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Trees and Ehrenfeucht–Fraïssé games[1]

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Abstract

Trees are natural generalizations of ordinals and this is especially apparent when one tries to find an uncountable analogue of the concept of the Scott-rank of a countable structure. The purpose of this paper is to introduce new methods in the study of an ordering between trees whose analogue is the usual ordering between ordinals. For example, one of the methods is the tree-analogue of the successor operation on the ordinals.

1. Introduction

Suppose T and S are trees. We write $T \leqslant S$ if there is a mapping $f : T \to S$ such that

$$x <_T y \to f(x) <_S f(y)$$

for all x and y in T. This quasi-ordering of trees arises naturally in infinitary model theory, especially in the investigation of Ehrenfeucht–Fraïssé games [7]. It has also been used to study non-well-founded inductive definitions [15]. For a survey of the role of $\leqslant$ in infinitary logic, see [28].

The purpose of this paper is to introduce new methods in the study of this ordering. Our first method is called σ-operation. For a partial order P, the tree σP is defined as the set of ascending sequences of elements of P, ordered by end-extension. This operation is originally due to Kurepa [10] and it was later used and extended by Todorčević [21, 26]. In model theory it was used by Hyttinen [6]. In Section 2 we

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demonstrate the versatility of the σ-operation in providing both universal elements and counterexamples to universality in various classes of structures. In Section 3 we use the structures $\sigma\mathbb{Q}$ and $\sigma\mathbb{R}$ to get model-theoretic applications of the σ-operation ($\mathbb{Q}$ and $\mathbb{R}$ are the linearly ordered sets of rationals and reals, respectively.)

Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two relational structures of cardinality $\aleph_1$ over the same countable vocabulary such that $\mathfrak{A} \not\cong \mathfrak{B}$. In [7] it is suggested that trees with no uncountable branches measure similarity of $\mathfrak{A}$ and $\mathfrak{B}$. This is accomplished via the following Ehrenfeucht–Fraïssé-game $EF_T(\mathfrak{A},\mathfrak{B})$: There are two players $\exists$ and $\forall$. Player $\forall$ moves pairs $z_\alpha=(x_\alpha,t_\alpha)$ where $x_\alpha \in A\cup B$ and $t_\alpha \in T$. Player $\exists$ moves elements y_α. The only rules of the game are

$$x_\alpha \in A \Rightarrow y_\alpha \in B,$$
$$x_\alpha \in B \Rightarrow y_\alpha \in B,$$
$$\alpha<\beta \Rightarrow s\, t_\alpha <_T t_\beta.$$

The elements x_α and y_β are said to *correspond* to each other in the game. Suppose a sequence $(x_\alpha,t_\alpha,y_\alpha)_{\alpha<\gamma}$ is played so that $(t_\alpha)_{\alpha<\gamma}$ is a maximal chain in T. Then the game ends. Player $\exists$ has won if the correspondence $x_\alpha \leftrightarrow y_\alpha$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

A tree T is a *Karp tree* of $(\mathfrak{A},\mathfrak{B})$ if $\exists$ has a winning strategy in $EF_T(\mathfrak{A},\mathfrak{B})$ but not in $EF_{\sigma T}(\mathfrak{A},\mathfrak{B})$. A tree T is a *Scott tree* of $(\mathfrak{A},\mathfrak{B})$ if $\forall$ has a winning strategy in $EF_{\sigma T}(\mathfrak{A},\mathfrak{B})$ but not in $EF_T(\mathfrak{A},\mathfrak{B})$. These concepts, introduced in [7], arises naturally if one tries to find an uncountable analogue of the concept of Scott-rank of a countable structure. A Scott tree T, which is also a Karp tree, is called a *determined* Scott tree. Such trees really capture the difference between the structures: for $S\leqslant T$, Player $\exists$ wins $EF_S(\mathfrak{A},\mathfrak{B})$, while for $\sigma T\leqslant S$, Player $\forall$ wins $EF_S(\mathfrak{A},\mathfrak{B})$. We exhibit in this paper both special and non-special determined Scott trees.

In Section 3 we complement results of [7, 5] by proving, among other things, that it is consistent relative to the consistency of a supercompact cardinal, that every Karp tree contains a subtree which is a Karp tree of cardinality $\leqslant 2^\omega$. We also prove in Section 3 that there is a pair $(\mathfrak{A},\mathfrak{B})$ of models of cardinality $\aleph_1$ such that the family of Karp trees of $(\mathfrak{A},\mathfrak{B})$ has antichains of size 2^{ω_1}.

A tree of height ω_1 is *persistent* if it has a non-empty subtree in which each node has extensions of all countable heights. This concept was introduced by Huuskonen [5], who also proved that the following tree is the $\leqslant$-smallest persistent tree: The tree T^0 consists of sequences

$$(t_0,\dots,t_n),$$

of countable ordinals ordered by

$$(t_0,\dots,t_n)\leqslant(s_0,\dots,s_m),$$

if $n\leqslant m$, $\forall i<n(t_i=s_i)$ and $t_n\leqslant s_n$. In Section 4 we use coherent sequences to construct – without assuming CH – models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\aleph_1$ that have a Scott tree of size $\aleph_1$. The previous constructions of non-trivial Scott trees of size $\aleph_1$ use CH.

Let $\mathbb{T}$ denote the class of all trees of cardinality ω_1 with no uncountable branches. The quasi-ordering $\leqslant$ gives some structure to $\mathbb{T}$. In Section 5 we investigate the exact nature of this structure. What kind of chains and antichains does it have? What kind of hierarchies can we isolate? Are there some particularly interesting subclasses, etc. We give a relatively complete picture of the ordering $\leqslant$ of $\mathbb{T}$ below T^0. Among trees above T^0 we find pairs of incomparable trees, which can be chosen to be Aronszajn, if CH is assumed, and Souslin, if $\diamondsuit$ is assumed.

In Section 6 we study the *Bottleneck Problem* and the *Comparability Problem* for trees $T \in \mathbb{T}$ such that $T^0 < T < \omega_1$. The Bottleneck Problem asks whether there is such a tree T^1 with similar behavior to the tree T^0 i.e., a tree T^1 such that $T^0 < T^1 < \omega_1$ and such that for every other tree $T \in \mathbb{T}$ either $T^1 \leqslant T$ or $\sigma T \leqslant T^1$. The Comparability Problem asks for a tree $T \in \mathbb{T}$ such that $T^0 < T < \omega_1$ and such that T is comparable with any other tree from $\mathbb{T}$. These problems of course make sense if we replace $\mathbb{T}$ by the class of all trees of height ω_1 and of size continuum. In this generality and if we require solutions to use no additional axioms of set theory, the two problems are still open. However, we shall prove that a strong form of Jensen's diamond principle $\diamondsuit$ (still true in the constructible universe; see [3]) gives a negative answer to both problems. For example, we shall use the strong diamond to construct for every tree $T \in \mathbb{T}$ with $T^0 < T < \omega_1$ a Souslin tree S such that $S \not\leqslant T$ (Theorem 44). A large part of Section 6 is devoted to a study of a particular class

$$T(A), \quad (A \subseteq \omega_1)$$

of trees of size continuum. These are the trees $T(A)$ of all closed ascending sequences of elements of A. It turns out that these trees give rise to a rich and interesting family of non-special trees, all incomparable with Aronszajn trees. Naturally, these trees are in $\mathbb{T}$ only if CH is assumed. The trees $T(A)$ were first studied in [18]. Later they were used in [7] to give examples of Scott trees. For stationary A the trees $T(A)$ are all above T^0 and it is interesting to note that the infimum of $T(A)$ and $T(B)$, with $A \cap B$ non-stationary, is exactly T^0. This property of the trees $T(A)$ makes them immediately relevant to the bottleneck and comparability problems, for it is clear that these two problems would have negative solutions if for every tree $T \in \mathbb{T}$ with $T^0 < T < \omega_1$ we can find a costationary set A such that $T(A) \not\leqslant T$. This is exactly what we are going to show using the strong diamond principle. This should be compared with results of Mekler and Shelah [13] who proved that the statement that the class of $T(A)$, A costationary, has an upper bound (below ω_1) is consistent and independent of ZFC$+\diamondsuit$.

Notation

We denote first-order structures by bold face letters $\mathfrak{A}, \mathfrak{B}$, etc. The universe of $\mathfrak{A}$ is A, that of $\mathfrak{B}$ is B, etc. There are some exceptions to this rule: We use mere light face capital letters $S, T, U, K, L, \ldots$ to denote trees and other partial orderings. In this case the ordering is denoted by $\leqslant_S, \leqslant_T, \ldots$. The class of successor ordinals is denoted by Succ.

Let T_i, $(i \in I)$, be a set of trees. Let r_i be the root of T_i. The disjoint union $\bigoplus_{i \in I} T_i$ is defined as the set $\{(t,i)\colon\ i \in I,\ t \in T_i\}$ endowed with the ordering

$$(t,i) \leqslant (t',i') \ \Leftrightarrow\ i = i' \text{ and } t \leqslant_{T_i} t'$$

and with the elements (r_i, i) identified. The infimum $T \otimes S$ of two trees T and S is the set of pairs (t,s), where $t \in T$, $s \in S$ and $ht_T(t) = ht_S(s)$, endowed with the canonical ordering

$$(t,s) \leqslant_{T \otimes S} (t',s') \ \Leftrightarrow\ t \leqslant_T t' \text{ and } s \leqslant_S s'.$$

The product $T \cdot T'$ of two trees is the set of all triples (g,t,t'), where $t \in T$, $t' \in T'$ and g maps every predecessor of t' in T' to a maximal branch of T, endowed with the ordering

$$\begin{aligned}(h,u,u') \leqslant (g,t,t') \ \Leftrightarrow\ & u' \leqslant_{T'} t' \text{ and} \\ & \forall v <_{T'} u' (h(v) = g(v)) \text{ and} \\ & (u' <_{T'} t' \text{ and } u \in g(u') \text{ or } (u' = t' \text{ and } u \leqslant_T t)).\end{aligned}$$

The idea is that going up a branch of $T \cdot T'$ entails going up a branch b of T' in such a way that for each node on b we go through some maximal branch of T. In a way, T is traversed T' many times. It turns out that it is not essential that the branches that the function g has as its values are maximal. Note that $T^0 = (\bigoplus_{\alpha < \omega_1} \alpha) \cdot \omega$.

2. The σ-operation and universal objects

Suppose $\mathfrak{A} = (A, R)$, $R \subseteq A^2$, is an arbitrary binary structure. Let $R_{\neq}$ be the relation $\{(x,y) \in R\colon\ x \neq y\}$. Let $\sigma\mathfrak{A}$ denote the tree of all sequences s such that $dom(s) \in Ord$, $rng(s) \subseteq A$ and

$$\forall \xi \in dom(s)\ \forall \eta \in dom(s)\ (\xi < \eta \rightarrow s(\xi) R_{\neq} s(\eta)).$$

The ordering of $\sigma\mathfrak{A}$ is by end-extension. The σ-operation provides a uniform approach to the failure of universality, as we demonstrate in this section.

Example 1. Let LO_1 be the class of linear orders without uncountable well-ordered or conversely well-ordered sub-orderings.

Does LO_1 have a universal element, that is, is there $K \in LO_1$ so that every element of LO_1 is isomorphic to a subordering of K? Notice that by the Erdös–Rado Theorem, every element K of LO_1 satisfies $|K| \leqslant 2^{\omega}$. We shall use the σ-operation to prove that LO_1 has no universal elements.

Example 2. Let $\mathbb{G}_1$ be the class of graphs G such that G contains no subgraph isomorphic to the clique K_{ω_1} of size $\aleph_1$, i.e. $K_{\omega_1} \not\leqslant G$, and the complement graph $\overline{G}$ also satisfies $K_{\omega_1} \not\leqslant \overline{G}$.

Is there a universal element in $\mathbb{G}_1$? Again by the Erdös–Rado Theorem, every G in $\mathbb{G}_1$ satisfies $|G| \leqslant 2^\omega$. We shall use the σ-operation to show that $\mathbb{G}_1$ has no universal elements.

If $\mathfrak{A} = (A, R)$ and $\mathfrak{B} = (B, S)$ are binary structures, we define $\mathfrak{A} \leqslant \mathfrak{B}$ to mean that there is a mapping $f : A \to B$ with the property

$$\forall a \in A \ \forall a' \in A \ (aR_{\neq}a' \to f(a)S_{\neq}f(a')).$$

If $\mathfrak{A} \leqslant \mathfrak{B}$, we say that $\mathfrak{A}$ is $\mathfrak{B}$-*embeddable*. Clearly, $\mathfrak{A} \leqslant \mathfrak{B}$ implies $\sigma\mathfrak{A} \leqslant \sigma\mathfrak{B}$.

Theorem 3 (Todorčević [21]). $\sigma(\mathfrak{A}) \not\leqslant \mathfrak{A}$.

