Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Journal of Computational and Applied Mathematics 195 (2006) 220–228

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Analytic and algebraic properties of canal surfaces ☆

Zhiqiang Xu[a,*], Renzhong Feng[b], Jia-guang Sun[c]

[a] *Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100080, China*

[b] *Department of Mathematics, Beijing University of Aeronautics and Astronautics, Beijing, China*

[c] *School of software, Tsinghua University, Beijing 100084, China*

Received 15 August 2004; received in revised form 24 March 2005

Abstract

The envelope of a one-parameter set of spheres with radii $r(t)$ and centers $\mathbf{m}(t)$ is a canal surface with $\mathbf{m}(t)$ as the spine curve and $r(t)$ as the radii function. This concept is a generalization of the classical notion of an offset of a plane curve. In this paper, we firstly survey the principle geometric features of canal surfaces. In particular, a sufficient condition for canal surfaces without local self-intersection is presented. Moreover, a simple expression for the area and Gaussian curvature of canal surfaces are given. We also consider the implicit equation $f(x, y, z) = 0$ of canal surfaces with the degree of $f(x, y, z)$ presented. By using the degree of $f(x, y, z)$, a low boundary of the degree of parameterizations representations of canal surfaces is presented.

Keywords: Canal surfaces; Self-intersections; Implicit equations

1. Introduction

A canal surface is defined as envelope of a nonparameter set of spheres, centered at a spine curve $\mathbf{m}(t)$ with radius $r(t)$. When $r(t)$ is a constant function, the canal surface is the envelope of a moving sphere

☆ This project was supported by National Natural Science Foundation of China (10401021) and China Postdoctoral Science Foundation.

* Corresponding author.

E-mail address: xuzq@lsec.cc.ac.cn (Z. Xu).

0377-0427/$ - see front matter

doi:10.1016/j.cam.2005.08.002

and is called a pipe surface. The defining equations for canal surfaces are (cf. [10])

$$\sum(t) : (x - \mathbf{m}(t))^2 - r(t)^2 = 0, \tag{1}$$

$$\sum{}'(t) : (x - \mathbf{m}(t)) \cdot \mathbf{m}'(t) + r(t)r'(t) = 0. \tag{2}$$

To guarantee that the canal surface is a smooth, nondegenerate surface, we assume that $\mathbf{m}(t)$ and $r(t)$ are smooth and satisfy the following nondegeneracy condition: $|\mathbf{m}'(t)|^2 > r'(t)^2$ (cf. [10]).

Canal surfaces have wide applications in CAGD, such as construction of blending surfaces, shape reconstruction, transition surfaces between pipes, robotic path planning, etc. (cf. [6,11,12]).

Most of the literature on canal surfaces within the CAGD context has been motivated by the observation that canal surfaces with rational spine curve and rational radius function is rational, and it is therefore natural to ask for methods which allow one to construct a rational parameterization of canal surfaces from its spine curve and radius function. In this paper we shall not be concerned with parameterization but rather with the certain fundamental geometric and algebraic characteristics of canal surfaces.

The canal surface can be considered as a generalization of the classical notion of an offset of a plane curve. In [4,5], the analysis and algebraic properties of offset curves are discussed in detail. In [3], do Carmo discussed some geometrical features of pipe surfaces. Moreover, by using pipe surfaces, do Carmo proved two very important theorems inn Differential Geometry concerning the total curvature of space curves, namely Fenchel's theorem and Fary–Milnor theorem. To the best of our knowledge, there is no literature concerned with the geometric or algebraic features of canal surfaces. In this paper, we shall discuss the fundamental geometric and algebraic properties for the canal surface. Some results are also useful for finding the lowest degree rational parameterization of canal surfaces.

The paper is organized as follows. In Section 2, we shall discuss geometry properties of canal surfaces. The first and second fundamental form of canal surfaces is presented. By using the first fundamental form, a sufficient condition for canal surfaces without local self-intersection is presented. By the second fundamental form, Gaussian curvature of canal surfaces is given. Moreover, a simple expression for the area of the canal surface is also discussed. In Section 3, algebraic properties of canal surfaces are discussed with the degree of the implicit equation of canal surfaces given. In Section 4, by using the results in Section 3, a low boundary of the degree of parameterization form of canal surfaces is presented. Furthermore, we prove the low boundary can be reached in some cases.

