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Determination of All Coherent Pairs

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A pair of quasi-definite linear functionals \( \{u_0, u_1\} \) on the set of polynomials is called a coherent pair if their corresponding sequences of monic orthogonal polynomials \( \{P_n\} \) and \( \{T_n\} \) satisfy a relation

\[
T_n = P_{n+1} - \sigma_n P_n, \quad n \geq 1,
\]

with \( \sigma_n \) non-zero constants. We prove that if \( \{u_0, u_1\} \) is a coherent pair, then at least one of the functionals has to be classical, i.e. Hermite, Laguerre, Jacobi, or Bessel. A similar result is derived for symmetrically coherent pairs.

1. INTRODUCTION

Several authors studied polynomials orthogonal with respect to a Sobolev inner product of the form

\[
\langle f, g \rangle_s = \int_a^b fg \, d\Psi_0 + \lambda \int_a^b f' g' \, d\Psi_1,
\]

where \( \Psi_0 \) and \( \Psi_1 \) are distribution functions and \( \lambda \geq 0 \). For a survey of the theory we refer the reader to [5] and [11].

In [4] Iserles et al. introduced the concept of the coherent pair, which proved to be a very fruitful concept. It reads as follows. Let \( \{P_n\} \) denote the monic orthogonal polynomial sequence (MOPS) with respect to \( d\Psi_0 \) and let \( \{T_n\} \) denote the MOPS with respect to \( d\Psi_1 \), then \( \{d\Psi_0, d\Psi_1\} \) is called a coherent pair if there exist non-zero constants \( \sigma_n \) such that

\[
T_n = \frac{P_{n+1}}{n+1} - \sigma_n \frac{P_n}{n}, \quad \text{for all } n \geq 1.
\]
Iserles et al. showed that if \( \{d\Psi_0, d\Psi_1\} \) is a coherent pair, then the sequence of polynomials \( \{S_n^*\} \) orthogonal with respect to the inner product (1.1) has an attractive structure. Put

\[
S_n^* = \sum_{m=1}^{n} \alpha_m^*(\lambda) P_m(x), \quad n \geq 1,
\]

then Iserles et al. showed that the normalization of \( P_n \) and \( S_n^* \) can be changed by multiplying these functions with suitable constants in such a way that the coefficients \( \alpha_m^* \) become independent of \( n \), apart from the leading coefficient \( \alpha_n^* \).

Write \( \alpha_m^* = \alpha_m^* \) for \( 1 \leq m \leq n-1 \), then \( \alpha_m^* \) is a polynomial in \( \lambda \) of degree \( m \) and the polynomials \( \alpha_m^*(\lambda) \) satisfy a three term recurrence relation. Moreover, if \( \{d\Psi_0, d\Psi_1\} \) is a coherent pair, then the \( \{S_n^*\} \) satisfy a four term recurrence relation; see [2]. It is easy to prove that when \( \{d\Psi_0, d\Psi_1\} \) is a coherent pair, \( \lambda \) is sufficiently large and \( n \geq 2 \), then \( S_n^* \) has \( n \) different, real zeros interlacing with the zeros of \( P_{n-1} \) and with those of \( T_{n-1} \); see [10]. Therefore it is interesting to investigate under what conditions \( \{d\Psi_0, d\Psi_1\} \) is a coherent pair.

Marcellán and Petronilho [7] studied this problem in a more general setting where \( u_0 \) and \( u_1 \) are quasi-definite linear functionals on the space of polynomials and the corresponding MOPS satisfy a relation of the form (1.2). They solved the problem completely for the case when one of the functionals \( u_0, u_1 \) is a classical one, i.e. Hermite, Laguerre, Jacobi, or Bessel. In a recent paper [6], Marcellán, Pérez, and Piñar showed that if \( \{u_0, u_1\} \) is a coherent pair of quasi-definite linear functionals, then both are semiclassical, i.e. there exist polynomials \( \phi_i, \psi_i \) \( (i = 0, 1) \) such that \( D(\phi_i u_0) = \psi_i u_1, \phi_i u_1 = \psi_i u_0 \), where \( D \) denotes the distributional differentiation. Moreover, they showed that there exist polynomials \( A \) and \( B \), such that \( Au_0 = Bu_1 \) with degree \( A \leq 3 \), degree \( B \leq 2 \).