Proof. Suppose $f : \sigma(\mathfrak{A}) \to A$ is a witness to $\sigma(\mathfrak{A}) \leqslant \mathfrak{A}$. Let $s : Ord \to A$ be defined by the condition $s(\alpha) = f(s|_\alpha)$. Then $s(\alpha) \in A$ for all $\alpha \in Ord$ and $\alpha < \beta$ implies $s(\alpha) \neq s(\beta)$, a contradiction. □

Kurepa [10] proved Theorem 3 for posets.

Corollary 4. *There is no universal element in the class of posets of cardinality* $\leqslant 2^\omega$ *with no increasing* ω_1*-sequence.*

Proof. Suppose P is an arbitrary element of the class. Then σP is in the class but $\sigma P \not\leqslant P$. Hence P is not universal. □

Corollary 5. *There is no universal element in the class* LO_1 *of Example* 1.

Proof. If $P \in LQ_1$, let P^* be σP with the lexicographic ordering. Then $P^* \in LO_1$ and $P^* \not\leqslant P$. □

Corollary 6. *There is no universal element in the class* $\mathbb{G}_1$ *of Example* 2.

Proof. Suppose $G \in \mathbb{G}_1$. Let $\sigma_1 G$ be the tree of sequences $t : \alpha \to G$ such that

$$\forall \xi < \alpha \ \forall \eta < \alpha (\xi < \eta \to t(\xi) E_G t(\eta)),$$

where $\alpha \in Ord$. Since $|G| \leqslant 2^\omega$ and $K_{\omega_1} \not\leqslant G$, $|\sigma_1 G| \leqslant 2^\omega$. Let $<^*$ be a linear ordering of G with $(G, <^*) \leqslant \mathbb{R}$. Let $<_{\text{lex}}$ be the lexicographic ordering of $\sigma_1 G$ determined by $\leqslant^*$. Let $\ll$ be a well-ordering of $\sigma_1 G$ extending the tree-ordering $\subseteq_{\text{end}}$. We are ready to define an edge-relation E on the universe $\sigma_1 G$. Let

$$sEt \leftrightarrow (s \ll t \leftrightarrow s <_{\text{lex}} t).$$

Now $(\sigma_1 G, E) \in \mathbb{G}_1$. To prove that G cannot be universal, it suffices to show that G cannot have a subgraph isomorphic to $(\sigma_1 G, E)$. Suppose it does. Then $(\sigma_1 G, \subseteq_{\text{end}}) \leqslant (G, E_G)$, a contradiction with Theorem 3. □

The above corollary shows that while the σ-operation often produces a candidate for the failure of universality, one may have to work more to find the right structure on the set $\sigma\mathfrak{A}$. For some applications in topology and Banach space theory, see [26].

Example 7 (Todorčević [21]). For pure sets A we get from Theorem 3

$$S(A) =_{\mathrm{d}f} \sigma((A, A^2)) \not\leqslant (A, A^2). \tag{1}$$

Example 7 says that there is no function f from the 1–1 sequences of elements of A into A such that

$$\forall s \forall t\, (s \subseteq_{\text{end}} t \to f(s) \neq f(t)).$$

For $A = \omega$ this means that $S(\omega)$ is not the union of ω antichains, i.e. it is non-special. Since elements of $S(A)$ canonically well-order their range, we have

$$S(A) \leqslant W(A) =_{\text{df}} (\{X \subseteq A\colon\ X \text{ is well-orderable}\}, \subseteq). \tag{2}$$

As a consequence of (1) and (2) we get $W(A) \not\leqslant (A, A^2)$. Tarski [16] proved that there is no 1–1 map $W(A) \to A$. Indeed, the idea behind the σ-operation can be utilized to yield a combination of Zermelo's and Cantor's theorems which is provable without AC: For every $f : W(A) \to A$ there are $X \subset Y \subseteq A$ with $f(X) = f(Y) \in Y \setminus X$. Namely, let

$$s(\alpha) = f(rng(s|_\alpha))$$

for $\alpha \in Ord$. Let β be the least β with $s(\beta) = s(\beta + 1)$. Let $X = rng(s|_\beta)$ and $Y = rng(s|_{\beta+1})$. Then $X \subset Y$ and $f(X) = f(Y) \in Y \setminus X$.

A tree T is *Hausdorff* if different nodes of a limit level of T have different sets of predecessors. An example of a Hausdorff tree is provided by

$$\sigma\mathbb{Q} =_{\mathrm{d}f} \sigma((\mathbb{Q}, <)).$$

Theorem 8. *The tree $\sigma\mathbb{Q}$ is a universal element in the class of $\mathbb{R}$-embeddable Hausdorff-trees of cardinality $\leqslant 2^\omega$.*

Proof. Let $N = \omega \setminus \{0\}$. Suppose T is in the class. For each $t \in T$ pick $b_t : \omega \to N$ one–one such that $t \neq s$ implies $b_t \neq b_s$. Let T' be the set of $t \in T$ with $ht_T(t) \in Succ$. The tree T' is clearly special. Let $a : T' \to \omega$ such that $a^{-1}(n)$ is an antichain for all n. For $t \in T$, let $Pr'(t)$ be the set of $s \in T'$ with $s \leqslant_T t$. Let $s_\beta(t)$, $\beta < \alpha_t$, be an increasing enumeration of the set $Pr'(t)$. Let $\hat{t} : \omega \cdot \alpha_t \to \omega \times \omega$ be defined by the equation

$$\hat{t}(\omega \cdot \beta + n) = (a(s_\beta(t)), b_{s_\beta(t)}(n)).$$

Now the mapping $t \mapsto \hat{t}$ is an isomorphic embedding of T into $S(\omega \times \omega)$. Thus it suffices to prove that $S(\omega)$ is isomorphic to a subtree of $\sigma\mathbb{Q}$. For this end, let Q_d be the set of $g:\omega\to\{0,1\}$ such that g is eventually zero, ordered lexicographically. If $s\in S(\omega)$, $s:\alpha\to\omega$, let $s^*:\alpha\to Q_d$ so that

$$s^*(\xi)(n)=1 \quad \text{iff } n\leqslant s(\xi)\ \&\ \exists\eta\leqslant\xi\ (n=s(\eta)).$$

Now s^* is strictly increasing, so $s^*\in\sigma Q_d$. The mapping $s\mapsto s^*$ is the desired isomorphic embedding $S(\omega)\to\sigma Q_d$. $\square$

Proposition 9. *σT is a compact subspace of $\mathbb{P}(T)$ ($\cong\{0,1\}^{\mathrm{T}}$).*

Proof. σT is closed and $\mathbb{P}(T)$ is compact. $\square$

This proposition indicates that the σ-operation may be used to kill universality in certain classes of compact spaces. For example, there is no universal space in the class of first countable compact Hausdorff spaces (see [26]). Todorčević has also proved that every closed subset of σT is its retract (see [26, Lemma 2]).

In the following, we use the non-speciality of the tree $\sigma\mathbb{Q}$ to give a new proof (due to Todorčević) of a result in combinatorics known as the *Elekes theorem*:

Theorem 10 (Elekes [4]). *For every decomposition $\mathbb{P}(\omega)=\bigcup_{n=1}^{\infty}P_n$ there exist n and $A,B,C\in P_n$, all different, with $A\cup B=C$.*

Proof. If we identify elements of $\sigma\mathbb{Q}$ with their ranges, we have $\sigma\mathbb{Q}\subseteq\mathbb{P}(\mathbb{Q})$. Suppose $\mathbb{P}(\mathbb{Q})=\bigcup_{n=1}^{\infty}P_n$. Since $\sigma\mathbb{Q}$ is non-special, there is n such that $T=\sigma\mathbb{Q}\cap P_n$ is non-special. Let T_0 be the set of $t\in T$ with $\sup t<\infty$ and $\sup t\notin\mathbb{Q}$. T_0 is also non-special. Let T_1 be the set of $t\in T_0$ for which there is a rational $q_t>\sup t$ such that there is no $u\in T_0$ with $t<u$ and $\sup u<q_t$. The function $t\mapsto q_t$ specializes T_1, hence the subtree $T_2=T_0\backslash T_1$ is non-special. In particular, there are $t_1,t_2\in T_2$ so that $t_1<t_2$. Let $C=rng(t_1)$. Let $q=\min(t_2\backslash t_1)$. Since $t_1\notin T_1$, there is $u\in T_0$ such that $t_1<u$ and $\sup u<q$. Let A be the range of u, and B the range of t_2. Then $A,B,C\in P_n$ and $A\cap B=C$. By considering complements, we get the claim for unions. $\square$

Todorčević has used a similar argument to give a proof of the following result (see [29, Theorem 2.26 and Corollary 2.27]): Let G be a compact topological group of some infinite weight θ. Then for every decomposition $G=\bigcup_{n=1}^{\infty}G_n$ there is n such that G_n contains a topological copy of $A(\theta)$, the one-point compactification of discrete space of size θ.

As a mathematical object $\sigma\mathbb{Q}$ is rather complicated. It is Π^1_1-complete as a subset of the Polish space $\mathbb{P}(\mathbb{Q})$ ($\cong\{0,1\}^{\mathbb{Q}}$). A simpler version $\pi\mathbb{Q}$ was introduced in [24]. The poset $\pi\mathbb{Q}$ has the set of all subsets of $\mathbb{Q}$ as its universe and its ordering is: $s<^*t$ iff s is an initial part of t and $\min(t\backslash s)$ exists. Thus $\pi\mathbb{Q}$ is a Borel structure. Naturally, $\sigma\mathbb{Q}$ is a substructure of $\pi\mathbb{Q}$ and $\pi\mathbb{Q}$ has no uncountable chains, but $\pi\mathbb{Q}$ is not a tree. It is just a pseudotree, i.e. the predecessors of a node form a linear order. But $\pi\mathbb{Q}$ is the union of $\aleph_0$ subtrees. To see this, fix a well-ordering $<_w$ of $\pi\mathbb{Q}$ and an enumeration

$\langle q_n\colon n<\omega\rangle$ of the rationals. For $t\in\pi\mathbb{Q}$ which has proper extensions in $\pi\mathbb{Q}$, let $s(t)$ denote the $<_w$-minimal such extension. For $n\in N$ let

$$\pi_n\mathbb{Q}=\{t\in\pi\mathbb{Q}\colon\ q_n=\min(s(t)\backslash t)\}.$$

Since $\pi_n\mathbb{Q}$, $(n\in N)$, cover $\{t\in\pi\mathbb{Q}\colon \sup(t)<\infty\}$, it suffices to show that each $\pi_n\mathbb{Q}$ is well founded (and therefore a tree). Otherwise some $\pi_n\mathbb{Q}$ contains an infinite decreasing sequence $t_1>t_2>\cdots$. Then $s(t_1)\geqslant_w s(t_2)\geqslant_w\cdots$. So the sequence must stabilize from some point j on. Let s be the constant value of $s(t_i)$, $i\geqslant j$. By the definition of $\pi_n\mathbb{Q}$ we have that $q_n=\min(s\backslash t_i)$ for all $i\geqslant j$ contradicting the assumption that $\{t_i\}$ is strictly decreasing.

3. On $\mathbb{R}$-embeddability of products of trees

Huuskonen [5] proved that if two trees T and S are $\mathbb{Q}$-embeddable, then so is their product $T\cdot S$. The situation is quite different with $\mathbb{R}$-embeddability. We show in Corollary 13 that the product $S\cdot T$ is $\mathbb{R}$-embeddable for all $\mathbb{R}$-embeddable S if and only if T is in fact $\mathbb{Q}$-embeddable.

Let $\sigma'(P)$ denote the tree of elements of $\sigma(P)$ of successor length. More generally, let Succ(T) denote the subtree of T consisting of nodes of T of successor height. Note that $\sigma'(P)\leqslant P$ whence $\sigma'(P)<\sigma(P)$.