2. Differential properties

Throughout this section, we assume the spine curve $\mathbf{m}(t)$ is regular, i.e. for any t, $|\mathbf{m}'(t)| \neq 0$. To simplify our exposition, we shall restrict ourselves to the spine curve parameterized by arc length s. The radii function is also considered as a function about s. By the nondegenerated condition, $r'(s)^2 < |\mathbf{m}'(s)|^2 = 1$. The canal surface can be parameterized using the Frenet trihedron $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ as follows:

$$C(s, v) = \mathbf{m}(s) + r(s)\left(\sqrt{(1 - r'(s)^2)}\cos(v)\mathbf{n} + \sqrt{(1 - r'(s)^2)}\sin(v)\mathbf{b} - r'(s)\mathbf{t}\right),$$

$$s \in [0, l],\ v \in [0, 2\pi),$$

where l is the total length of the spine curve. When $C_s(s_0, v_0) \times C_v(s_0, v_0) = 0$, the point (s_0, v_0) is called a singular point of C. Similar with [9], we refer to this singularity as local self-intersection. A canal surface without local self-intersection is spoken of as a regular canal surface. The condition for local self-intersection of a pipe surface has been discussed widely (cf. [3,7,9]). In [11], Shani and Ballard described a method for preventing local self-intersection of a generalized cylinder. However, to the best of our knowledge, there is no literature which deals with the problem of local self-intersection of a canal surface. In this section, by using the first fundamental form, a necessary condition for canal surfaces without local self-intersection is presented.

Recall the Frenet equations for a spine curve

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \kappa\mathbf{n}, \quad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = -\kappa\mathbf{t} + \tau\mathbf{b}, \quad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -\tau\mathbf{n}, \tag{3}$$

where, $\mathbf{t}$ is the unit tangent vector, $\mathbf{n}$ is the unit normal vector, $\mathbf{b}$ is the unit binormal vector, τ is the torsion, κ is the curvature of the spine curve.

For the convenience of discussion, let $g := r\sqrt{1 - r'^2}$, $h := rr'$. By using Frenet equation, we can obtain the first fundamental form of canal surfaces

$$E = C_s \cdot C_s = (1 - \kappa g \cos v - h')^2 + (g\tau + h\kappa \sin v)^2 + (g' - h\kappa \cos v)^2, \tag{4}$$

$$F = C_s \cdot C_v = g^2\tau + gh\kappa \sin v, \tag{5}$$

$$G = C_v \cdot C_v = g^2. \tag{6}$$

Hence,

$$|C_s \times C_v|^2 = EG - F^2 = g^2((1 - \kappa g \cos v - h')^2 + (h\kappa \cos v - g')^2). \tag{7}$$

Lemma 1. *When* $1-\kappa(s_0)g(s_0)\cos v_0 - h'(s_0) = 0$, $h(s_0)\kappa(s_0)\cos v_0 - g'(s_0) = 0$, $s_0 \in [0, l]$, $v_0 \in [0, 2\pi)$.

Proof. Since $r'(s)^2 < |\mathbf{m}'(s)|^2 = 1$, $g(s) = r(s)\sqrt{1 - r'^2(s)} \neq 0$, $\forall s \in [0, l]$. By $1 - \kappa(s_0)g(s_0)\cos v_0 - h'(s_0) = 0$, $\kappa(s_0)\cos v_0 = \frac{1-h'(s_0)}{g(s_0)}$. Moreover, it is not difficult for proving $h(s) - h(s)h'(s) - g(s)g'(s) \equiv 0$, $\forall s \in [0, l]$. Hence, when $1 - \kappa(s_0)g(s_0)\cos v_0 - h'(s_0) = 0$, $h(s_0)\kappa(s_0)\cos v_0 - g'(s_0) = h(s_0)\frac{1-h'(s_0)}{g(s_0)} - g'(s_0) = (h(s_0) - h(s_0)h'(s_0) - g(s_0)g'(s_0))/g(s_0) = 0$. The theorem holds. □

