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Göllnitz–Gordon identities and parity questions

Sun Kim [a], Ae Ja Yee [b]

[a] Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

[b] Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

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ABSTRACT

Parity has played a role in partition identities from the beginning. In his recent paper, George Andrews investigated a variety of parity questions in partition identities. At the end of his paper, he then listed 15 open problems. The purpose of this paper is to provide solutions to the first three problems from his list, which are related to the Göllnitz–Gordon identities and their generalizations.

1. Introduction

Parity has played a role in additive number theory, in particular partition identities, from the beginning. In his recent paper [4], Andrews made a thorough study of parity questions arising from partition identities.

Most likely, the first theorem in the history of partitions is Euler's famous discovery that the number of partitions of a positive integer n into distinct parts equals the number of partitions of n into odd parts. Equivalently in terms of generating functions, for $|q| < 1$, [3, p. 5, Eq. (1.2.5)]

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}.$$

Euler's partition identity involves the parity of integers.

Gordon [7,8] and Göllnitz [5,6] independently considered parity as follows:

Theorem 1.1 (*First Göllnitz–Gordon Identity*)**.** *The number of partitions of n into distinct non-consecutive parts with no even parts differing by exactly 2 equals the number of partitions of n into parts $\equiv 1, 4$, or $7 \pmod 8$.*

The famous Rogers–Ramanujan identities do not immediately involve parity. However, several results related to the Rogers–Ramanujan identities concern parity. In particular, many q-series identities from Ramanujan's Lost Notebook raise parity questions.

E-mail addresses: sunkim2@illinois.edu (S. Kim), yee@math.psu.edu (A.J. Yee).

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These examples initiated the thorough examination of parity in partition identities by Andrews [4]. At the end of his paper [4], he listed 15 open problems, most of which ask for combinatorial and bijective proofs.

The purpose of this paper is to provide solutions to the first three problems of Andrews, which involve the celebrated Rogers–Ramanujan–Gordon Theorem [1,8].

Theorem 1.2 (*Rogers–Ramanujan–Gordon Identities*)**.** *For* $1 \le a \le k$*, let* $B_{k,a}(n)$ *be the number of partitions of n of the form*

$$b_1 + b_2 + \cdots + b_j,$$

where $b_i \ge b_{i+1}$, $b_i - b_{i+k-1} \ge 2$*, and at most* $a - 1$ *of the* b_i *are equal to* 1*. Let* $A_{k,a}$ *be the number of partitions of n into parts* $\not\equiv 0, \pm a \pmod{2k+1}$*. Then for all* $n \ge 0$*,*

$$A_{k,a}(n) = B_{k,a}(n).$$

We now add parity restrictions.

Theorem 1.3 (*Andrews*)**.** *Suppose* $k \ge a \ge 1$ *are integers with* $k \equiv a \pmod 2$*. Let* $W_{k,a}(n)$ *denote the number of those partitions enumerated by* $B_{k,a}(n)$ *with the added restriction that even parts appear an even number of times. If k and a are both even, let* $G_{k,a}(n)$ *denote the number of partitions of n in which no odd part is repeated and no even part* $\equiv 0, \pm a \pmod{2k+2}$*. If k and a are both odd, let* $G_{k,a}(n)$ *denote the number of partitions of n into parts that are neither* $\equiv 2 \pmod 4$ *nor* $\equiv 0, \pm a \pmod{2k+2}$*. Then for all* $n \ge 0$*,*

$$W_{k,a}(n) = G_{k,a}(n).$$

It follows from comparison of Theorem 1.3 with the Göllnitz–Gordon identity in Theorem 1.1 that $W_{3,3}(n)$ is equal to the number of partitions of n into parts that differ by at least 2 and by more than 2 if the parts are even. Finding a bijective proof of this partition identity is the first problem in the list of Andrews [4]. The second problem is to show bijectively that $W_{3,1}(n)$ is equal to the number of partitions of n into parts (each > 1) that differ by at least 2 and by more than 2 if the parts are even.

A generalization of the Göllnitz–Gordon identities, the first of which is stated in Theorem 1.1, has been accomplished by Andrews [2] in the same manner that the Rogers–Ramanujan–Gordon identities stated in Theorem 1.2 generalize the celebrated Rogers–Ramanujan identities.