Lemma 11. *If S is $\mathbb{R}$-embeddable and T is $\mathbb{Q}$-embeddable, then $S\cdot T$ is $\mathbb{R}$-embeddable.*

Proof. Since T is $\mathbb{Q}$-embeddable, σT is $\mathbb{R}$-embeddable. For $t\in T$, let t^+ be the ascending sequence of $u\in T$ with $u\leqslant t$, and let t^- be the ascending sequence of $u\in T$ with $u<t$. Suppose $f:S\to(0,1)$ and $g:\sigma T\to(0,1)$ are strictly increasing. Suppose $u\in S\cdot T$. Thus $u=(h,s,t)$, where $s\in S$ and $t\in T$. Let $F(u)=g(t^-)+f(s)(g(t^+)-g(t^-))$. It is easy to see that $F:S\cdot T\to\mathbb{R}$ is strictly increasing. □

Theorem 12. *Suppose T is a tree with the following property: There is a linear order L with a suborder L_0 which is dense in L such that $T\leqslant L$ but $T\nleqslant L_0$. Then the following holds for all trees S_1 and S_2:*

$$S_1\leqslant S_2\ \Leftrightarrow\ S_1\cdot T\leqslant S_2\cdot T.$$

Proof. Suppose $S_1\cdot T\leqslant S_2\cdot T$ but $S_1\nleqslant S_2$. Let f be a strictly increasing mapping $S_1\cdot T\to S_2\cdot L$. We derive a contradiction by constructing a strictly increasing $f':T\to L_0$. For this end, let a be the root of S_1 and $t\in T$. Let us suppose $f'(u)$ is defined already for all predecessors u of t in T. We assume that while $f'(u)$ was defined, also a maximal branch B_u of S_1 was defined. Let $g(u)=B_u$ whenever $u\leqslant t$. Let $f((g,a,t))=(g',b,t')$. If $c\in S_1$ extends a, then $f((g,c,t))=(h_1,h_2(c),h_3(c))$ extends (g',b,t') in $S_2\cdot L$. If $h_3(c)=t'$ for all $c\in S_1$, we have a strictly increasing $h_2:S_1\to S_2$, contrary to $S_1 \not\leqslant S_2$. Thus there is some $c \in S_1$ with $h_3(c) > t'$. Let $f'(t) \in L_0$ with $t' < f'(t) < h_3(c)$. Let B_t be an extension of c to a maximal branch of S_1. Clearly, $f' : T \to L_0$ is strictly increasing. This end the proof. $\square$

We get immediately a hierarchy of product trees above the tree $\sigma'\mathbb{R}$. For example,

$$\mathbb{R} < 2 \cdot \sigma'\mathbb{R} < 3 \cdot \sigma'\mathbb{R} < \cdots .$$

Corollary 13. *Suppose T is a tree. Then the following conditions are equivalent*:
(1) $T \leqslant \mathbb{Q}$,
(2) $2 \cdot T \leqslant \mathbb{R}$,
(3) $\forall S \leqslant \mathbb{R} (S \cdot T \leqslant \mathbb{R})$.

Proof. Assume $2 \cdot T \leqslant \mathbb{R}$. Then $T \leqslant Succ(2 \cdot T) \leqslant \mathbb{Q}$. Conversely, if $T \leqslant \mathbb{Q}$ and $S \leqslant \mathbb{R}$ then,

$$Succ(S \cdot T) \leqslant Succ(S) \cdot T \leqslant \sigma'\mathbb{Q} \cdot \sigma'\mathbb{Q} \leqslant \mathbb{Q}$$

and hence $S \cdot T \leqslant \mathbb{R}$. $\square$

Corollary 14. *Suppose S and T are trees. The following conditions are equivalent*:
(1) $S \cdot T \leqslant \mathbb{R}$,
(2) $(2 \leqslant S \leqslant \mathbb{R}$ *and* $T \leqslant \mathbb{Q})$ *or* $(S \leqslant 1$ *and* $T \leqslant \mathbb{R})$.

Proof. (2) implies (1): If $S \leqslant \mathbb{R}$ and $T \leqslant \mathbb{Q}$, then $S \cdot T \leqslant \mathbb{R}$ by Lemma 11.

(1) implies (2): Suppose $S \cdot T \leqslant \mathbb{R}$. Then in particular $S \leqslant \mathbb{R}$. Thus $2 \cdot T \leqslant S \cdot T \leqslant \mathbb{R}$, whence by Corollary 13, $T \leqslant \mathbb{Q}$. $\square$

We may conclude from the above analysis that the product $S \cdot T$, where S is a non-trivial special tree, is never between the trees $\sigma'\mathbb{Q}$ and $\sigma'\mathbb{R}$ in the quasi-order $\leqslant$ of trees. More exactly, if S is a tree such that $2 \leqslant S \leqslant \mathbb{Q}$ and T is an arbitrary tree. Then

$$S \cdot T \leqslant \mathbb{Q} \Leftrightarrow S \cdot T \leqslant \mathbb{R}.$$

4. Scott and Karp trees

The existence of Scott and Karp trees, as well as their mutual relationship, is established by the following result:

Theorem 15 (Hyttinen and Väänänen [7]). *Karp trees and Scott trees exist for any pair $(\mathfrak{A}, \mathfrak{B})$ of non-isomorphic structures of cardinality $\aleph_1$ and of the same vocabulary. If T_1 is a Karp tree and T_2 a Scott tree of $(\mathfrak{A}, \mathfrak{B})$, then $T_1 \leqslant T_2$.*

Apart from this there does not seem to be much order in the families of Scott and Karp trees. See [7, 5] for earlier results in this direction.

Our first concern in this section is the cardinality of Karp trees. It was pointed out in [7] that any Scott tree contains a subtree which is a Scott tree of cardinality $\leqslant 2^{\omega}$. Huuskonen [5] showed that the same need not be true of Karp trees. We show that the same may be true of Karp trees. Let

$$k_0$$

be the least θ, if any exist, such that if $\mathfrak{A}$ and $\mathfrak{B}$ are models of cardinality $\leqslant 2^{\omega}$ and T is a Karp tree of $(\mathfrak{A}, \mathfrak{B})$, then there is a subtree T_0 of T such that T_0 is a Karp tree of $(\mathfrak{A}, \mathfrak{B})$ and $|T_0| < \theta$. This number is a kind of Löwenheim number for Karp trees. It is closely related to the following number introduced in [25]:

$$r_0$$

is the least θ, if any exist, such that if T is any non-special tree, then there is a non-special subtree T_0 of T of cardinality $< \theta$. Thus $r_0 \geqslant \aleph_2$. The statement $r_0 = \aleph_2$ is called the *Rado conjecture* (RC).

Theorem 16 (Todorčević [25, 26]).
(1) *RC implies* $2^{\aleph_0} \leqslant \aleph_2$.
(2) *RC implies* $MA_{\aleph_1}$ *is false.*
(3) *RC implies Chang's conjecture.*
(4) *RC is consistent relative to the consistency of a supercompact cardinal.*
(5) *The Singular Cardinal Hypothesis holds above* r_0.
(6) $\square_\kappa$ *fails for* $\kappa \geqslant r_0$.

We will use the following result Tuuri [27] to establish a connection between r_0 and k_0 at least when the continuum hypothesis is assumed.

Theorem 17 (Tuuri [27]). *There are structures* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of cardinality* 2^{ω} *so that the following conditions are equivalent for any tree* T:
(i) T *is* $\mathbb{R}$*-embeddable if and only if* $\exists$ *has a winning strategy in* $EF_T(\mathfrak{A}, \mathfrak{B})$.
(ii) $\sigma\mathbb{R} \leqslant T$ *if and only if* $\forall$ *has a winning strategy in* $EF_T(\mathfrak{A}, \mathfrak{B})$.

Proof. We present the proof for completeness. Our proof is an adaption of the original unpublished proof of Tuuri and is presented with his kind permission.

Let P_0 be the set of functions $f : \mathbb{R} \to 3$. If $f \in P_0$, let $\mathrm{Supp}(f) = \{r \in \mathbb{R}\colon f(r) \neq 0\}$. Let

$$P = \{f \in P_0\colon \mathrm{Supp}(f) \text{ is a well-ordered subset of } \mathbb{R}\}.$$

If $d \in 3$, let $\#(d)$ be defined by $\#(d) = 0$ if $d = 0$ and $\#(d) = 1$ otherwise. If $f \in P$ let $\#(f)$ be the function $\#(f)(r) = \#(f(r))$. Let F be the set of functions f with $dom(f)$ an open initial segment of $\mathbb{R}$ and $ran(f) \subseteq \{0, 1\}$. For $f \in F$ we define $(1 - f)(r) = 1 - f(r)$, when $r \in dom(f)$. The construction that follows depends on certain decisions at limit stages and these decisions are not canonically determined by earlier stages of the construction. For this reason we introduce a decision-making function m. Let $m : F \to 2$ be a function so that $m(f)$ is arbitrarily chosen, subject to the conditions that if $f \in F$ is eventually constant d, then $m(f) = d$, and for all f, $m(1-f) = 1 - m(f)$, and if f and g eventually agree, then $m(f) = m(g)$.

Our models will have $P \cup \mathbb{R}$ as the universe, an auxiliary predicate E, and a unary predicate U the interpretation of which makes the models different. The predicate U is interpreted by defining two "smashed" versions $f^{\mathfrak{A}}$ and $f^{\mathfrak{B}}$ of every $f \in P$. These are defined by induction along Supp(f). At successor stages the idea is to use the smash function # to let $f(r)$ generate a value $\#(f(r))$ or $1 - \#(f(r))$, according to what decisions have been made before, for $f^{\mathfrak{A}}(r')$ and $f^{\mathfrak{B}}(r')$, $r' < r$. At limit stages we use the function m.

Suppose $f \in P$ and $\mathrm{Supp}(f) = \langle x_\alpha : \alpha < \beta \rangle$ in increasing order. Suppose $\mathfrak{C} \in \{\mathfrak{A}, \mathfrak{B}\}$. We define $f^{\mathfrak{C}}(x)$ by cases. If g is a function with $dom(g) \subseteq \mathbb{R}$, denote the restriction of g to $(-\infty, x)$ by $g|_x$ and the restriction to $(-\infty, x]$ by $g|_{\leqslant x}$.

If $x \in \mathbb{R}$ and $\mathrm{Supp}(f) = \emptyset$ or $x < x_0$, we let $f^{\mathfrak{C}}(x) = 0$ if $\mathfrak{C} = \mathfrak{A}$ and $F^{\mathfrak{C}}(x) = 1$ otherwise. For other x we let

$$f^{\mathfrak{C}}(x) = \begin{cases} \#(f(x)) & \text{if } x = x_\alpha, \text{ and } m(f^{\mathfrak{C}}|_x) = 0, \\ 1 - \#(f(x)) & \text{if } x = x_\alpha, \text{ and } m(f^{\mathfrak{C}}|_x) = 1, \\ f^{\mathfrak{C}}(x_\alpha) & \text{if } x_\alpha < x < x_{\alpha+1}, \\ m(f^{\mathfrak{C}}|_x) & \text{if } x = \sup\limits_{\alpha < \nu} x_\alpha < x_\nu, \nu = \bigcup \nu, \\ f^{\mathfrak{C}}(x_\alpha) & \text{if } x > \sup\limits_{\alpha < \beta} x_\alpha \text{ and } \beta = \alpha + 1, \\ m(f^{\mathfrak{C}}|_{x^*}) & \text{if } x \geqslant x^* = \sup\limits_{\alpha < \beta} x_\alpha \text{ and } \beta = \bigcup \beta. \end{cases}$$

Now we are ready to define the models needed in the theorem:

$$\mathfrak{A} = \langle P \cup \mathbb{R}, E, \{f \in P: (f^{\mathfrak{A}} \text{ is eventually } 0\}\rangle,$$
$$\mathfrak{B} = \langle P \cup \mathbb{R}, E, \{f \in P: (f^{\mathfrak{B}} \text{ is eventually } 0\}\rangle,$$

where $E = \{\langle f, g, r\rangle : f, g \in P, r \in \mathbb{R} \text{ and } f|_r = g|_r\}$.

Claim. *$\forall$ has a winning strategy in $EF_{\sigma\mathbb{R}}(\mathfrak{A}, \mathfrak{B})$.*

The strategy of $\forall$ is the following. Player $\forall$ will play elements h^0_α and h^1_α together with s^0_α and $s^1_\alpha = s^0_\alpha \frown \langle r_\alpha \rangle$ in $\sigma\mathbb{R}$. We construe round α of the game as consisting of actually two rounds. First, $\forall$ plays h^0_α and s^0_α, and after this h^1_α and s^1_α. The responses of $\exists$ are, respectively, k^0_α and k^1_α. As part of his strategy $\forall$ then chooses one of h^0_α and h^1_α, say $h^{d_\alpha}_\alpha$, to be denoted by a_α, if the move $h^{d_\alpha}_\alpha$ was made in $\mathfrak{A}$ and the corresponding $k^{d_\alpha}_\alpha$ in $\mathfrak{B}$ will then be denoted by b_α. If the move $h^{d_\alpha}_\alpha$ was made in $\mathfrak{B}$, then $h^{d_\alpha}_\alpha$ will be denoted by b_α and the corresponding $k^{d_\alpha}_\alpha$ by a_α. Eventually, a limit ordinal δ will emerge such that $\sup_{\alpha<\delta} r_\alpha = \infty$ and then the unions $f = \bigcup_\alpha a_\alpha$ and $f' = \bigcup_\alpha b_\alpha$ will be elements of P. The point is that $\forall$ takes care that $f^{\mathfrak{A}}$ and $f'^{\mathfrak{B}}$ will not be both eventually 0, so when he finally plays f forcing $\exists$ (because of the predicate E) to play f', he has won the game.

Suppose a_α, r_α and b_α for $\alpha<\beta$ have been played as above and $\sup_{\alpha<\beta}(r_\alpha)<\infty$. Part of the strategy of $\forall$ is to maintain the condition $a_\alpha^{\mathfrak{A}}|_{\leqslant r_\alpha}=1-b_\alpha^{\mathfrak{B}}|_{\leqslant r_\alpha}$.