Moreover, we have the following theorem:

Theorem 1. *When* $r(s) < \frac{2-(r^2(s))''}{2|\kappa|}$, *the canal surface has no local self-intersection.*

Proof. By the first fundamental form, the canal surface has local self-intersection if and only if $1 - \kappa g \cos v - h' = 0$, $h\kappa \cos v - g' = 0$. By Lemma 1, we only need to consider $1 - \kappa g \cos v - h' = 0$. Since $\cos v$ varies between -1 and 1, there will be no local self-intersection if $|\kappa|g + h' < 1$. By $\sqrt{1 - r'^2} \leqslant 1$, $|\kappa|g + h' = |\kappa|r\sqrt{1 - r'^2} + \frac{1}{2}(r^2)'' \leqslant |\kappa|r + \frac{1}{2}(r^2)''$. Hence, when $|\kappa|r + \frac{1}{2}(r^2)'' < 1$, $|\kappa|g + h' < 1$, i.e. when $r(s) < \frac{2-(r^2(s))''}{2|\kappa|}$, $|\kappa|g + h' < 1$. The theorem holds. □

In fact, a sufficient condition for canal surfaces without local self-intersection is presented in Theorem 1. Moreover, we have the following interesting corollaries.

Corollary 1. *When* $r(s) = as + b$, *if* $r < \frac{1-a^2}{|\kappa|}$, *the canal surface has no local self-intersection.*

Corollary 2. *When* $r(s) = \sqrt{as + b}$, *if* $r < \frac{1}{|\kappa|}$, *the canal surface has no local self-intersection.*

By Theorem 1, it is easy for proving Corollaries 1 and 2. Hence, we omit its proofs.

We turn to an examination of certain global characteristics of a canal surface, deriving Gaussian curvature and a simple expressions for the area of a canal surface.

The developable surface plays a important role in CAGD. A natural question is when the canal surface is developable. It is well known that, at regular points, the Gaussian curvature of a developable surface is identically zero. Hence, to answer the question, we have to compute the Gaussian curvature of canal surfaces. To compute the Gaussian curvature of a canal surface is really tough work. Firstly, we compute the second fundamental form. Let **N** denote the normal vector on the canal surface, then

$$\mathbf{N} = \frac{(g' - \kappa h \cos v) \cdot \mathbf{t} + (\kappa g \cos v + h' - 1) \sin v \cdot \mathbf{b} + (\kappa g \cos v + h' - 1) \cos v \cdot \mathbf{n}}{\sqrt{(g' - \kappa h \cos v)^2 + (\kappa g \cos v + h' - 1)^2}}.$$

Hence, we can obtain the second fundamental form of canal surfaces

$$\begin{aligned} L = C_{ss} \cdot \mathbf{N} = & \frac{1}{\sqrt{(g' - kh \cos v)^2 + (kg \cos v + h' - 1)^2}} \\ & \times (-(2h' - 1)k(\kappa g \cos v + h' - 1) \cos v - k^2 g(\kappa g \cos v + h' - 1)\cos^2 v \\ & - 2(g' - kh \cos v)kg' \cos v + k(g' - kh \cos v)g\tau \sin v \\ & + (g' - kh \cos v)(k^2 h - h'') - (\kappa g \cos v + h' - 1)(\tau^2 g - g'') \\ & - \tau(\kappa g \cos v + h' - 1)\kappa h), \end{aligned}$$

$$\begin{aligned} M = C_{sv} \cdot \mathbf{N} = & \frac{1}{\sqrt{(g' - kh \cos v)^2 + (\kappa g \cos v + h' - 1)^2}} \\ & \times ((g' - kh \cos v)kg \sin v - \tau g(\kappa g \cos v + h' - 1)), \end{aligned}$$

$$N = C_{vv} \cdot \mathbf{N} = -\frac{g(\kappa g \cos v + h' - 1)}{\sqrt{(g' - kh \cos v)^2 + (\kappa g \cos v + h' - 1)^2}}.$$