It is the aim of the present paper to solve the problem completely and to determine all coherent pairs \( \{u_0, u_1\} \) of quasi-definite linear functionals. We will prove that at least one of the functionals \( u_0, u_1 \) has to be classical, so all coherent pairs of functionals \( \{u_0, u_1\} \) are already determined in [7] (apart from some special cases which are not mentioned in [7]).

We show that there are only two cases:

(i) The functional \( u_0 \) is classical; there exist polynomials \( \phi, \psi, \rho \), degree \( \phi \leq 2 \), degree \( \psi = \rho = 1 \), such that \( D(\phi u_0) = \psi u_0 \) and \( \phi u_0 = \rho u_1 \).

(ii) The functional \( u_1 \) is classical; there exist polynomials \( \phi, \psi, \rho \), degree \( \phi \leq 2 \), degree \( \psi = \rho = 1 \), such that \( D(\phi u_1) = \psi u_1 \) and \( \phi u_0 = \rho u_1 \).
We remark that it is possible that both $u_0$ and $u_1$ are classical. In Section 2 we give the basic definitions, notations, and known results on functionals and coherent pairs of functionals. In Section 3 we show that every coherent pair $\{u_0, u_1\}$ belongs to case (i) or to case (ii). Moreover, we give all coherent pairs for linear functionals which can be represented by distribution functions.

In Section 4 the functionals are symmetric. A pair of symmetric functionals $\{u_0, u_1\}$ is called a symmetrically coherent pair if their corresponding MOPS $\{P_n\}$ and $\{T_n\}$ satisfy a relation

$$T_n = \frac{P_{n+1}}{n+1} - \sigma_n \frac{P_{n-1}}{n-1}, \quad \text{for } n \geq 2,$$

with $\sigma_n$ non-zero constants. We prove that if $\{u_0, u_1\}$ is a symmetrically coherent pair, then at least one of the functionals has to be classical. Moreover, a division in two cases as for the coherent pairs is given.

2. BASIC DEFINITIONS AND RESULTS

Let $u$ denote a linear functional defined on the space of polynomials $\mathcal{P}$. A sequence of monic polynomials $\{P_n\}$ is called a monic orthogonal polynomial sequence (MOPS) with respect to $u$ if

(i) degree $P_n = n$, $n = 0, 1, 2, ...$

(ii) $\langle u, P_n P_m \rangle = 0$, $n \neq m$, $n, m = 0, 1, 2, ...$

(iii) $\langle u, P_n^2 \rangle = \alpha_n \neq 0$, $n = 0, 1, 2, ...$

There exists a MOPS with respect to $u$ if and only if $u$ is quasi-definite; see [3], Ch. I §3. In that case the MOPS is unique. In the sequel we always suppose the functionals to be quasi-definite.

The MOPS $\{P_n\}$ satisfies a three-term recurrence relation of the form (see [3], p. 18)

$$P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 1,$$

with $\gamma_n \neq 0$ for $n \geq 1$, $P_0(x) \equiv 1$, $P_1(x) = x - \beta_0$.

If $A$ is a polynomial and $u$ a functional, then $Au$ is defined by

$$\langle Au, p \rangle = \langle u, Ap \rangle, \quad p \in \mathcal{P}.$$

If the polynomial $A$ is not the zero-polynomial and degree $A = n$, then we can write

$$A = \sum_{k=0}^{n} c_k P_k,$$
with \( c_n \neq 0 \), so \( \langle Au, P_n \rangle = c_n p_n \neq 0 \). This implies that if \( A \) and \( B \) are polynomials with \( Au = Bu \), then \( \langle (A - B) u, p \rangle = 0 \) for all \( p \in \mathcal{P} \) and \( A - B \) has to be the zero-polynomial, i.e. \( A = B \).