Theorem 1.4 (*Andrews*)**.** *Let a and k be integers with* $0 < a \le k$*. Let* $C_{k,a}(n)$ *be the number of partitions of n into parts which are neither* $\equiv 2 \pmod 4$ *nor* $\equiv 0, \pm(2a-1) \pmod{4k}$*. Let* $D_{k,a}(n)$ *denote the number of partitions of n of the form* $n = \sum_{i\ge 1} f_i i$ *with* $f_1 + f_2 \le a - 1$ *and for all* $i \ge 1$*,*

$$f_{2i-1} \le 1 \quad \text{and} \quad f_{2i} + f_{2i+1} + f_{2i+2} \le k - 1,$$

where f_i *denotes the number of appearances of i in the partition. Then* $C_{k,a}(n) = D_{k,a}(n)$*.*

By comparing Theorems 1.3 and 1.4, we see that

$$W_{2k-1,2a-1}(n) = D_{k,a}(n). \tag{1.1}$$

In the third problem of Andrews, it is asked to prove (1.1) bijectively. Here, we emphasize that the first two problems of Andrews' list are special cases of (1.1), when $k = 2$ and $a = 1, 2$. Thus it would be sufficient to solve only the third problem. However, we can establish more direct bijections for those two simpler cases. So, in Section 2, we first prove combinatorially that

1. $W_{3,3}(n)$ is equal to the number of partitions of n into parts that differ by at least 2 and by more than 2 if the parts are even, namely $W_{3,3}(n) = D_{2,2}(n)$, and
2. $W_{3,1}(n)$ is equal to the number of partitions of n into parts (each > 1) that differ by at least 2 and by more than 2 if the parts are even, namely $W_{3,1}(n) = D_{2,1}(n)$.
   In Section 3, by applying a similar idea inductively, we prove the generalization combinatorially.
3. $W_{2k-1,2a-1}(n) = D_{k,a}(n)$.

2. Problems 1 and 2

We first consider the first problem of Andrews.

Theorem 2.1. *For any positive integer n,*

$$W_{3,3}(n) = D_{2,2}(n).$$

Proof. Let $\pi = (\pi_1, \ldots, \pi_m)$ with $\pi_i \le \pi_{i+1}$, be a partition counted by $W_{3,3}(n)$. By the definition of $W_{3,3}(n)$, we see that each part can be repeated at most twice and all the even parts appear exactly twice. We represent the partition π by an array with two rows (counted from bottom to top), where the first and second rows consist of the first and second copies of the parts, respectively and each column has the same parts. For instance, if $\pi = (2, 2, 4, 4, 7, 9, 14, 14, 23, 23, 33)$ is counted by $W_{3,3}(135)$, then we write π as follows.

$$\begin{array}{ccccccc} 2 & 4 & & & 14 & 23 & \\ 2 & 4 & 7 & 9 & 14 & 23 & 33 \end{array}$$

We note that since $\pi_{i+2} - \pi_i \ge 2$ and even parts appear twice, the parts appearing only in the first row are odd and the parts from the first row differ by at least 2. Let $(\tau_1, \ldots, \tau_l)$ be the parts appearing in the first row. For each i with $1 \le i \le l$, subtract $2i-1$ from τ_i and add the parts in the same column. In the above example, we have $(\tau_1, \ldots, \tau_l) = (2, 4, 7, 9, 14, 23, 33)$, and we obtain

$$\begin{array}{ccccccc} 2 & 4 & & & 14 & 23 & \\ 1 & 1 & 2 & 2 & 5 & 12 & 20 \\ \hline 3 & 5 & 2 & 2 & 19 & 35 & 20 \end{array}$$

We note that the sums of two parts from the same column are odd and the parts appearing only in the first row are even. Besides, since the parts from the second row differ by at least 2, all the odd parts in the resulting partition are distinct. Finally, we rearrange the parts in weakly increasing order and add $2i - 1$ to the ith part for each $1 \le i \le l$. Then, the parts of the resulting partition differ by at least two and even parts differ by more than 2. Hence, the resulting partition is counted by $D_{2,2}(n)$. In the example, we obtain

$$\begin{array}{ccccccc} 2 & 2 & 3 & 5 & 19 & 20 & 35 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 \\ \hline 3 & 5 & 8 & 12 & 28 & 31 & 48 \end{array}$$

and we see that $(3, 5, 8, 12, 28, 31, 48)$ is counted by $D_{2,2}(135)$.