It is well known that Gaussian curvature $K = \frac{LN - M^2}{EG - F^2}$. Hence,

$$\begin{aligned} K = \frac{LN - M^2}{EG - F^2} = & \frac{g(\kappa g \cos v + h' - 1)}{g^2((\kappa g \cos v + h' - 1)^2 + (g' - \kappa h \cos v)^2)^2} \\ & \times (\kappa(\kappa g \cos v + h' - 1)(\kappa g \cos v + 2h' - 1) \cos v + 2(g' - \kappa h \cos v)\kappa g' \cos v \\ & + \tau\kappa(g' - \kappa h \cos v)g \sin v - (g' - \kappa h \cos v)(\kappa^2 h - h'') - g''(\kappa g \cos v + h' - 1) \\ & + (\kappa g \cos v + h' - 1)\tau\kappa h - (g' - \kappa h \cos v)^2 \kappa^2 g \sin v^2/(\kappa g \cos v + h' - 1)). \end{aligned} \tag{8}$$

Hence, we have

Theorem 2. *The regular canal surface is developable if and only if the canal surface is a cylinder or cone.*

Proof. Obviously, when it is a cylinder or cone, the canal surface is developable.

The regular canal surface is developable iff Gaussian curvature $K \equiv 0$, i.e. $LN - M^2 \equiv 0$. By (8), when $LN - M^2 \equiv 0$

$$\begin{aligned}
&\tau\kappa(\kappa g \cos v + h' - 1)(g' - \kappa h \cos v) g \sin v \\
&\quad \equiv (\kappa g \cos v + h' - 1)(\kappa(\kappa g \cos v + h' - 1)(\kappa g \cos v + 2h' - 1)\cos v \\
&\qquad + 2(g' - \kappa h \cos v)\kappa g' \cos v - (g' - \kappa h \cos v)(\kappa^2 h - h'') \\
&\qquad - g''(\kappa g \cos v + h' - 1) + (\kappa g \cos v + h' - 1)\tau\kappa h \\
&\qquad - (g' - \kappa h \cos v)^2 \kappa^2 g \sin v^2/(\kappa g \cos v + h' - 1)).
\end{aligned} \tag{9}$$

Square both sides of (9). By $\sin^2 v = 1 - \cos^2 v$, the left and right side of (9) are polynomial about $\cos v$ with degree 6 and 8, respectively. According to properties of polynomial, the coefficient of the highest term of right side of (9) is identical to 0. Hence, $\kappa^4 g^2(g^2 + h^2) \equiv 0$. Since $g \neq 0$, we have $\kappa \equiv 0$.

Substituting $\kappa \equiv 0$ into (8), we have

$$(h' - 1)(g'h'' - g''(h' - 1)) \equiv 0. \tag{10}$$

When $\kappa \equiv 0$, it is easy to prove that if $h'(s_0) - 1 = 0$, then for any v, $EG - F^2 = 0$ at (s_0, v), where $s_0 \in [0, l]$. Hence, when $h'(s_0) - 1 = 0$, the canal surface is not regular. According to the assumption of canal surfaces being regular, we have

$$g'h'' - g''(h' - 1) \equiv 0. \tag{11}$$

It is easy to verify that $g'' = (\frac{3r'r'' + rr'''}{h' - 1} + \frac{r''}{r'(1 - r'^2)})g'$. Using these results, Eq. (11) is simplified as

$$-(h' - 1)\frac{r''}{r'(1 - r'^2)} g' \equiv 0. \tag{12}$$

From above analysis, we know that $h' - 1 \not\equiv 0$. By $g' = r'(1 - h')/\sqrt{1 - r'^2}$, if $g' \equiv 0$, then $r' \equiv 0$. Hence, when $g' \equiv 0$, r is constant. If $r'' \equiv 0$, r is linear about s or constant. Hence, from (12), r is constant or a linear function about s. Since $\kappa \equiv 0$, the spine curve is a line. When r is constant and linear function about s, the canal surface is a cylinder and cone, respectively. Hence, the theorem holds. □

In the following theorem, a simple expression for the area of a canal surface is given.