Case 1: *β is a successor* $\alpha+1$. Since $a_\alpha^{\mathfrak{A}}|_{\leqslant r_\alpha}=1-b_\alpha^{\mathfrak{B}}|_{\leqslant r_\alpha}$ there is a smallest $r>r_\alpha$ such that $a_\alpha(r)\neq 0$ or $b_\alpha(r)\neq 0$, for otherwise $\exists$ has lost the game already. Now $\forall$ plays h_β^0 and h_β^1, choosing carefully $h_\beta^0(r)$ and $h_\beta^1(r)$, together with s_β^0 and $s_\beta^1=s_\beta^0\frown\langle r_\beta\rangle$, $r_\beta=r$, in such a way that, to avoid immediate loss, $\exists$ has to play k_β^0 and k_β^1 so that $a_\beta^{\mathfrak{A}}|_{\leqslant r_\beta}=1-b_\beta^{\mathfrak{B}}|_{\leqslant r_\beta}$.

Case 2: *β is limit*. Let $r_\beta=\sup_{\alpha<\beta}r_\alpha$. Now $\forall$ plays h_β^0 and h_β^1, choosing carefully $h_\beta^0(r_\beta)$ and $h_\beta^1(r_\beta)$, together with $s_\beta^0=\bigcup_{\alpha<\beta}s_\alpha^1$ and $s_\beta^1=s_\beta^0\frown\langle r_\beta\rangle$ in such a way that, to avoid immediate loss, $\exists$ has to play k_β^0 and k_β^1 so that $a_\beta^{\mathfrak{A}}|_{\leqslant r_\beta}=1-b_\beta^{\mathfrak{B}}|_{\leqslant r_\beta}$

Claim. *$\exists$ has a winning strategy in $EF_{\sigma'\mathbb{R}}(\mathfrak{A},\mathfrak{B})$.*

Suppose h_α and $r_\alpha\in\mathbb{R}$ have been played by $\forall$ and k_α by $\exists$ for $\alpha<\beta$, and $r_\beta=\sup_{\alpha<\beta}\sup(s_\alpha)<\infty$. Let $a_\alpha=h_\alpha$ if h_a was played in $\mathfrak{A}$ and $a_\alpha=k_\alpha$ otherwise. Similarly, let $b_\alpha=h_\alpha$ if h_a was played in $\mathfrak{B}$ and $b_\alpha=k_\alpha$ otherwise. The strategy of $\exists$ is to keep $a_\alpha^{\mathfrak{A}}(r_\alpha)=b_\alpha^{\mathfrak{B}}(r_\alpha)$ and $a_\alpha(x)=b_\alpha(x)$ for $x>r_\alpha$. If $\exists$ can keep playing like this he wins since then $a_\alpha^{\mathfrak{A}}$ is eventually 0 if and only if $b_\alpha^{\mathfrak{B}}$ is. $\square$

Note. In the above proof we showed that $\sigma\mathbb{Q}$ is both a Karp and a Scott tree of $(\mathfrak{A},\mathfrak{B})$. Thus $\sigma\mathbb{Q}$ is *a determined Scott tree* of $(\mathfrak{A},\mathfrak{B})$.

Theorem 18. $r_0\leqslant k_0$.

Proof. Suppose T is non-special. Then $T_0=T\otimes\sigma\mathbb{Q}$ is non-special and $\mathbb{R}$-embeddable. Thus $\exists$ has a winning strategy in $EF_{T_0}(\mathfrak{A},\mathfrak{B})$, where $\mathfrak{A}$ and $\mathfrak{B}$ are from Theorem 17. In fact, T_0 is a Karp tree of $(\mathfrak{A},\mathfrak{B})$, for if $\exists$ had a winning strategy in $EF_{\sigma T_0}(\mathfrak{A},\mathfrak{B})$, then $\sigma T_0\leqslant\mathbb{R}$ and T_0 would be special. Let T_1 be a subtree of T_0 so that T_1 is a Karp tree of $(\mathfrak{A},\mathfrak{B})$ and $|T_1|<k_0$. Then $\exists$ does not have a winning strategy in $EF_{\sigma T_1}(\mathfrak{A},\mathfrak{B})$, whence $\sigma T_1\not\leqslant\mathbb{R}$ and hence T_1 is non-special. $\square$

Corollary 19. (1) *$k_0=\aleph_2$ implies RC, and hence $2^{\aleph_0}\leqslant\aleph_2$ and Chang's conjecture.*
(2) *The singular cardinal hypothesis holds above k_0 and $\square_\kappa$ fails for $\kappa\geqslant k_0$.*

We may draw the conclusion, due to [5], that k_0 does not exist in L. In fact, this corollary shows that some quite large cardinals are needed to prove the consistency of the existence of k_0. With this remark it is interesting to note:

Theorem 20 (Huuskonen [5]). *If κ is strongly compact, then $k_0\leqslant\kappa$.*

To get the consistency of $k_0=\aleph_2$, we may modify appropriately the proof of [25, Theorem 3.3(3)], where the consistency of $r_0=\aleph_2$ was proved.

Theorem 21. *The statement $k_0 = \aleph_2$ is consistent relative to the consistency of the existence of a supercompact cardinal.*

Proof. Suppose λ is a supercompact and GCH holds. Let P be the Levy-collapse of λ to $\aleph_2$. Conditions of P are partial countable functions p such that

$$dom(p) \subseteq \lambda \times \omega_1, rng(p) \subseteq \lambda,$$

$$(\alpha, \xi) \in dom(p) \Rightarrow p(\alpha, \xi) < \alpha.$$

Let P_ν consist of $p \in P$ so that $dom(p) \subseteq \nu \times \omega_1$ and let P^ν consist of $p \in P$ so that $dom(p) \subseteq (\lambda - \nu) \times \omega_1$. Then $P \cong P_\nu \times P^\nu$. Let G be P-generic. It is well known that $V[G] \models \lambda = \aleph_2 + GCH$ (see e.g. [8]). We prove $V[G] \models k_0 = \aleph_2$.

Let $\mathfrak{A}, \mathfrak{B} \in V$ be structures of the same vocabulary and with ω_1 as their universe. We shall later deal with the case $\mathfrak{A}, \mathfrak{B} \in V[G]$. Let $T, \tau \in V[G]$ so that
($\star$1) $V[G] \models T$ is a Karp tree of $(\mathfrak{A}, \mathfrak{B})$.
($\star$2) $V[G] \models \tau$ is a winning strategy of $\exists$ in $EF_T(\mathfrak{A}, \mathfrak{B})$.
We may assume that the domain $dom(T)$ of T is a cardinal $\theta > \lambda$. We can think of τ as a function:

$$\tau : \omega_1 \to \bigcup_{\alpha < \omega_1} {}^{\alpha}\omega_1.$$

We shall first assume $\tau \in V$ and come back later to the more general case that $\tau \in V[G]$. Let $\tilde{T}$ be a forcing name for T. We may assume $\tilde{T} \subseteq \theta \times \mathbb{P}$. By supercompactness there is a transitive M and an elementary embedding $j : V \to M$ such that
($\star$3) $j(\alpha) = \alpha$ for $\alpha < \lambda$, $j(\mathfrak{A}) = \mathfrak{A}, j(\mathfrak{B}) = \mathfrak{B}$ and $j(\tau) = \tau$.
($\star$4) $j(\lambda) > \theta$.
($\star$5) ${}^{\theta}M \subseteq M$.
($\star$6) $P, \mathfrak{A}, \mathfrak{B}, T \in M$.
In such a situation we know that
($\star$7) $M \models j(\mathbb{P}) \cong \mathbb{P} \times j(\mathbb{P})^\lambda$.
($\star$8) $M[G] \models j(\mathbb{P})^\lambda$ is countably closed.
It is well known that we can extend j to an elementary

$$j^* : V[G] \to M[G'],$$

where G' is $j(P)$-generic over M, and furthermore,

$$M[G'] = M[G][H],$$

where H is $j(\mathbb{P})^\lambda$-generic over $M[G]$. Since $dom(T) = \theta, {}^{\theta}M \subseteq M$ and $\tilde{T} \in M$, we have $T \in M[G]$. Now the restriction of j to θ gives rise to an isomorphic copy $T_0 \in M[G]$ of T which is a subtree of $j^*(T)$ of cardinality $< j(\lambda)$.

Claim. $M[G'] \models T_0$ *is a Karp tree of* $(\mathfrak{A}, \mathfrak{B})$.

Let us a define a strategy τ' of $\exists$ in $EF_{T_0}(\mathfrak{A},\mathfrak{B})$ in $M[G]$ by letting $\exists$ copy the T_0-moves of $\forall$ to T and then use τ. By ($\star$2),

$$M[G] \models \tau' \text{is a winning strategy of } \exists \text{ in } EF_{T_0}(\mathfrak{A},\mathfrak{B}).$$

Then by ($\star$8),

$$M[G'] \models \tau' \text{ is a winning strategy of } \exists \text{ in } EF_{T_0}(\mathfrak{A},\mathfrak{B}).$$

Secondly, suppose

$$M[G'] \models \exists \text{ has a winning strategy in } EF_{\sigma T_0}(\mathfrak{A},\mathfrak{B}).$$

By ($\star$8) we get with an easy construction

$$M[G] \models \exists \text{ has a winning strategy } \sigma \text{ in } EF_{\sigma T_0}(\mathfrak{A},\mathfrak{B}).$$

Again by ($\star$5)

$$V[G] \models \sigma \text{ is a winning strategy of } \exists \text{ in } EF_{\sigma T_0}(\mathfrak{A},\mathfrak{B})$$

contrary to ($\star$1). This ends the proof of the claim. We have established

$$M[G'] \models (j^*(T) \text{ has a subtree of size } < j(\lambda) \text{ which is a Karp tree of } (\mathfrak{A},\mathfrak{B})).$$

By the fact that j^* is elementary,

$$V[G] \models (T \text{ has a subtree of size } < \lambda \text{ which is a Karp tree of } (\mathfrak{A},\mathfrak{B})).$$

Finally, if $\mathfrak{A},\mathfrak{B},\tau \in V[G]$, then by the λ^+-c.c. of $\mathbb{P}$ there is $v < \lambda$ so that

$$V[G] = V[G_1][G_2],$$

where G_1 is $\mathbb{P}_v$-generic, G_2 is $\mathbb{P}^v$-generic over $V[G_1]$ and $\mathfrak{A},\mathfrak{B},\tau \in V[G_1]$. In this case λ is still supercompact in $V[G_1]$ because $|\mathbb{P}_v| < \lambda$. Hence we may carry out the above argument inside $V[G_1]$. □

In the next part of this section we present a new construction of Karp trees. The question we consider arises from the following result:

Theorem 22 (Hyttinen and Väänänen [7]). *There is a pair* $(\mathfrak{A},\mathfrak{B})$ *of models of cardinality* $\aleph_1$ *such that the family of Scott trees of* $(\mathfrak{A},\mathfrak{B})$ *has antichains of size* 2^{ω_1}.

We shall prove the same result for Karp trees. At first we recall some results from [18]. Let NS_T denote the set of all $E \subseteq \omega_1$ for which there is a regressive mapping $f : T|E \to T$ such that $f^{-1}(s)$ is special for all $s \in T$ (see [18]). NS is the non-stationary ideal on ω_1.

Proposition 23 (Todorčević [18]). (i) $NS_{\omega_1} = NS$.
(ii) *NS_T is a normal ideal.*

(iii) $NS_T \cup NS_U \subseteq NS_{T \otimes U}$.
(iv) $T \leqslant U \Rightarrow NS_U \subseteq NS_T$.
(v) NS_T *is trivial if and only if* T *is special.*

Theorem 24. *There is a pair* $(\mathfrak{A}, \mathfrak{B})$ *of models of cardinality* 2^{ω} *such that the family of Karp trees of* $(\mathfrak{A}, \mathfrak{B})$ *has antichains of size* 2^{ω_1}.