Now, we show that the process is reversible. Let $\sigma = (\sigma_1, \ldots, \sigma_l)$ with $\sigma_i \le \sigma_{i+1}$, be a partition counted by $D_{2,2}(n)$. We first subtract $2i - 1$ from σ_i to obtain σ'. Since the even parts of σ differ by at least 4, σ' has distinct odd parts. For example, if $\sigma = (3, 5, 8, 12, 28, 31, 48)$, then $\sigma' = (2, 2, 3, 5, 19, 20, 35)$. Now, we rearrange the parts of σ' to obtain $w = (w_1, \ldots, w_l)$ as follows. In order to select w_i from the parts of σ', we consider the remaining parts of σ' after removing $w_1, \ldots, w_{i-1}$ from σ', and choose the smallest odd and even parts among them, say σ'_o and σ'_e, respectively. If $(\sigma'_o - (2i-1))/2 \le \sigma'_e$, then let $w_i = \sigma'_o$, and otherwise, let $w_i = \sigma'_e$. We continue this process until we determine all of $w_1, \ldots, w_l$ (if we use all of the odd parts or even parts of σ', then just arrange the remaining parts in weakly increasing order). In the same example, we have $\sigma'_o = 3$ and $\sigma'_e = 2$. Since $(3 - 1)/2 \le 2$, we have $w_1 = 3$. For w_2, we have $\sigma'_o = 5$ and $\sigma'_e = 2$. Since $(5 - 3)/2 \le 2$, we have $w_2 = 5$. Similarly, since $\sigma'_o = 19$, $\sigma'_e = 2$ and $(19 - 5)/2 > 2$, we have $w_3 = 2$. By continuing this, we have $w = (3, 5, 2, 2, 19, 35, 20)$.

Now, if w_i is odd, then we split it into two parts $(w_i + (2i - 1))/2$ and $(w_i - (2i - 1))/2$, whose difference is $2i - 1$. We write w as an array with two rows (counted from bottom to top), where the first and second rows of ith column are $(w_i - (2i - 1))/2$ and $(w_i + (2i - 1))/2$ if w_i is odd, and if w_i is even, then place it in the first row of the ith column. Thus, in the example we have

$$\begin{array}{ccccccc} 2 & 4 & & & 14 & 23 & \\ 1 & 1 & 2 & 2 & 5 & 12 & 20 \end{array}$$

Finally, we add $2i - 1$ to the ith part in the first row. Then, the columns with two parts will have the same parts. Since the parts in the first row differ by at least two and the parts appearing only in the first row are odd, the resulting partition is counted by $W_{3,3}(n)$. From the example, we obtain the following array.

$$\begin{array}{ccccccc} 2 & 4 & & & 14 & 23 & \\ 2 & 4 & 7 & 9 & 14 & 23 & 33 \end{array}$$

Note that the resulting partition is $(2, 2, 4, 4, 7, 9, 14, 14, 23, 23, 33)$, which is counted by $W_{3,3}(135)$. □

Next, we consider the second problem.

Theorem 2.2. *For any positive integer n,*

$$W_{3,1}(n) = D_{2,1}(n).$$

Proof. By the definition, 1 is not allowed in any partitions counted by $W_{3,1}(n)$, and both 1 and 2 are not allowed in any partitions counted by $D_{2,1}(n)$. Thus, in the proof of Theorem 2.1, we add the constraints that the parts of π are larger than 1; the remainder of the proof is the same. Let τ be the parts appearing in the first row of the two line array representation of π, and let τ' be the sequence of integers obtained after subtracting $2i - 1$ from each τ_i and adding the parts in columns. Since each τ_i is larger than 1 and the τ_i differ by at least 2, $\tau_i' \geq 2$. Thus rearranging the parts of τ' and adding back $2i - 1$ to the ith part of the resulting sequence gives a sequence of integers larger than 2. This completes the proof. □