Theorem 3. *If the canal surface $C(s, v)$ is regular on $s \in [0, l]$, $v \in [0, 2\pi)$, the total area A is given by*

$$A = \pi \left| \int_0^l r(r^2)'' - 2r \, ds \right|. \tag{13}$$

Proof. It is well known that $A=\int_0^l\int_0^{2\pi}\sqrt{EG-F^2}\,\mathrm{d}v\,\mathrm{d}s$. By (7), $EG-F^2=g^2((1-\kappa g\,\cos v-h')^2+(h\kappa\cos v-g')^2)$. Substituting $g=r\sqrt{1-r'^2}$, $h=rr'$ into $EG-F^2$, a simple expressions of $EG-F^2$ is presented, i.e. $EG-F^2=(r^2\sqrt{1-r^2}\kappa\,\cos v+rr''+r'^2-1)^2$. Hence

$$A=\int_0^l\int_0^{2\pi}|r^2\sqrt{1-r^2}\kappa\cos v+rr''+r'^2-1|\,\mathrm{d}v\,\mathrm{d}s. \tag{14}$$

According to the assumption of the canal surface being regular, $EG-F^2$ must be of constant sign on $s\in[0,l]$, $v\in[0,2\pi)$, and the right-hand side of (14) may thus be re-written in the form

$$\begin{aligned}A&=\int_0^l\int_0^{2\pi}|r^2\sqrt{1-r^2}\kappa\cos v+rr''+r'^2-1|\,\mathrm{d}v\,\mathrm{d}s\\&=\left|\int_0^l\int_0^{2\pi}r^2\sqrt{1-r^2}\kappa\cos v+rr''+r'^2-1\,\mathrm{d}v\,\mathrm{d}s\right|\\&=\left|2\pi\int_0^l rr''+r'^2-1\,\mathrm{d}v\,\mathrm{d}s\right|=\pi\left|\int_0^l r(r^2)''-2r\,\mathrm{d}s\right|.\qquad\square\end{aligned} \tag{15}$$

3. Algebraic properties

In Section 2, we have analyzed the geometric features of canal surfaces with any spine curve $\mathbf{m}(t)$ and radii function $r(t)$. In most practical applications, only spine curve and radii function parameterized by polynomial or rational functions are employed. We are interested in the properties of the canal surfaces with such spine curves and radii functions. In this paper, we shall only discuss the canal surface with the polynomial spine curve and radii function. In fact, the algebraic properties of canal surfaces with the rational spine curve and radii function can be discussed by using similar method.

Suppose $\mathbf{m}(t)=(m_1(t),m_2(t),m_3(t))$ and $m_i(t)$ and $r(t)$ are polynomial functions with coefficients in $\mathbb{R}$. In general, the defining equations of canal surfaces are

$$\sum(t):(x-m_1)^2+(y-m_2)^2+(z-m_3)^2-r^2=0, \tag{16}$$

$$\sum{}'(t):(x-m_1)m_1'+(y-m_2)m_2'+(z-m_3)m_3'+rr'=0. \tag{17}$$

To guarantee the canal surface is of nondegeneracy, we assume $m_1'(t)^2+m_2'(t)^2+m_3'(t)^2>r'(t)^2$, $t\in\mathbb{R}$. Obviously, the plane $\sum'(t)$ is perpendicular to the derivative vector m'. But, if τ is a value of the parameter such that $m_1'(\tau)=m_2'(\tau)=m_3'(\tau)=r(\tau)r'(\tau)=0$, Eq. (17) will degenerate to an identity at this value, which holds for any (x,y,z). This defect can be remedied as follows. Let $\phi(t)$ be the greatest common divisor of $m_1'(t)$, $m_2'(t)$, $m_3'(t)$ and $r(t)r'(t)$, i.e.

$$\phi(t)=\mathrm{GCD}(m_1'(t),m_2'(t),m_3'(t),r(t)r'(t)). \tag{18}$$

If $\phi(t)$ is not a constant, we can compute $p_1(t)$, $p_2(t)$, $p_3(t)$ and $q(t)$ through polynomial divisions

$$q(t)=\frac{r(t)r'(t)}{\phi(t)},\quad p_i(t)=\frac{m_i'(t)}{\phi(t)},\quad i=1,2,3. \tag{19}$$

