z)}{3} \left(\frac{1}{\alpha - \frac{8\pi^2 \varepsilon}{\mu^2}}\right)\right\}.$$  

It should be noted that we cannot naively take the limit $\varepsilon \to 0$ in the expression (25) since $\phi(z)$ is then diverging. This equation shows that $\beta(z)$ is non-vanishing for any value of $z$ and therefore this pole is actually avoided by the evolution of the fermion mass and there is no problem to reconstruct the physical theory at $z = 0$. It is remarkable that the mechanism by which this happens, a running fermion mass, is actually the same how the Landau pole is claimed to be avoided in lattice QED by spontaneous symmetry breakdown of the chiral symmetry [9].

6. Conclusion

To conclude, we stress the analogies between the momentum dependence traced by the usual renormalization procedures and the dependence on the amplitude of quantum fluctuations. We showed that the leading order scaling law corresponding to small amplitudes (i) agrees with the traditional scaling law of the U.V. regime and (ii) is independent of the choice of the gauge as shown by Eqs. (17). The present work is the first step towards an extension of the renormalization scheme based on the field amplitude for non-Abelian gauge theories where a suitable gradient expansion should provide us with a deeper insight into the dynamics of the I.R. regime.

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