3. Problem 3

Theorem 3.1. *Let $k \geq 2$ be an integer. Then, for any positive integer n,*

$$W_{2k-1,2k-1}(n) = D_{k,k}(n).$$

Proof. By Theorem 2.1, it suffices to prove the case when $k > 2$. Let $\pi = (\pi_1, \ldots, \pi_m)$ with $\pi_i \leq \pi_{i+1}$, be a partition counted by $W_{2k-1,2k-1}(n)$. By the definition of $W_{2k-1,2k-1}(n)$, we see that each part can be repeated at most $2k - 2$ times and each even part appears an even number of times. We represent the partition π by an array with two rows (counted from left to right and bottom to top), where the parts of each column are the same. For example, let

$$\pi = (2, 2, 3, 3, 4, 4, 5, 5, 7, 7, 7, 10, 10, 12, 12, 13, 13, 14, 14, 15, 17, 20, 20, 25),$$

which is counted by $W_{5,5}(244)$. Then we write π as follows.

$$\begin{array}{cccccccccccccc} 2 & 3 & 4 & 5 & 7 & & 10 & 12 & 13 & 14 & & & 20 & \\ 2 & 3 & 4 & 5 & 7 & 7 & 10 & 12 & 13 & 14 & 15 & 17 & 20 & 25 \end{array}$$

We now add the parts in the same column. In the example, we obtain

$$\begin{array}{cccccccccccccc} 4 & 6 & 8 & 10 & 14 & 7 & 20 & 24 & 26 & 28 & 15 & 17 & 40 & 25 \end{array}$$

We consider the even parts $(\tau_1, \ldots, \tau_l)$. By the definition of $W_{2k-1,2k-1}(n)$, we see that $\tau_{i+k-1} - \tau_i > 2$. Let τ_c be the largest τ_i such that $\tau_{i+k-2} - \tau_i \leq 2$. Let υ be the smallest odd part to the right of τ_c. If $\tau_c + 1 < \upsilon$, then we stop. If $\tau_c + 1 \geq \upsilon$, then we subtract 2 from υ and put it right before τ_c, and add 2 to τ_c and rearrange the τ_i for $i \geq c$ in weakly increasing order. So, we have the parts $(\nu - 2, \tau_{c+1}, \ldots, \tau_{c+k-2}, \tau_c + 2)$. In the example, $\tau_c = 26$ and $\upsilon = 15$. Since $\tau_c + 1 \geq \upsilon$, the array becomes

$$\begin{array}{cccccccccccccc} 4 & 6 & 8 & 10 & 14 & 7 & 20 & 24 & 13 & 28 & 28 & 17 & 40 & 25 \end{array}$$

Here, note that if we let $\nu' = \nu - 2$ and $\tau_c' = \tau_c + 2$, then

$$\tau_c'/2 - \nu' \leq 1, \tag{3.1}$$

since $\nu - \tau_c/2 \geq 2$.

By an abuse of notation, we denote by τ_c the largest τ_i for $i \geq c$ such that $\tau_{i+k-2} - \tau_i \leq 2$ and let v be the smallest odd part to the right of τ_c. We repeat the same process until there is no odd part to the right of τ_c or $\tau_c + 1 < v$. By performing the process with $\tau_c = 28$ and $v = 17$, we obtain

4 6 8 10 14 7 20 24 13 15 28 30 40 25

We then obtain

4 6 8 10 14 7 20 24 13 15 23 30 30 40

Let τ_c be the second largest τ_i such that $\tau_{i+k-2} - \tau_i \leq 2$. We repeat the entire process with τ_c. We continue this process running through all such τ_i with $\tau_{i+k-2} - \tau_i \leq 2$ from largest to smallest. In the example, we obtain

4 6 5 10 10 14 20 24 13 15 23 30 30 40

and then we have

3 6 6 10 10 14 20 24 13 15 23 30 30 40

Finally, we rearrange the parts in weakly increasing order to form a partition σ. In the example, $\sigma = (3, 6, 6, 10, 10, 13, 14, 15, 20, 23, 24, 30, 30, 40)$, which is counted by $D_{3,3}(244)$.

By the choice of τ_c, we note that consecutive even integers $2i, 2i + 2$ in σ can occur at most $k - 1$ times. Furthermore, since we subtracted 2 from v if $\tau_c + 1 \geq v$, the number of consecutive integers $2i, 2i+1, 2i+2$ cannot exceed $k-1$. Therefore, we see that σ is indeed a partition counted by $D_{k,k}(n)$.