The distributional derivative \( Du \) of the functional \( u \) is defined by

\[
\langle Du, p \rangle = - \langle u, p' \rangle, \quad p \in \mathcal{P}.
\]

It is easy to check that we have for an arbitrary polynomial \( \varphi \):

\[
D(\varphi u) = \varphi' u + \varphi \, Du.
\]

A functional \( u \) is called classical if it satisfies a relation

\[
D(\varphi u) = \psi u,
\]

with \( \varphi \) and \( \psi \) polynomials, degree \( \varphi \leq 2 \), degree \( \psi = 1 \).

The classical functionals and corresponding orthogonal polynomial sequences are the following ones, see [9], up to a linear transformation of the variable.

(i) degree \( \varphi = 0 \): Hermite polynomials \( \{H_n\} \) with \( \varphi(x) = 1 \), \( \psi(x) = -2x \).

(ii) degree \( \varphi = 1 \): Laguerre polynomials \( \{L_n^{(\alpha)}\} \) with \( \alpha \not\in \{-1, -2, \ldots\} \), \( \varphi(x) = x \), \( \psi(x) = -x + x + 1 \).

(iii) degree \( \varphi = 2 \) and \( \varphi \) has two different roots: Jacobi polynomials \( \{P_n^{(\alpha, \beta)}\} \) with \( \alpha, \beta, \alpha + \beta + 1 \not\in \{-1, -2, \ldots\} \), \( \varphi(x) = 1 - x^2 \), \( \psi(x) = -(\alpha + \beta + 2)x + \beta - \alpha \).

(iv) degree \( \varphi = 2 \) and \( \varphi \) has a double root: Bessel polynomials \( \{J_n^{(\alpha)}\} \) with \( \alpha \not\in \{-2, -3, \ldots\} \), \( \varphi(x) = x^2 \), \( \psi(x) = (x + 2)x + 2 \).

Finally we remark that for a quasi-definite functional \( u \) a relation

\[
D(\varphi u) = cu,
\]

with \( \varphi \) a non-zero polynomial and \( c \) a constant cannot be satisfied, since \( c = 0 \) would imply \( \langle u, 1 \rangle = 0 \) and \( c \neq 0 \) would imply \( \langle \varphi u, p \rangle = 0 \) for all \( p \in \mathcal{P} \).

In the sequel we will use the following definition and notations: \( u_0 \) and \( u_1 \) denote quasi-definite linear functionals on \( \mathcal{P} \), \( \{P_n\} \) the MOPS with respect to \( u_0 \), \( \{T_n\} \) the MOPS with respect to \( u_1 \),

\[
\langle u_0, P_n^2 \rangle = p_n \neq 0, \quad n = 0, 1, 2, \ldots
\]

\[
\langle u_1, T_n^2 \rangle = t_n \neq 0, \quad n = 0, 1, 2, \ldots
\]
The pair \( \{u_0, u_1\} \) is called a coherent pair if there exist non-zero constants \( \sigma_n \) such that

\[
T_n = \frac{P_{n+1}}{n+1} - \sigma_n \frac{P_n}{n} \quad \text{for } n \geq 1. \tag{2.1}
\]

For a coherent pair we introduce the polynomials

\[
C_n = \sigma_n \frac{T_n}{P_n} \frac{T_{n-1}}{P_{n-1}}, \quad n = 1, 2, \ldots \tag{2.2}
\]

Then the leading coefficient of \( C_n \) is \( \sigma_n / \sigma_n \neq 0 \).

The following basic proposition is due to Marcellán, Pérez, Piñar \([6]\).

**Proposition 1.** Let \( \{u_0, u_1\} \) denote a coherent pair, then

\[
n \frac{P_n}{P_{n+1}} u_0 = D (C_n u_1) \quad \text{for } n \geq 1.
\]

**Corollary 1.** Let \( \{u_0, u_1\} \) denote a coherent pair. Then

\[
\varphi Du_1 = \pi u_1, \quad \varphi u_0 = Bu_1, \quad \pi u_0 = B D u_1,
\]

with

\[
\varphi = 2 \frac{P_2}{P_1} C_1 - \frac{P_1}{P_1} C_2, \quad \pi = -2 \frac{P_2}{P_2} C_1 + \frac{P_1}{P_1} C_2, \quad B = C_1 C_2 - C_1 C_2,
\]

where degree \( \varphi \leq 3 \), degree \( \pi \leq 2 \), degree \( B = 2 \).