Proof. We use the models $\mathfrak{A}$ and $\mathfrak{B}$ of Theorem 17. For the Karp trees we use the following construction. If $A \subseteq \omega_1$, let

$$\sigma_A \mathbb{Q}$$

consist of those $s \in \sigma\mathbb{Q}$ for which

$$\forall \delta \in dom(s) \cap Lim \left(\delta \in A \text{ or } s(\delta) = \sup_{\xi < \delta} s(\xi) \right).$$

We shall need the following three properties of these trees:
(Q0) $NS_{\sigma_A \mathbb{Q}} = NS_{\omega_1} | A$. So in particular, $\sigma_A \mathbb{Q}$ is special if and only if A is non-stationary.
(Q1) If $A \cap B$ is non-stationary, then $\sigma_A \mathbb{Q} \otimes \sigma_B \mathbb{Q}$ is special.
(Q2) If $A \backslash B$ is stationary, then $\sigma_A \mathbb{Q} \nleqslant \sigma_B \mathbb{Q}$.
To prove (Q0) first assume $E \in NS_{\omega_1} | A$. Let C be a cub disjoint from $E \cap A$. The equation

$$f(t) = t | \max(ht(t) \cap C)$$

defines a regressive map on $\sigma_A \mathbb{Q} | (E \backslash C)$ so that $f^{-1}(s)$ is special for all s. Hence $E \backslash C \in NS_{\sigma_A \mathbb{Q}}$. The tree $\sigma_A \mathbb{Q} | (E \cap C)$ is special, as the increasing mapping

$$h(t) = \begin{cases} \frac{\sup(t)}{\sup(t)+1} & \text{if } \sup(t) \in \mathbb{Q}, \\ 1 & \text{otherwise} \end{cases}$$

from the tree to the rationals demonstrates. Now the equation $g(s) = \emptyset$ for $s \in \sigma_A \mathbb{Q} | (E \cap C)$ defines a regressive mapping such that $g^{-1}(\emptyset)$ is special. Hence $E \cap C \in NS_{\sigma_A \mathbb{Q}}$. Summa summarum, $E \in NS_{\sigma_A \mathbb{Q}}$. For the converse, suppose $E \in NS_{\sigma_A \mathbb{Q}}$. We show that $E \cap A$ is non-stationary. Suppose $E \cap A$ were stationary. Let f be a regressive mapping

$$f : \sigma_A \mathbb{Q} | E \to \sigma_A \mathbb{Q}$$

such that $f^{-1}(s)$ is special for all $s \in \sigma_A \mathbb{Q} | E$. Choose a countable $M \prec H(\lambda)$ for a large λ so that $f, A, E, \mathbb{Q}, \sigma\mathbb{Q} \in M$ and $\delta = M \cap \omega_1 \in E \cap A$. Let $R_n (n \in \mathbb{N})$ enumerate all antichains of $\sigma_A \mathbb{Q} | E$ which belong to M. Working in M we build a sequence $t_0 < t_1 < \cdots$ of elements of $M \cap \sigma_A \mathbb{Q} | E$ and a sequence $q_0 > q_1 > \cdots$ of rationals such that $\sup t_i < q_i$ and such that no proper end extension t of t_i with $\sup t < q_i$ is in R_i. To find t_i we first try to properly extend t_{i-1} to an element t of R_i with $\sup t < q_{i-1}$. If this works, then we let $t_i = t$ and choose $q_i < q_{i-1}$ with $\sup t_i < q_i$. Otherwise we choose an arbitrary proper end-extension t_i of t_{i-1} with $\sup t_i < q_{i-1}$ and choose $q_i < q_{i-1}$ so that $\sup t_i < q_i$. Additionally, we can arrange so that $\bigcup_{i<\omega} dom(t_i) = \delta$.

Let $t_\infty = \bigcup_i t_i$, and let $s = f(t_\infty)$. Then s is an initial segment of some t_i and hence $s \in M$. Therefore $f^{-1}(s) \in M$. Since $f^{-1}(s)$ is special, it is a union of countably many antichains and hence it contains an antichain R_i in M with $t_\infty \in R_i$. This contradicts the fact that no proper extension of t_i is in R_i.

For (Q1), suppose $A \cap B$ is non-stationary. We show that $NS_{\sigma_A \mathbb{Q} \otimes \sigma_B \mathbb{Q}}$ is trivial. By Proposition 23(iii) it suffices to show that the ideal

$$NS_{\sigma_A \mathbb{Q}} \cup NS_{\sigma_B \mathbb{Q}} = NS_{\omega_1}|A \cup NS_{\omega_1}|B,$$

is trivial. Since

$$\omega_1 \backslash A \in NS_A \cup NS_B \quad \text{and} \quad \omega_1 \backslash B \in NS_A \cup NS_B,$$

we have

$$\omega_1 \backslash (A \cap B) \in NS_A \cup NS_B.$$

Since also $A \cap B \in NS$, the ideal $NS_{\omega_1}|A \cup NS_{\omega_1}|B$ is trivial.

Now the proof of (Q2). Suppose $A \backslash B$ is stationary. Let $f : \sigma_A \mathbb{Q} \to \sigma_B \mathbb{Q}$ be strictly increasing. We may assume f is level-preserving. It follows that $s \mapsto (s, f(s))$ is a strictly increasing mapping $\sigma_A \mathbb{Q} \to \sigma_A \mathbb{Q} \otimes \sigma_B \mathbb{Q}$. Thus,

$$NS_{\sigma_A \mathbb{Q}} \supseteq NS_{\sigma_A \mathbb{Q} \otimes \sigma_B \mathbb{Q}} \supseteq NS_{\sigma_B \mathbb{Q}}.$$

Since, $\omega_1 \backslash B \in NS_{\sigma_B \mathbb{Q}}$, we get $\omega_1 \backslash B \in NS_{\sigma_A \mathbb{Q}}$, whence $A \backslash B$ is non-stationary, a contradiction.

Using (Q1) and (Q2) we can now prove:

(Q3) If A and B are disjoint stationary sets, then $\sigma_A \mathbb{Q}$ and $\sigma_B \mathbb{Q}$ are Karp trees of $(\mathfrak{A}, \mathfrak{B})$ but $\sigma_A \mathbb{Q} \otimes \sigma_B \mathbb{Q}$ is not.

Since $\sigma_A \mathbb{Q} \leqslant \sigma \mathbb{Q} \leqslant \mathbb{R}$, we can infer from Theorem 17 that $\exists$ has a winning strategy in $EF_{\sigma_A \mathbb{Q}}(\mathfrak{A}, \mathfrak{B})$. If $\exists$ had a winning strategy in $EF_{\sigma \sigma_A \mathbb{Q}}(\mathfrak{A}, \mathfrak{B})$, then $\sigma \sigma_A \mathbb{Q} \leqslant \mathbb{R}$, whence $\sigma_A \mathbb{Q}$ is special contrary to (Q1). This shows that $\sigma_A \mathbb{Q}$ is indeed a Karp tree of $(\mathfrak{A}, \mathfrak{B})$. The tree $T = \sigma_A \mathbb{Q} \otimes \sigma_B \mathbb{Q}$ satisfies by (Q1) $\sigma T \leqslant \mathbb{R}$ whence T cannot be a Karp tree of $(\mathfrak{A}, \mathfrak{B})$.

Now we can finish the proof of the theorem. Let $A_\alpha, \alpha < 2^{\omega_1}$, be a family of stationary subsets of ω_1 so that $A_\alpha \backslash A_\beta$ is stationary for $\alpha \neq \beta$. The trees $\sigma_{A_\alpha} \mathbb{Q}$ are Karp trees of $(\mathfrak{A}, \mathfrak{B})$ and they are mutually non-comparable by $\leqslant$, as we noted in (Q2). $\square$

Note that another construction of Karp trees T_1 and T_2 such that $T_1 \otimes T_2$ is not a Karp tree, was presented in [5] by Huuskonen. His Karp trees were special, whereas ours are non-special. It is not known if the models in Theorem 24 can be chosen to be of cardinality $\aleph_1$.

5. An application of coherent sequences

In this section we use coherent sequences to construct two models $\mathfrak{A}$ and $\mathfrak{B}$ such that $(\mathfrak{A}, \mathfrak{B})$ has a Scott tree which is a special Aronszajn tree. Earlier constructions gave non-special Scott trees of size 2^{ω} only [5, 7]. We also use coherent sequences to construct an uncountable family of non-equivalent Aronszajn trees in $\mathbb{T}$ without using CH. [4]

A sequence $a = \langle A_\alpha\colon \alpha < \omega_1\rangle$ is called *coherent* (see [23]) if it satisfies
(H1) $A_\alpha \subseteq \alpha$ for $\alpha < \omega_1$,
(H2) $A_\alpha \triangle (A_\beta \cap \alpha)$ is finite for $\alpha < \beta < \omega_1$.
A coherent sequence may be quite trivial, like the sequence $e = \langle \emptyset\colon \alpha < \omega_1\rangle$. For any coherent a we define

$$T(a) = \{s\colon dom(s) = \alpha < \omega_1, rng(s) \subseteq \{0,1\},$$
$$(\alpha \cap s^{-1}(1)) \triangle A_\alpha \text{ is finite}\}.$$

We can make $T(a)$ a tree by defining

$$s \leqslant s' \Leftrightarrow dom(s) \subseteq dom(s') \quad \text{and} \quad \forall \xi \in dom(s)(s(\xi) = s'(\xi)).$$

For any a the tree $T(a)$ is a normal tree of height ω_1 (and hence of size $\aleph_1$). Naturally, $T(a)$ may have uncountable branches, like $T(e)$, for example.

Two coherent sequences a and b form a *gap* if there is no $X \subseteq \omega_1$ which *splits* a and b, i.e.

$$\forall \alpha < \omega_1 (|A_\alpha \setminus X| < \omega \text{ and } |B_\alpha \cap X| < \omega).$$

Let $-a = \langle \alpha \setminus A_\alpha\colon \alpha < \omega_1\rangle$.

Lemma 25. *$T(a)$ is Aronszajn if and only if a and $-a$ form a gap. If $T(a) \cong T(b)$ and $T(a)$ is Aronszajn, then a and b form a gap.*

Proof. Suppose $h\colon T(a) \cong T(b)$. Suppose X splits a and b, that is, for all $\alpha < \omega_1$

$$|A_\alpha \setminus X| < \omega \quad \text{and} \quad |B_\alpha \cap X| < \omega.$$

If δ is a limit ordinal, there is $f(\delta) < \delta$ so that $A_\delta \cap [f(\delta), \delta[\subseteq X$ and

$$h(\chi_{A_\delta}) \cap [f(\delta), \delta[\cap X = \emptyset,$$

where χ_{A_δ} is the characteristic function of A_δ. By the Pressing Down Lemma there is $\gamma < \omega_1$ so that $f(\delta) = \gamma$ for δ in a stationary set S. We may assume $A_\delta \cap \gamma = A_{\delta'} \cap \gamma$ and $h(\chi_{A_\delta}) \cap \delta = h(\chi_{A_{\delta'}}) \cap \gamma$ for $\delta, \delta' \in S$. The set $\{\chi_{A_\delta}\colon \delta \in S\}$ cannot be a branch in $T(a)$. Hence there are $\delta \neq \delta' \in S$ so that $\chi_{A_\delta}(\alpha) \neq \chi_{A_{\delta'}}(\alpha)$ for some α. Then necessarily $\alpha \geqslant \gamma$,

[4] [14, Proposition 5] claims that Aronszajn trees are equivalent assuming $MA_{\aleph_1}$, but the proof contains an error.

whence $\alpha \in X$. Since h is an isomorphism, $h(\chi_{A_\delta})(\alpha) \neq h(\chi_{A_{\delta'}})(\alpha)$ whence $\alpha \notin X$, a contradiction. The claim is proved. □

One of the standard constructions of an Aronszajn tree essentially consists of a construction of a coherent sequence a so that a and $-a$ form a gap (see e.g. [9]). So we know there are coherent sequences a and b which form a gap. On the other hand, there are also coherent sequences a and b which do not form a gap. For example, e does not form a gap with any a because $X = \emptyset$ splits e and any other coherent sequence a. A less trivial example of a pair which does not form a gap is obtained by taking an a such that $T(a)$ forms an Aronszajn tree and looking at its copy $2a$ inside the even ordinals and also its copy $2a+1$ inside the odd ordinals, i.e. the sequences

$$2a = \langle \{2\xi:\ \xi \in A_\alpha\}:\ \alpha < \omega_1 \rangle,$$

$$2a+1 = \langle \{2\xi+1:\ \xi \in A_\alpha\}:\ \alpha < \omega_1 \rangle.$$

In [22, Section 6] there is a canonical construction which gives for every real $r \in \{0,1\}^\omega$ a coherent sequence a_r such that the family

$$T(a_r) \quad (r \in \{0,1\}^\omega)$$

includes the whole spectrum of trees starting from special Aronszajn trees, including Souslin trees (when r is a Cohen real) and ending with the ones which contain uncountable branches (which are all isomorphic to $T(e)$). Suppose now a and b are arbitrary coherent sequences.