Now, we show that the process is reversible. For a partition $\sigma = (\sigma_1, \ldots, \sigma_l)$ with $\sigma_i \leq \sigma_{i+1}$, counted by $D_{k,k}(n)$, let τ be the partition consisting of only even parts of σ. Then, by the definition of $D_{k,k}(n)$, we have $\tau_{i+k-1} - \tau_i > 2$. Let τ_c be the smallest τ_i such that $\tau_i - \tau_{i-k+2} \leq 2$, and let v be the largest odd part to the left of τ_c. If $\tau_c/2 - 1 > v$, then we stop. If $\tau_c/2 - 1 \leq v$, then we add 2 to v and put it right after τ_c, and subtract 2 from τ_c and rearrange the τ_i for $i \leq c$ in weakly increasing order. So, we have $(\tau_c - 2, \tau_{c-k+2}, \ldots, \tau_{c-1}, \nu + 2)$. For example, if $\sigma = (3, 6, 6, 10, 10, 13, 14, 15, 20, 23, 24, 30, 30, 40)$, then $\tau_c = 6$ and $v = 3$. Since $\tau_c/2 - 1 \leq v$, we obtain

4 6 5 10 10 13 14 15 20 23 24 30 30 40

We note that if $\nu' = \nu + 2$ and $\tau_c' = \tau_c - 2$, then

$$\tau_c' + 1 \geq \nu', \tag{3.2}$$

since $\tau_c \geq \nu + 3$.

By an abuse of notation, we denote by τ_c the smallest τ_i for $i \leq c$ such that $\tau_i - \tau_{i-k+2} \leq 2$ and let v be the largest odd part to the left of τ_c. We repeat the process until we run out of the odd parts to the left of τ_c or $\tau_c/2 - 1 > v$. In the example, $\tau_c = 6$ and there are no odd parts to its left, so we stop. Let τ_c be the second smallest τ_i such that $\tau_i - \tau_{i-k+2} \leq 2$. We repeat the entire process with τ_c. In the example, $\tau_c = 10$ and $v = 5$. Since $\tau_c/2 - 1 \leq v$, we obtain

4 6 8 10 7 13 14 15 20 23 24 30 30 40

We continue this process through all such τ_i with $\tau_i - \tau_{i-k+2} \leq 2$ from smallest to largest. In the example, we obtain

4 6 8 10 7 13 14 15 20 24 28 30 25 40

and then we have

4 6 8 10 7 13 14 20 24 28 28 17 25 40

Finally, we obtain

$$4 \quad 6 \quad 8 \quad 10 \quad 7 \quad 14 \quad 20 \quad 24 \quad 26 \quad 28 \quad 15 \quad 17 \quad 25 \quad 40$$

We split each even part of the resulting array into halves and rearrange them with the odd parts in weakly increasing order to form a partition π. In the example,

$$\pi = (2, 2, 3, 3, 4, 4, 5, 5, 7, 7, 7, 10, 10, 12, 12, 13, 13, 14, 14, 15, 17, 20, 20, 25).$$

By the choice of τ_c, two consecutive integers $i, i+1$ obtained by splitting even parts can occur at most $2k-2$ times. Furthermore, since we subtracted 2 from τ_c and added 2 to υ if $\tau_c/2 - 1 \leq \upsilon$, two consecutive integers $i, i+1$ can occur in π at most $2k-2$ times. Therefore, π is a partition counted by $W_{2k-1,2k-1}(n)$.

Each step of this process is indeed the inverse of the map from partitions counted by $W_{2k-1,2k-1}(n)$ to partitions counted by $D_{k-1,k-1}(n)$. We started with the largest τ_c and the smallest υ to its right. Meanwhile, in the reverse process, we chose the smallest τ_c and the largest υ to its left. The criteria for the process match condition (3.2) and the criteria for the inverse process match condition (3.1). □

We now consider the general case.

Theorem 3.2. *Let $k \geq a \geq 1$ be integers. Then, for any positive integer n,*

$$W_{2k-1,2a-1}(n) = D_{k,a}(n).$$

Proof. By the definitions, at most $2a-2$ of the parts are equal to 1 in any partitions counted by $W_{2k-1,2a-1}(n)$, and at most $a-1$ of the parts are equal to 1 or 2 in any partitions counted by $D_{k,a}(n)$. Thus, in the proof of Theorem 3.1, we add the constraints that 1 can occur at most $2a-2$ times in π. Then, we can see that the number of 1 and 2 in σ cannot exceed $a-1$. The remaining portion of the proof is the same, so we omit the details. □

Remark. The bijection for Problem 3 can be specialized to the solution of Problems 1 and 2 when $k = 2$.

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