**Proof.** Proposition 1 with \( n = 1 \) and \( n = 2 \) reads:

\[
\frac{P_1}{P_1} u_0 = C_1 u_1 + C_1 D u_1, \tag{2.6}
\]

\[
2 \frac{P_2}{P_2} u_0 = C_2 u_1 + C_2 D u_1. \tag{2.7}
\]

Elimination of \( u_0 \) gives the first result, elimination of \( D u_1 \), the second one and elimination of \( u_1 \) the last one. The coefficient of \( x^n \) in the polynomial
3. DETERMINATION OF COHERENT PAIRS

In this section we suppose that \( \{u_0, u_1\} \) is a coherent pair and we use the notations of Section 2. We will prove that at least one of the functionals \( u_0, u_1 \) is classical. The polynomial \( B \) defined in (2.5) is of degree 2 and therefore has two zeros \( \zeta_1 \) and \( \zeta_2 \). We will prove that if \( \zeta_1 = \zeta_2 \), then \( u_0 \) is classical (Theorem 1) and if \( \zeta_1 \neq \zeta_2 \), then \( u_1 \) has to be classical (Theorem 2).

If the polynomial \( B \) in (2.5) has a double zero, then the situation is simple.

**Theorem 1.** Let \( \{u_0, u_1\} \) denote a coherent pair of quasi-definite linear functionals. Suppose that the polynomial \( B \) in (2.5) has a double zero \( \xi \). Then

(i) \( u_0 \) is classical with

\[
D(\hat{\varphi} u_0) = \psi u_0
\]

for some polynomials \( \hat{\varphi}, \psi \), degree \( \hat{\varphi} \leq 2 \), degree \( \psi = 1 \);

(ii) \( \hat{\varphi} u_0 = \left( \frac{\sigma_1 \sigma_2}{t_1 t_2} \right) (x - \xi) u_1 \).

**Proof.** From (2.5) we obtain

\[
0 = B'(\xi) = C_1(\xi) \ C_3(\xi).
\]

Hence \( C_1(\xi) = 0 \). Then applying again (2.5) we have \( 0 = B(\xi) = - C_1(\xi) \ C_2(\xi) \), so \( C_2(\xi) = 0 \). Then (2.3) implies \( \varphi(\xi) = 0 \). Write \( \varphi(x) = (x - \xi) \ \hat{\varphi}(x) \).

Since \( C_1(\xi) = C_2(\xi) = 0 \), the polynomial \( C_1 \) divides \( C_2 \). Then the elimination of \( D u_1 \) from (2.6) and (2.7) can be done in such a way, that one arrives at

\[
\hat{\varphi} u_0 = \frac{\sigma_1 \sigma_2}{t_1 t_2} (x - \xi) u_1.
\]

Then using (2.6),

\[
D(\hat{\varphi} u_0) = \frac{\sigma_1 \sigma_2}{t_1 t_2} D((x - \xi) u_1) = \frac{\sigma_2}{t_2} D(C_1 u_1) = \frac{\sigma_2}{t_2} \frac{P_1}{p_1} u_0 = \psi u_0,
\]

where \( \psi \) is a polynomial of degree 1, i.e. \( u_0 \) is classical.