Theorem 26. *Suppose a and $-a$ form a gap. Then $T(a)$ is a Scott tree of $(T(e), T(a))$.*

Proof. The winning strategy of $\forall$ in $EF_{\sigma T(a)}(T(e), T(a))$ is easy to construct, since $T(e)$ has an uncountable branch whereas $T(a)$ is Aronszajn. Player $\forall$ chooses an uncountable branch of $T(e)$ and plays, move by move, elements of the branch in ascending order. At the same time he has to submit moves in $\sigma T(a)$. The idea is to use the sequence of the previous moves of $\exists$ in $T(a)$ as the next move of $\forall$ in $\sigma T(a)$. Thus $\forall$ can always continue playing the game after $\exists$ has moved, and eventually $\exists$ comes to the end of a countable maximal branch in $T(a)$. Thus, to prove the theorem it suffices to show that $\forall$ does not have a winning strategy in $EF_{T(a)}(T(e), T(a))$. Suppose ρ were such a strategy. Let $u \in T(a)$ be arbitrary. Let us play $EF_{T(a)}(T(e), T(a))$ so that $\forall$ follows ρ and $\exists$ plays as follows. Suppose $\forall$ plays on round α an element t_α of $T(a)$ and an element x_α of $T(a)$ (or $T(e)$). Now $\exists$ responds with y_α such that $dom(y_\alpha) = dom(x_\alpha)$ and for all $\xi \in dom(y_\alpha)$:

$$y_\alpha(\xi) = x_\alpha(\xi) + u(\xi) \pmod 2$$

Naturally, this is only possible if $dom(x_\alpha) \leqslant dom(u)$. But playing in this way $\exists$ cannot otherwise lose. To see that this is the case, suppose $x_\beta \leqslant x_\alpha$ where $\beta < \alpha$. Thus for

$\xi \in dom(x_\beta)$

$$x_\beta(\xi) = x_\alpha(\xi), t_\beta(\xi) = t_\alpha(\xi), u_\beta(\xi) = u_\alpha(\xi)$$

and hence

$$y_\beta(\xi) = x_\beta(\xi) + t_\beta(\xi) + u_\beta(\xi) = x_\alpha(\xi) + t_\alpha(\xi) + u_\alpha(\xi) = y_\alpha(\xi).$$

We also have to check that $y_\alpha \in T(b)$ (we are assuming $x_\alpha \in T(a)$). Let $\delta = dom(y_\alpha)$. Let $I \subseteq \delta$ be finite so that

$$\xi \in \delta \setminus I \Rightarrow x_\alpha(\xi) = \chi_{A_\delta}(\xi), t_\alpha(\xi) = \chi_{A_\delta}(\xi), u_\alpha(\xi) = \chi_{B_\delta}(\xi).$$

Then for $\xi \in \delta \setminus I$ we have

$$y_\alpha(\xi) = x_\alpha(\xi) + t_\alpha(\xi) + u_\alpha(\xi) = \chi_{A_\delta}(\xi) + \chi_{A_\delta}(\xi) + \chi_{B_\delta}(\xi) = \chi_{B_\delta}(\xi).$$

Since $\forall$ is playing a winning strategy, he will eventually move $t^*_\alpha \in T(a)$ and $x^*_\alpha \in T(a)$ (or $x^*_\alpha \in T(e)$) so that $dom(x^*_\alpha) > dom(u)$. Let us denote the least such α by θ_u. By the Pressing Down Lemma, there is a club $C \subseteq \omega_1$ such that for all $u \in T(a)$, if $dom(u) \in C$, then $\theta_u \geqslant dom(u)$. If $U = \langle u_\xi \colon \xi < \delta \rangle$ is in $\sigma T(a)$ such that $dom(u_\xi) \in C$ for all $\xi < \delta$ then $\langle t^*_{u_\xi} \colon \xi < \delta \rangle$ is an ascending chain in $T(a)$. Moreover, there is a sequence of rounds of $EF_{T(a)}(T(e), T(a))$ in which $\forall$ plays ρ, makes the moves $\langle t^*_{u_\xi} \colon \xi < \delta \rangle$ among his moves in $T(a)$, and has not beaten $\exists$ yet. Therefore, the sequence $\langle t^*_{u_\xi} \colon \xi < \delta \rangle$ has an upper bound t_U in $T(a)$. If U' extends U in $\sigma T(a)$, then $t_{U'}$ extends t_U in $T(a)$. This leads easily to a contradiction with Theorem 3. □

Theorem 26 gives a Scott tree that is special. The Scott tree constructed in the proof of Theorem 17 was non-special.

Theorem 27. *For every $\alpha < \omega_1$ there is a sequence $\langle T_\beta \colon \beta < \alpha \rangle$ of Aronszajn trees of the form $T(a)$ such that $T_\beta < T_\gamma$ for all $\beta < \gamma < \omega_1$.*

Proof. Let $\zeta < \omega_1$ be given. We may assume that $\omega^\zeta = \zeta$. Every ordinal α can be uniquely expressed as $\alpha = \zeta \cdot \gamma + \beta$, where $\beta < \zeta$. Let $a = \langle A_\xi \colon \xi < \omega_1 \rangle$ be a coherent sequence so that $(a, -a)$ forms a gap and $A_\xi \subseteq \mathrm{Succ}$ for each $\xi < \omega_1$. Let for each $\alpha < \zeta$

$$A^\alpha_\xi = \{\zeta \cdot \gamma + \beta \cdot \alpha < \xi : \zeta \cdot \gamma + \beta \in A_\xi, \beta < \zeta\}.$$

Let $a^\alpha = \langle A^\alpha_\xi \colon \xi < \omega_1 \rangle$. Note that $(a^\alpha, -a^\alpha)$ forms still a gap so we get Aronszajn trees by letting $T_\alpha = T(a^\alpha)$. Suppose there were a strictly increasing $f : T_{\alpha'} \to T_\alpha$ with $\alpha < \alpha' < \omega_1$. We may assume f is level-preserving. For every $\xi < \omega_1$ there is a finite set X_ξ such that $f(\chi_{A^{\alpha'}_\xi}) = \chi_{A^\alpha_\xi} \bmod X_\xi$. By the Pressing Down Lemma there is a stationary set $S \subseteq \omega_1$ such that X_ξ is constant X for $\xi \in S$. Let $\xi < \theta$ be ordinals in $S \setminus X$. Let $\zeta \cdot \gamma + \beta \cdot \alpha' = \min(A^{\alpha'}_\xi \triangle A^{\alpha'}_\theta)$. Since $\zeta \cdot \gamma + \beta \in \mathrm{Succ}$, we have $\beta \neq 0$ and hence $\zeta \cdot \gamma + \beta \cdot \alpha < \zeta \cdot \gamma + \beta \cdot \alpha'$. We may assume $\zeta \cdot \gamma + \beta \cdot \alpha \notin X$ and therefore

$\zeta \cdot \gamma + \beta \cdot \alpha \in A_\xi^\alpha \triangle A_\theta^\alpha$. This means that $\chi_{A_\xi^{\alpha\prime}}$ and $\chi_{A_\theta^{\alpha\prime}}$ split higher than their images under f, a contradiction. $\square$

6. The ordering of trees

The second number class is a lower level analogue of $\mathbb{T}$, for we may identify an ordinal α with the tree B_α of finite descending chains of elements of α. The tree B_α has no infinite branches and $\alpha \leqslant \beta \Leftrightarrow B_\alpha \leqslant B_\beta$. Using these trees, the structure of $\{T \in \mathbb{T} : T$ has no infinite branches$\}$ is easy to describe. It is that of $\langle \omega_2, < \rangle$ (up to $\equiv$).

Let us then consider trees of countable height. Naturally, we may restrict to the case of limit height, as in successor height there is always just one tree, up to equivalence.

Theorem 28. *For every countable limit ordinal $\delta > \omega$ there is a family $\mathbb{F}_\delta$ of size $2^{\aleph_1}$ of elements of $\mathbb{T}$ of height δ such that $T \nleqslant T'$ for every two different elements T and T' of $\mathbb{F}_\delta$.*

Proof. We shall first consider the special case that $\delta = \omega + \omega$. For limit $v < \omega_1$ let $f_v : \omega \to v$ be an increasing cofinal map. For $S \subseteq \omega_1 \cap Lim$, let $T(s)$ be the tree consisting of all $f_v|_n, v \in S, n < \omega$, and all sequences $f_v\frown s$, where $s \in B_v$, and B_v is the tree of sequences $\langle \alpha_0, \ldots, \alpha_n \rangle$ so that $v > \alpha_0 > \cdots > \alpha_n$.

Claim. *If $S \backslash S'$ is stationary, then $T(S) \nleqslant T(S')$.*

Suppose $H : T(s) \to T(S')$ is increasing. If $v \in S$, let $h(v)$ be the unique $\zeta \in S'$ such that $f_\zeta \leqslant H(f_v)$. The map H gives rise to an increasing map $B_v \to B_\zeta$, whence $v \leqslant \zeta$, that is, $v \leqslant h(v)$. Let $M \, prec \, H(\aleph_2)$ be a countable structure with $S, S', h, H \in M$ and $\delta = \omega_1 \cap M \in S \backslash S'$. Then $\gamma = h(\delta) > \delta$. Let $m < \omega$ so that $f_\gamma(m) \geqslant \delta$. Since

$$f_\delta \,|\, 0 < \cdots < f_\delta \,|\, (m+1)$$

are in M, also

$$H(f_\delta|0) < \cdots < H(f_\delta \,|\, (m+1))$$

are in M. Moreover, $f_\delta \,|\, n < f_\delta$ implies $H(f_\delta \,|\, n) < f_\gamma$. Therefore

$$H(f_\delta \,|\, (m+1))(m) = f_\gamma(m) \geqslant \delta,$$

a contradiction, because $H(f_\delta \,|\, (m+1))(m) \in M$, but $\delta \notin M$. The Claim is proved.

For $\{S_\alpha \colon \alpha < 2^{\omega_1}\}$ so that $S_\alpha \backslash S_\beta$ is stationary for all $\alpha \neq \beta$, we have the family $\mathbb{F}_{\omega+\omega} = \{T(S_\alpha) \colon \alpha < 2^{\omega_1}\}$ with the desired property that $T \nleqslant T'$ for $T \neq T' \in \mathbb{F}_{\omega+\omega}$.

Let us now return to the general case of a limit ordinal $\delta < \omega_1$. Let B_α^v be obtained from B_α by inserting a chain of length $f_v(n+1) - f_v(n)$ between any nodes

$$\langle \alpha_0, \ldots, \alpha_n \rangle \quad \text{and} \quad \langle \alpha_0, \ldots, \alpha_{n+1} \rangle$$

of B_α. Clearly, $\alpha \leqslant \beta \Leftrightarrow B_\alpha^\nu \leqslant B_\beta^\nu$. Let $T^\delta(S)$ be the tree of sequences $f_\nu \mid n$, $\nu \in S$, and the sequences $f_\nu^\frown s$, where $s \in B_\nu^\delta$. Now $T^\delta(S)$ has height δ and still. If $S \setminus S'$ is stationary, then $T^\delta(S) \nleqslant T^\delta(S')$. The required family $\mathbb{F}_\delta$ consists of $T^\delta(S_\alpha), \alpha < 2^{\omega_1}$, with $\{S_\alpha \colon \alpha < 2^{\omega_1}\}$ as above. □

Theorem 29. *For every countable limit ordinal $\delta > \omega$ there is a family $\mathbb{F}'_\delta = \{T_\alpha \colon \alpha < \omega_1\}$ of elements of $\mathbb{T}$ of height δ such that $T_\alpha < T_\beta$ for every $\alpha < \beta < \omega_1$.*

Proof. Let $\{S_\alpha \colon \alpha < \omega_1\}$ be a family of subsets of ω_1 so that $S_\alpha \supseteq S_\beta$ and $S_\alpha \setminus S_\beta$ is stationary for $\alpha > \beta$. Let T_α be $T^\delta(S_\alpha)$ (see the proof of Theorem 28). □

The non-persistent trees are a special subclass of $\mathbb{T}$ which can be analyzed in terms of the elements of $\mathbb{T}$ of countable height. For $T \in \mathbb{T}$ define

$$d(T) = \{t \in T : \forall \alpha < \omega_1 \, \exists t' \in T (t \leqslant t' \wedge ht(t') \geqslant \alpha\},$$

$$d_\nu(T) = \bigcap_{\alpha < \nu} d(d_\alpha(T)) \qquad (d_0(T) = T),$$

$$\textit{p-rank}\ (T) = \text{least } \alpha \text{ such that } d_{\alpha+1}(T) = \emptyset.$$

This concept occurs already in [5] but in a different form.

Clearly, a tree T is persistent if and only if $d_\infty(T) \neq \emptyset$. Let T_α^0 be the tree $(\bigoplus_{\alpha<\omega_1} \alpha) \cdot B_\alpha$ and T^0 the tree $(\bigoplus_{\alpha<\omega_1} \alpha) \cdot \omega$. The following result follows from Lemma 20 of [5]:

Proposition 30. *Let T be a tree. Then*
(1) *p-rank*$(T) \geqslant \alpha \Leftrightarrow T_\alpha^0 \leqslant T$.
(2) *p-rank*$(T) < \alpha \Leftrightarrow T < T_\alpha^0 \Leftrightarrow \sigma T \leqslant T_\alpha^0$.
(3) *T persistent* $\Leftrightarrow T^0 \leqslant T$.
(4) *T non-persistent* $\Leftrightarrow T \leqslant T^0 \Leftrightarrow \sigma T \leqslant T^0$.

Thus, we have the sequence

$$T_0^0 < T_1^0 < \cdots < T_\alpha^0 < \cdots < T^0 \qquad (\alpha < \omega_2)$$

of length ω_2 of trees in $\mathbb{T}$ that are comparable by $\leqslant$ to all other trees in $\mathbb{T}$. Between these trees, that is, on persistency rank levels, we have a rich structure as the following analysis reveals.