We now substitute in place of (17), and obtain

$$p_1(t)(x - m_1(t)) + p_2(t)(y - m_2(t)) + p_3(t)(z - m_3(t)) + q(t) = 0. \tag{20}$$

Let τ denote a root of $\phi(t)$. In fact, Eq. (20) defines the planar at $(m_1(\tau), m_2(\tau), m_3(\tau))$ as the proper limit of the planar $\sum'(t)$ at regular curve points $(m_1(t), m_2(t), m_3(t))$ as $t \to \tau$.

Let $\widehat{\phi}(t)=\mathrm{GCD}(m_1'(t), m_2'(t), m_3'(t))$. Obviously, $\phi(t)$ is a factor of $\widehat{\phi}(t)$. If τ is a value of the parameter such that $\phi(\tau) \neq 0$ and $\widehat{\phi}(\tau)=0$, then Eq. (17) will become inconsistent. Hence, in this case, we cannot find a solution of (16) and (17) in Euclid space. But in complex project space, the solution of (16) and (17) is corresponding to the circular point at infinity. The homogeneous coordinates of solutions is $(x_0, y_0, z_0, 0)$, where $x_0^2 + y_0^2 + z_0^2 = 0$.

In fact, when $\tau \in \mathbb{R}$, by $r'(\tau)^2 < m_1'(\tau)^2 + m_2'(\tau)^2 + m_3'(\tau)^2$, if $\widehat{\phi}(\tau) = 0$ then $\phi(\tau) = 0$. Hence, if only consider the real parameter value, we do not worry about equation appearing inconsistent.

Now for a prescribed canal surface, we form the following polynomials in t:

$$P(t, x, y, z) = (x - m_1)^2 + (y - m_2)^2 + (z - m_3)^2 - r(t)^2 = 0, \tag{21}$$

$$Q(t, x, y, z) = (x - m_1)p_1(t) + (y - m_2)p_2(t) + (z - m_3)p_3(t) + q(t) = 0. \tag{22}$$

Obviously, the implicit equation of the canal surface with spine curve $m(t)$ and radii function $r(t)$ is given by the resultant

$$f(x, y, z) = \mathrm{Res}_t(P(t, x, y, z), Q(t, x, y, z)), \tag{23}$$

where $\mathrm{Res}_t(P, Q)$ denotes the Sylvester resultant of P and Q with respect to t.

Suppose the degrees of $\mathbf{m}(t)$, $r(t)$ and $\phi(t)$ are n, m and v, respectively. Motivated by Corollary 2.3 in [5] we have

Theorem 4. *The canal surface $f(x, y, z) = 0$ given by (23) is of degree $4\max(m, n) - 2 - 2v$, where n, m and v are the degrees of $\mathbf{m}(t)$, $r(t)$ and $\phi(t)$, respectively.*

Proof. First, we consider the case where $m \geqslant n$. Obviously, $\deg_t P$ and $\deg_t Q$ are $2m$ and $2m - 1 - v$, respectively. Hence, the Sylvester determinant for (23) is of dimension $4m - 1 - v$. Let $\rho_{i,j}(x, y, z)$ be a typical entry in the Sylvester determinant for the resultant (23).

According to the property of the determinant, the expansion of the Sylvester determinant for (22) and (21) may be written as

$$f(x, y, z) = \sum_{\sigma \in S} (-1)^{\mathrm{sign}(\sigma)} \prod_{i=1}^{4m-1-v} \rho_{i\sigma_i}(x, y), \tag{24}$$

where S is the set of all permutations of the sequence of integer $\{1, \ldots, 4m - 1 - v\}$, σ is a typical member in S which maps i to σ_i, and $(-1)^{\mathrm{sign}(\sigma)} = 1$ when σ is an even permutation, otherwise $(-1)^{\mathrm{sign}(\sigma)} = -1$.

Note that when $1 \leqslant i \leqslant 2m - 1 - v$ and $j \leqslant i$, $1 \leqslant i \leqslant 2m - 1 - v$ and $j > i + 2m$, $2m - v \leqslant i \leqslant 4m - 1 - v$ and $j < i + 1 - (2m - v)$, $2m - v \leqslant i \leqslant 4m - 1 - v$ and $j > i$, $\rho_{ij}(x, y, z) = 0$,

when $1 \leqslant i \leqslant 2m - 1 - v$ and $i \leqslant j \leqslant i + n - 1$, $2m - v \leqslant i \leqslant 4m - 1 - v$ and $i - (2m - v) + 1 \leqslant j \leqslant i + n - (2m - v)$, $\rho_{ij}(x, y, z)$ is a constant,

when $1 \leqslant i \leqslant 2m-1-v$ and $j = i + 2m$ $\deg(\rho_{ij}) = 2$,
otherwise, $\deg(\rho_{ij}) = 1$.

We are interested in the permutation σ which contributes a nonzero product to the sum (24). This contribution may be expressed in terms of the five sub-products:

$$\pi_1(\sigma) = \prod_{1 \leqslant i \leqslant 2m-1-v, \sigma_i = i+2m} \rho_{i\sigma_i}(x, y, z),$$

$$\pi_2(\sigma) = \prod_{1 \leqslant i \leqslant 2m-1-v, i+n \leqslant \sigma_i \leqslant i+2m-1} \rho_{i\sigma_i}(x, y, z),$$

$$\pi_3(\sigma) = \prod_{2m-v \leqslant i \leqslant 4m-1-v, i-(2m-v)+n+1 \leqslant \sigma_i \leqslant i} \rho_{i\sigma_i}(x, y, z),$$

$$\pi_4(\sigma) = \prod_{1 \leqslant i \leqslant 2m-1-v, i \leqslant \sigma_i \leqslant i+n-1} \rho_{i\sigma_i}(x, y, z),$$

$$\pi_5(\sigma) = \prod_{2m-v \leqslant i \leqslant 4m-1-v, i-(2m-v)+1 \leqslant \sigma_i \leqslant i-(2m-v)+n} \rho_{i\sigma_i}(x, y, z).$$

Since both π_4 and π_5 are constant, the degree of π_4, π_5 equals 0. Let $n_k(\sigma)$ be the number of terms in the kth sub-product $\pi_k(\sigma)$. We have

$$n_1(\sigma) + n_2(\sigma) + n_3(\sigma) \leqslant 2m - 1 - v. \tag{25}$$

The degree of the product corresponding to the permutation σ is of degree $2n_1(\sigma) + n_2(\sigma) + n_3(\sigma)$.

Obviously, $2n_1(\sigma) + n_2(\sigma) + n_3(\sigma) \leqslant 2(n_1(\sigma) + n_2(\sigma) + n_3(\sigma)) \leqslant 4m - 2 - 2v$. The maximum degree can be achieved when $n_1(\sigma) = 2m - 1 - v$, $n_2(\sigma) = n_3(\sigma) = 0$. In fact, we can find such σ reaching the maximum degree. Hence, the degree of $f(x, y, z)$ is precisely $4m - 2 - 2v$.

When $m < n$, by using similar method we can prove $\deg\ f(x, y, z) = 4n - 2 - 2v$. Hence, the theorem holds. □

4. A low boundary of degree of parameterization of canal surfaces

A rational surface is defined by $\Phi(t, u) = (a(t, u)/\mathrm{d}(t, u), b(t, u)/\mathrm{d}(t, u), c(t, u)/\mathrm{d}(t, u))$, where a, b, c and d are polynomials in t, u with degree (n_t, n_u), i.e. n_t and n_u are the highest degrees of the parametric equation with respect to t and u, respectively. It may happen that there are values of t, u for which $a(t, u) = b(t, u) = c(t, u) = d(t, u) = 0$. These are known as base points in contemporary algebraic geometry. By [1], the implicit degree of $\Phi(t, u)$ is $2n_t n_u - \rho$, where ρ is the number of base points.