Let $\mathbb{P}_\alpha$ be the family of $\equiv$-equivalence classes $[T]$ of $T \in \mathbb{T}$ with p-rank$(T) = \alpha$.

Theorem 31. $(\mathbb{P}_0, \leqslant) \cong (\mathbb{P}_\alpha, \leqslant)$ *for all $\alpha < \omega_2$.*

Proof. If $T \in \mathbb{P}_0$, let T^* be obtained from T by replacing every maximal node t of T by the elements $t^\frown s, s \in T_\alpha^0$. Now, $d_\alpha(T^*) = T$. Thus $T \mapsto T^*$ gives rise to a map of $\mathbb{P}_0$ into $\mathbb{P}_\alpha$. If $[U] \in \mathbb{P}_\alpha$, then by definition $d_{\alpha+1}(U) = \emptyset$, so $[d_\alpha(U)] \in \mathbb{P}_0$. Moreover,

$d_\alpha(U)^* \equiv U$. It is also obvious that $T \leqslant U \to T^* \leqslant U^*$. Thus the mapping $[T] \mapsto [T^*]$ is the required isomorphism. □

Corollary 32. *The family of elements of* $\mathbb{T}$ *of p-rank* α *has antichains of size* $2^{\aleph_1}$ *and chains of length* ω_1.

We have observed that the trees in $\mathbb{T}$ divide into two classes: those T with $T < T^0$ (the non-persistent ones) and those T with $T^0 \leqslant T$ (the persistent ones). We have established hierarchies in the first class that rather well reflect the structure of $\mathbb{T}$ under $\leqslant$. When we move on to the second class, the situation is much more complicated.

A tree of height and cardinality ω_1 which is persistent is called an $\aleph_1$-tree. We shall study the following disjoint classes of $\aleph_1$-trees:

- The class of special Aronszajn trees.
- The class of Souslin trees.
- The class of special trees with no Aronszajn subtrees.
- The class of non-special trees with no Aronszajn subtrees.

Theorem 33. *Assume* $2^\omega < 2^{\omega_1}$. *Then there are special Aronszajn trees* T *and* T' *such that* $T \not\leqslant T'$ *and* $T' \not\leqslant T$.

Proof. Let $a = \langle A_\alpha \colon \alpha < \omega_1 \rangle$ be a coherent sequence such that $a, -a$ form a gap. Let S be the Aronszajn tree $T(a)$ and $S_\alpha = \{s \in T(a) \colon \mathrm{dom}(s) < \alpha\}$. For any $\sigma \in 2^{<\omega_1}$ we define a tree T_σ of height $\mathrm{len}(\sigma)$ so that no increasing $h : S_{\mathrm{len}(\sigma)} \to T_\sigma$ can be extended to both $T_{\sigma\frown 0}$ and $T_{\sigma\frown 1}$. We define T_σ so that if $\sigma \in 2^{\omega_1}$, then $T_\sigma = \bigcup_{\alpha<\omega_1} T_{\sigma|\alpha}$ is an Aronszajn tree. Now we shall use: □

Weak diamond principle: If $2^\omega < 2^{\omega_1}$ then for all $F : (2^\omega)^{<\omega_1} \to 2$ there is $g : \omega_1 \to 2$ such that for all $h : \omega_1 \to 2^\omega$ the set $\{\alpha < \omega_1 : g(\alpha) = F(h|_\alpha)\}$ is stationary [3].

If $h : S_\alpha \to T_\sigma$ or $h : T_\sigma \to S_\alpha$ is increasing and $\mathrm{len}(\sigma) = \alpha$, let

$$F(h,i) = \begin{cases} 0 & \text{if } (i=0, h : S_\alpha \to T_\sigma \text{ and} \\ & h \text{ extends to an increasing map } S_{\alpha+1} \to T_{\sigma\frown 1}) \\ & \text{or } (i=1, h : T_\sigma \to S_\alpha \text{ and} \\ & h \text{ extends to an increasing map } T_{\sigma\frown 1} \to S_{\alpha+1}) \\ 1 & \text{otherwise.} \end{cases}$$

By the Weak Diamond Principle, there is $g : \omega_1 \to 2$ such that for all h and i, $\{\alpha < \omega_1 : g(\alpha) = F(h|_\alpha, i)\}$ is stationary.

Claim. $S \not\leqslant T_g$ *and* $T_g \not\leqslant S$.

Suppose $h : S \to T_g$ is order-preserving. Let $i = 0$. Let α be such that $g(\alpha) = F(h|_\alpha, i)$ and $h|_\alpha$ is an increasing map $S_\alpha \to T_{g|_\alpha}$. Then $h|_\alpha$ extends to increasing $h|_{\alpha+1} : S_{\alpha+1} \to T_{g|\alpha}(g(\alpha))$. If $g(\alpha) = 1$, then $F(h|_\alpha, 0) = g(\alpha) = 0$. Hence $g(\alpha) = 0$. But then $h|_\alpha$

extends to both

$$S_{\alpha+1} \longrightarrow T_{g|\alpha^\frown 0} \text{ and } S_{\alpha+1} \longrightarrow T_{g|\alpha^\frown 1},$$

a contradiction. The case that $T_g \leqslant S$ is ruled out similarly, using $i=1$. $\square$

Remark. A result like the above theorem was proved by Avraham and Shelah [1] for the ordering $\leqslant'$ of trees by homomorphic embedding. The following result strengthens an earlier result of Lindström [12] about the ordering $\leqslant'$.

Theorem 34. *Assume $\diamond$. Then there are Souslin trees T and T' such that $T \nleqslant T'$ and $T' \nleqslant T$.*

Proof. The proof is similar to the proof of Theorem 33, and therefore omitted. $\square$

Theorem 35. *There is a family $\mathbb{G}$ of size $2^{\aleph_1}$ of $\aleph_1$-trees such that $T \nleqslant T'$ and $T' \nleqslant T$ for any distinct members T and T' of $\mathbb{G}$.*

Proof. For $\delta \leqslant \omega_1 \cap Lim$ let $f_\delta : \omega \to \delta$ be cofinal and $C_\delta = rng(f_\delta)$. We assume $f_\delta(0)=0$ always. For any $S \subseteq \omega_1 \cap Lim$ let $P(S)$ be the tree of sequences $s \in \omega_1^{<\omega_1}$ such that
(P1) $\xi \leqslant \eta < dom(s) \Rightarrow \xi \leqslant s(\xi) \leqslant s(\eta)$.
(P2) $rng(s)$ is finite or $|\, rng(s) \triangle C_\sigma \,| < \omega$ for some $\delta \in S$.
The tree $P(\emptyset)$ has been first introduced for the purpose of giving a proof of Theorem 3.3 in [17]. One of its properties, listed in Theorem 3.3 of [17], shows that it can serve as another example of the minimal persistent tree. So, in particular, $P(\emptyset) \equiv T^0$, where T^0 is the tree $(\bigoplus_{\alpha<\omega_1} \alpha) \cdot \omega$ introduced above.

Claim. If $S \backslash S'$ *is stationary, then* $P(S) \nleqslant P(S')$.

Let $f : P(S) \to P(S')$ be strictly increasing. Let $M \prec H(\aleph_2)$ be countable so that $f, S, S' \in M$ and $\delta = M \cap \omega_1 \in S \backslash S'$. We define an element t of $P(S)$ as follows: $dom(t) = \delta$ and $t(\xi) = f_\delta(n)$ for the least n with $f_\delta(n) \geqslant \xi$. Let $s = f(t) \in P(S')$. We may assume f preserves height. Clearly, $ht(t) = \delta$, whence $ht(s) = \delta$. Since $\delta \notin S'$, there is $\beta < \delta$ so that $s(\xi)$ has a fixed value γ for $\xi \in [\beta, \delta)$. By (P1), $\delta \leqslant \gamma$. For $n < \omega$, let $t_n = t \,|\, f_\delta(n)$. Then $\langle t_n\colon n < \omega \rangle$ is a sequence of elements of $P(S) \cap M$ converging to t. Hence $\langle f(t_n)\colon n < \omega \rangle$ is a sequence of elements of $P(S') \cap M$ converging to s. Let $m < \omega$ such that $s \,|\, (\beta+1) \leqslant f(t_M)$. But then $\gamma = \max(rng(f(t_m))) \in M$, a contradiction. $\square$

Remark. (1) The proof of the above claim actually gives the stronger result that if $S \setminus S'$ is stationary then for every cub $C \subseteq \omega_1$ we have $P(S)|_C \nleqslant P(S')$. In particular, $P(S)$ and $P(S')$ are not isomorphic on a cub.

(2) The above trees $P(S)$ are special. To see this we define the following regressive map π on elements of limit height of $P(S)$. If $s \in P(S)$ has infinite range, let $\pi(s) = \emptyset$

(= the root of $P(S)$). If $s \in P(S)$ has finite range, there is $\beta < dom(s)$ so that $s(\xi)$ is constant γ for $\xi \in [\beta, dom(s))$. In this case we let $\pi(s) = s|_{\beta+1}$. The pre-image of $\emptyset$ under π is an antichain, hence special. The pre-image of any other element is countable, hence special. So $P(S)$ is special by the Pressing Down Lemma for trees of [18].

(3) The trees $P(S)$ do not have Aronszajn subtrees. This is proved for $P(\emptyset)$ in [17] but the proof works for all $P(S)$. It is an easy consequence of this that $T \not\leqslant P(S)$ for all Aronszajn trees T.

7. The incomparability problem

If $A \subseteq \omega_1$, let

$$T(A) = \{s : s \text{ is a continuous ascending sequence of elements of } A\}.$$

These trees have cardinality 2^ω, unless $ht(T(A)) \leqslant \omega$. So they belong to the family $\mathbb{T}$ only if we assume CH. These trees are mainly interesting in the case that A is bistationary. For example, if A is stationary, then for every $\alpha < \omega_1$ there is a continuous ascending sequence of length α of elements of A (see e.g. [8, page 60]), whence $ht(T(A)) = \omega_1$. On the other hand, we do not want A to contain a cub, because then $T(A)$ has an uncountable branch.

Proposition 36. (i) *If $A \cap B$ is non-stationary, then $T(A) \otimes T(B) \leqslant T^0$, In particular, $T(A) \otimes T(B)$ is special.*
(ii) *$T(A)$ is special if and only if A is non-stationary.*
(iii) *If $A \setminus B$ is stationary, then $T(A) \not\leqslant T(B)$.*

Proof. (i) Suppose $C \subseteq Lim$ is a cub disjoint with $A \cap B$. To prove $T(A) \otimes T(B) \leqslant T^0$, it suffices to describe a winning strategy of Player II in the following game: Two Players I and II pick elements of the trees. During a round of the game Player I picks from $T(A) \otimes T(B)$ and II picks from T^0. Both pick in ascending order. Player II wins if he can keep playing. Player I starts the game and plays first at limit stages. During the game Player II decides to call some future rounds *critical*. It turns out that after ω critical rounds have been played, Player I cannot move and loses. So we can talk about the nth critical round. Round number 0 is the 0th critical round. Suppose δ rounds have been played.

Case 1: Round δ is the nth critical round. Let c_n be the minimal element of $C \setminus \{\delta\}$ which is greater than all ordinals played so far. The move of Player II on this round is $(g_n, (0, c_n), n)$. Round c_n is declared the $(n+1)$th critical round. If $n = 0$, then $g_n = \emptyset$. Otherwise, $g_n(i) = g_{n-1}(i)$ for $i < n-1$ and $g_n(n-1) = \{(\xi, c_{n-1}) \colon \xi < c_{n-1}\}$.

Case 2: $\delta = \gamma + 1$. Suppose the previous critical round was the nth critical round. Suppose Player II played $(f, (\xi, c_n), n)$ on round γ. His move on this round will be $(f, (\xi + 1, c_n), n)$.

Case 3: $\delta = \cup\delta$ is not critical. Suppose the previous critical round was the nth critical round. Suppose Player II played (f, ξ_γ, n) on round γ, for $c_n \leqslant \gamma < \delta$. His move on this round will be $(f, (\xi + 1, c_n), n)$.

Case 4: Round δ is the limit of critical rounds. Suppose Player I plays $(t,s) \in T(A) \otimes T(B)$. Let $c = \sup\{c_n\colon n < \omega\}$. Since C is club, $c \in C$, and therefore $c \notin A \cap B$. By construction, $c \in rng(t) \cap rng(s)$, a contradiction. So the move of Player I was not legal, and Player II has won.

(ii) If A is non-stationary, then $T(A)$ is special by (i). Suppose then A is stationary but $T(A)$ is the union of the antichains B_n, $n < \omega$. Choose a countable $M \prec H(\lambda)$ for a large λ so that $A, T(A), \{B_n\colon n < \omega\}, B_n(n < \omega) \in M$ and $\delta = M \cap \omega_1 \in A$. Working in M we can build a sequence $t_0 < t_1 < \cdots$ of elements of $M \cap T(A)$ such that $\bigcup_{i<\omega} dom(t_i) = \delta$ and no proper end-extension t of t_i with is in B_i. Let $t_\infty = \bigcup_i t_i$, and let $t_\infty \in B_n$. This contradicts the fact that no proper extension of t_n is in B_n.

(iii) Suppose $A \setminus B$ is stationary but $f : T(A) \to T(B)$ is strictly increasing. We may assume that f is level preserving. Choose a countable $M \prec H(\lambda)$ for a large λ so that $A, T(A), B, T(B), f \in M$ and $\delta = M \cap \omega_1 \in A \setminus B$. Working in M we can build a sequence $t_0 < t_1 < \cdots$ of elements of $M \cap T(A)$ such that $\bigcup_{i<\omega} ot(t_i) = \delta$ and $\max(t_i) < ot(t_{i+1})$. Let $t_\infty = \bigcup_i t_i \cup \{\delta\}$. By construction, $t_\infty \in T(A)$ but $f(t_\infty) \notin T(B)$, a contradiction. □

Corollary 37. *If A and B are stationary sets such that $A \cap B$ is non-stationary, then $T(A) \otimes T(B) \equiv T^0$.*

Proof. Clearly, $T(A)$ and $T(B)$ are persistent, whence $T^0 \leqslant T(A) \otimes T(B)$. The rest follows form the previous theorem. □

Proposition 38. *If $A \subseteq \omega_1$ is costationary, then $T(A)$ contains no Aronszajn subtrees.*

Proof. This follows from the following more general result. □

Proposition 39. *If $A \subseteq \omega_1$ is bistationary and T is Aronszajn, then $T(A) \nleqslant T$ and $T \nleqslant T(A)$.*

Proof. Suppose $T(A) \leqslant T$, and $f : T(A) \to T$ is strictly increasing. Choose a countable $M \prec H(\lambda)$ for a large λ so that $A, T(A), T, f \in M$ and $\delta = M \cap \omega_1 \in A$. Let $\{s_n\colon n < \omega\}$ be an enumeration of $T_{\delta+1}$. Working in M we can build a sequence $t_0 < t_1 < \cdots$ of elements of $M \cap T(A)$ such that $\bigcup_{i<\omega} ot(t_i) = \delta$, t_i cannot be extended to s_i, and $\max(t_i) < ot(t_{i+1})$. Let $t_\infty = \bigcup_i t_i \cup \{\delta\}$. By construction, $t_\infty \in T(A)$ but $f(t_\infty) \notin T$, a contradiction. The other claim is proved similarly. □

Is it true that for every tree T with $T^0 < T < \omega_1$ there is a costationary $A \subseteq \omega_1$ such that $T(A) \nleqslant T$? Not necessarily: By a result of Mekler and Shelah [13] the existence of a tree T such that $T^0 < T < \omega_1$ and $T(A) \leqslant T$ for all costationary $A \subseteq \omega_1$, is consistent and independent of $ZFC + GCH + \diamondsuit$. This leads to the natural question whether some strong form of $\diamondsuit$ still true in the constructible universe decides this problem.

We shall prove below (Theorem 48) that if $V=L$, then for every tree T with $T^0<T<\omega_1$ there is a costationary $A\subseteq\omega_1$ such that $T(A)\not\leq T$ and $T\not\leq T(A)$. Our proof also gives, assuming $V=L$, for every tree T with $T^0<T<\omega_1$ a Souslin tree S such that $S\not\leq T$ (Theorem 44).

If $T=\langle\omega_1,<_T\rangle$ is a tree and $\delta<\omega_1$, we use $T|\delta$ to denote the tree $\langle\delta,<_T\cap\delta\times\delta\rangle$.

Definition 40. We use the symbol $\diamondsuit^+$ to denote the following assumption: There exist transitive p.r. closed sets M_α, $\alpha<\omega_1$, such that for all $X\subseteq\omega_1$ there exists a club $C_X\subseteq\omega_1$ such that

$$\forall\delta\in C_X(\{X\cap\delta, C_X\cap\delta\}\subseteq M_\delta). \tag{3}$$

Suppose we are given a tree $T=\langle\omega_1,<_T\rangle$. We use the symbol $\diamondsuit^+_T$ to denote the following assumption: There exist transitive p.r. closed sets M_α, $\alpha<\omega_1$, such that for all $X\subseteq\omega_1$ there exists a club $C_X\subseteq\omega_1$ such that (3) holds and the set

$$D_T=\{\delta\in C_T : M_\delta\models \text{“}\delta=\omega_1 \text{ \& } T|_\delta \text{ has no uncountable branches”}\}$$

is stationary.

Theorem 41. *If $\diamondsuit^+_T$ is true then $T(\omega_1\backslash D_T)\not\leq T$.*

Proof. We need two lemmas:

Lemma 42. *We can choose $X\mapsto C_X$ in the definition of $\diamondsuit^+_T$ such that additionally*:
(a) *If X is club in ω_1, then $\forall\delta\in C_X(X\cap\delta$ is a club in $\delta)$.*
(b) *If X is club in ω_1, then $\forall\delta\in C_X\cap D_T(C_X\cap\delta$ is a club in $\delta)$.*

Proof. Let $X\mapsto C_X$ satisfy $\diamondsuit^+_T$ and define $X\mapsto\overline{C}_X$ which agrees with $X\mapsto C_X$ if X is non-club. If X is club, then $\overline{C}_X=\lim X\cap\lim C_X$. □

So assume from now on (a) and (b). Let

$$B_T=\{\delta\in D_T\cap C_{D_T} : M_\delta\models \text{“}D_T\cap\delta \text{ is non-stationary”}\}.$$

Lemma 43. *B_T is stationary.*

Proof. Let X be a given club in ω_1. We need to show that $X\cap B_T\neq\emptyset$. Let δ be the minimal point of $C_X\cap C_{D_T}\cap D_T$. It suffices to prove $\delta\in X\cap B_T$. By (a), $\delta\in X$. By (b), $M_\delta\models$ “$(C_X\cap\delta)\cap(C_{D_T}\cap\delta)$ is a club in δ disjoint from $D_T\cap\delta$”. So $M_\delta\models$ “$D_T\cap\delta$ is non-stationary”. i.e. $\delta\in B_T$. □

Proof of Theorem 41. Suppose $f:T(\omega_1\backslash D_T)\to T$ is strictly increasing. Pick $\delta\in C_f\cap B_T$. Then $M_\delta\models$ “$f_\delta=f|_\delta:T(\omega_1\backslash D_T)|_\delta\to T|_\delta$ is strictly increasing”. Now $\delta\in B_T\subseteq D_T$, so $M_\delta\models$ “$\delta\subseteq\omega_1$ and $T|_\delta$ has no uncountable chains”. Pick a set $E\in M_\delta$ such that

$$M_\delta\models \text{“}E \text{ is a club in } \omega_1 \text{ disjoint from } D_T\text{”}.$$

Let $b = \{f_\delta(E \cap (\xi + 1)) \colon \xi < \delta\}$. Note that $E \cap (\xi + 1) \in T(\omega_1 \setminus D_T)|\delta$. Then $M_\delta \models$ "b is an uncountable chain in $T|_\delta$", a contradiction. □

Theorem 44. *If $\diamondsuit_T^+$, then there is a Souslin tree S such that $S \not\leqslant T$.*

Proof. Construct $S = \langle \omega_1, <_S \rangle$ recursively. Let $S_\alpha = [\omega\alpha, \omega\alpha + \omega)$. The construction is made by diagonalizing over a given $\diamondsuit_T^+$ sequence $(M_\alpha)_{\alpha < \omega_1}$ which satisfies $\{M_\alpha \colon \alpha < \delta\} \in M_\delta$. If $\delta \in D_T$ then one extends a δ-chain of $S|_\delta$ only if it intersects all dense open subsets $A \subseteq S|_\delta$ with $A \in M_\delta$. For other δ choose to extend all δ-branches of $S|_\delta$ which are in M_δ. To prove $S \not\leqslant T$, suppose $f : S \to T$ is strictly increasing. Pick $\delta \in C_{D_T} \cap C_f \cap B_T \cap C_S$. We show that $M_\delta \models$ "$S|_\delta$ has an uncountable branch", contradicting $\delta \in D_T$. Since $M_\delta \models$ "$D_T \cap \delta$ is not stationary", we can pick a set $E \in M_\delta$ such that $M_\delta \models$ "E club disjoint from $D_T \cap \delta$". In M_δ, recursively pick a chain $t_\alpha (\alpha \in E)$ of $S|_\delta$. Let $\delta_0 = \min(E)$ and $t_{\delta_0} = \min(S_{\delta_0})$. If γ^+ is $\min(E \setminus \gamma)$, let t_{γ^+} be the minimal extension of t_γ to an element of S_{γ^+}. At $\lambda \in \lim E$ let b be the branch determined by $t_\gamma, \gamma \in E \cap \lambda$. This branch is in M_λ because we extended all branches. We may assume $E \cap \lambda \in M_\lambda$. □

Definition 45. Let M_δ $(\delta < \omega_1)$ be a $\diamondsuit^+$-sequence. It is called *Π_n^1-reflecting* if for every Π_n^1-sentence ϕ and for every structure $\langle \omega_1, \omega, R_i \rangle_{i=1}^\infty$ with

$$\langle \omega_1, \omega, R_i \rangle_{i=1}^\infty \models \phi$$

there exists $\delta < \omega_1$ such that

$$M_\delta \models \text{“}\phi \text{ is true in } \langle \delta, \omega, R_i|_\delta \rangle_{i=1}^\infty\text{”}.$$

$\diamondsuit^+(\Pi_n^1)$ is the assumption that there is a Π_n^1-reflecting sequence. $\diamondsuit^+(\Pi_{<\omega}^1)$ is the assumption that there is a sequence as above which is Π_n^1-reflecting for all $n < \omega$.

Proposition 46. *$\diamondsuit^+(\Pi_1^1)$ implies $\diamondsuit_T^+$ for all trees $T = \langle \omega_1, <_T \rangle$ without uncountable branches.*

Proof. If $T = \langle \omega_1, <_T \rangle$ is a tree, we can find a Π_1^1-sentence ϕ such that $(\omega_1, \omega, T) \models \phi$ if and only if T has no ω_1-branches. □

Note. If M_δ $(\delta < \omega_1)$ is a $\diamondsuit^+$-sequence which is also a $\diamondsuit_T^+$-sequence for every tree $T = \langle \omega_1, <_T \rangle$ without uncountable branches, then M_δ $(\delta < \omega_1)$ is Π_1^1-reflecting.

Theorem 47 (See Devlin [2]). *$V = L$ implies $\diamondsuit^+(\Pi_{<\omega}^1)$.*

Theorem 48. ($\diamondsuit^+(\Pi_1^1)$). *For every tree T with $|T| = \aleph_1$, either $T \leqslant T^0$ or $\omega_1 \leqslant T$ or there is a stationary $D \subseteq \omega_1$ such that $T(D) \not\leqslant T$ and $T \not\leqslant T(D)$.*

Proof. Assume $\omega_1 \not\leqslant T$. By Theorem 41 there is a stationary D such that $T(D) \not\leqslant T$. If $T \not\leqslant T(D)$, we are done. So we may assume $T \leqslant T(D)$. If $T \leqslant T(\omega_1 \setminus D)$, then

$T \leqslant T(D) \otimes T(\omega_1 \backslash D) \equiv T^0$, by Proposition 36(i). If $T \nleqslant T(\omega_1 \backslash D)$, then $T(\omega_1 \backslash D) \nleqslant T$ since $T(\omega_1 \backslash D) \nleqslant T(D)$ by Proposition 36(iii). □

A summary of the structure of $(\mathbb{T}/\equiv, \leqslant)$.

- First, we have a copy of ω_2, formed by the ω_2 well-founded trees, i.e. trees without infinite branches.
- Next, we have ω_2 copies of $(\mathbb{P}_0/\equiv, \leqslant)$ on top of each other, with a copy of ω between them. The copies of $(\mathbb{P}_0/\equiv, \leqslant)$ correspond to trees of countable limit height. The structure $(\mathbb{P}_0/\equiv, \leqslant)$ has antichains of length 2^{ω_1} and chains of length ω_1.
- Next, we have ω_2 levels, corresponding to trees of different persistency-ranks. Each level has antichains of length 2^{ω_1} and chains of length ω_1.
- Next, we have the equivalence class of T^0, the smallest persistent tree. This tree is the biggest tree that we know is comparable with every other tree.
- Finally, we have the rich family of equivalence classes of $\aleph_1$-trees. In this class we have antichains of length 2^{ω_1} and chains of length ω_1. Assuming CH, there are two classes which themselves are rich, namely the class of equivalence classes of Aronszajn trees and the class of equivalence classes of trees $T(A)$ with $A \subseteq \omega_1$ bistationary.
- Assuming CH, there is no biggest equivalence class in $\mathbb{T}$.

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