It is well known that canal surfaces with rational spine curve and rational radius function are rational [10]. To be precise, they admit real rational parameterization of their real components. There are plenty of literature to discuss the parameterization of canal surfaces (cf. [2,8–10]). Denote the rational parametric form of canal surfaces by $\Phi(t, u)$. The degree of $\Phi(t, u)$ with respect to t and u is denoted by n_t and n_u, respectively. Since the low degree representations are important for practical use, in [10], degree reductions are discussed. Hence, a natural question is raised: how to find the lowest degree of parameterization of canal surfaces? The question is also described by the following: given a canal surface with a rational spine curve and radii function, how to find the minimum n_t and n_u?

Theorem 5. *Given a canal surface with a polynomial spine curve* $\mathbf{m}(t) = (m_1(t), m_2(t), m_3(t))$ *and radii function* $r(t)$, *we suppose the degrees of* $\mathbf{m}(t)$ *and* $r(t)$ *are* n *and* m, *respectively. Let* $\Phi(t, u)$ *be rational parameterization form of the canal surface. Then we have* $n_t n_u \geqslant 2\max(m, n) - 1 - v$, *where* v *is the degree of* $\mathrm{GCD}(m_1'(t), m_2'(t), m_3'(t), r(t)r'(t))$.

Proof. By Theorem 4, the implicit degree of the canal surface is $4\max(m, n) - 2 - 2v$. By [1], we have $2n_t n_u - \rho = 4\max(m, n) - 2 - 2v$, where ρ is the number of base points. Since $\rho \geqslant 0$, $2n_t n_u \geqslant 4\max(m, n) - 2 - 2v$, we have $n_t n_u \geqslant 2\max(m, n) - 1 - v$. □

In general, the degree of u in $\Phi(t, u)$ is 2. Obviously, the minimum n_u is also 2. Then we have

Corollary 3. *When* n_u *is selected as* 2, *under the condition of Theorem* 5, $n_t \geqslant \lceil \max(m, n) - (1 + v)/2 \rceil$.

We believe that the low boundary presented in Corollary 3 is sharp. A future research direction is to construct a rational parameterization of canal surfaces whose degree reaches the low boundary.

References

[1] E. Chionh, Base points, resultants and the implicit representation of rational surfaces, Ph.D. Thesis, Department of Computer Science, University of Waterloo, Canada, 1990.
[2] H.C. Cho, H.I. Choi, S.-H. Kwon, et al., Clifford algebra, Lorentzian geometry, and rational parametrization of canal surfaces, Comput. Aided Geom. design 4 (2004) 327–339.
[3] P.M. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Englewood Cliffs, NJ, 1976.
[4] R.T. Farouki, C.A. Neff, Analytic properties of plane offset curves, Comput. Aided Geom. Design 7 (1990) 83–99.
[5] R.T. Farouki, C.A. Neff, Algebraic properties of plane offset curves, Comput. Aided Geom. Design 7 (1990) 101–127.
[6] R.T. Farouki, R. Sverrissor, Approximation of rolling-ball blends for free-form parametric surfaces, Comput.-Aided Design 28 (1996) 871–878.
[7] E. Kreyszig, Differential Geometry, University of Toronto Press, Toronto, 1959.
[8] G. Landsmann, J. Schicho, F. Winkler, The parametrization of canal surfaces and the decomposition of polynomials into a sum of two squares, J. Symbolic Comput. 32 (2001) 119–132.
[9] T. Maekawa, N.M. Patrikalakis, T. Sakkalis, Guoxin Yu, Analysis and applications of pipe surfaces, Comput. Aided Geom. Design 15 (1988) 437–458.
[10] M. Peternell, H. Pottmann, Computing rational parametrizations of canal surfaces, J. Symbolic Comput. 23 (1997) 255–266.
[11] U. Shani, D.H. Ballard, Splines as embeddings for generalized cylinders, Comput. Vision Graphics Image Process. 27 (1984) 129–156.
[12] L. Wang, M.C. Leu, D. Blackmore, Generating sweep solids for NC verification using the SEDE method, in: Proceedings of the Fourth Symposium on Solid Modeling and Applications, Atlanta, Georgia, 14–16 May 1997, pp. 364–375.