If \( B \) in (2.5) has two different zeros, the analysis is more complicated. We first derive some auxiliary results.
It follows from Proposition 1 and Corollary 1, that for \( n \geq 1 \),

\[
\frac{P_n^m B u_1}{P_n} = n \frac{P_n^m}{P_n} \varphi u_0 = \varphi D(C_n u_1)
\]

\[
= \varphi C_n u_1 + \varphi C_n D u_1 = (\varphi C_n + C_n \pi) u_1.
\]

Hence

\[
\frac{P_n^m B}{P_n} = C_n \varphi + C_n \pi, \quad n \geq 1. \tag{3.1}
\]

**Lemma 1.** Suppose that \( \xi \) is such that \( B(\xi) = 0 \), \( \varphi(\xi) \neq 0 \). Then there exists a \( k \), independent of \( n \), \( k \neq 0 \), such that

\[
C_n(\xi) + k C_n'(\xi) = 0 \quad \text{for all} \quad n \geq 1.
\]

**Proof.** Substitution of \( \xi \) in (3.1) gives

\[
C_n(\xi) \varphi(\xi) + C_n(\xi) \pi(\xi) = 0, \quad n \geq 1.
\]

Consider the relation for \( n = 1 \). Then \( C_1 = \sigma_1/t_1 \neq 0 \) and \( \varphi(\xi) \neq 0 \) imply \( \pi(\xi) \neq 0 \).

Hence

\[
C_n(\xi) + k C_n'(\xi) = 0 \quad \text{for all} \quad n \geq 1,
\]

with

\[
k = \frac{\varphi(\xi)}{\pi(\xi)} \neq 0.
\]

**Lemma 2.** Suppose that there exist \( \xi_1, \xi_2 \), \( k_1 \neq 0 \), \( k_2 \neq 0 \) such that

\[
C_n(\xi_1) + k_1 C_n'(\xi_1) = 0 \quad \text{and} \quad C_n(\xi_2) + k_2 C_n'(\xi_2) = 0,
\]

for all \( n \geq 1 \). Then \( \xi_1 = \xi_2 \) and \( k_1 = k_2 \).

**Proof.** Using the definition of \( C_n \) in (2.2) we obtain for \( \xi_j \), \( j = 1, 2 \),

\[
\sigma_n \left\{ T_n(\xi_j) + \frac{T_n'(-\xi_j)}{t_n} \right\} = \frac{T_{n-1}(\xi_j)}{t_{n-1}} + k_j \frac{T_{n-1}'(-\xi_j)}{t_{n-1}}.
\]
Put
\[
    h_n^{(j)}(\xi_j) = \frac{T_n(\xi_j)}{t_n} + k_j \frac{T_n^j(\xi_j)}{t_n}, \quad n \geq 0, \quad j = 1, 2, \tag{3.2}
\]
then
\[
    \sigma_n h_n^{(j)}(\xi_j) = h_n^{(j)}(\xi_j), \quad n \geq 1, \quad j = 1, 2. \tag{3.3}
\]
Note that
\[
    h_0^{(j)}(\xi_j) = \frac{1}{t_0} \neq 0,
\]
and (3.3) implies \( h_n^{(j)}(\xi_j) \neq 0 \) for all \( n \geq 0 \). Dividing the relations (3.3) for \( j = 1 \) and \( j = 2 \) we obtain
\[
    \frac{h_n^{(1)}(\xi_1)}{h_n^{(2)}(\xi_2)} = \frac{h_{n-1}^{(1)}(\xi_1)}{h_{n-1}^{(2)}(\xi_2)}, \quad n \geq 1,
\]
and by repeated application
\[
    \frac{h_n^{(1)}(\xi_1)}{h_n^{(2)}(\xi_2)} = \frac{h_0^{(1)}(\xi_1)}{h_0^{(2)}(\xi_2)} = 1,
\]
or
\[
    h_n^{(1)}(\xi_1) = h_n^{(2)}(\xi_2) \quad \text{for all} \quad n \geq 0.
\]
But now (3.2) gives
\[
    T_n(\xi_1) + k_1 T_n^j(\xi_1) = T_n(\xi_2) + k_2 T_n^j(\xi_2) \quad \text{for all} \quad n \geq 0. \tag{3.4}
\]
It follows that every polynomial \( p \) satisfies
\[
    p(\xi_1) + k_1 p'(\xi_1) = p(\xi_2) + k_2 p'(\xi_2).
\]
Choose \( p(x) = (x - \xi_1)^n \) then
\[
    (\xi_2 - \xi_1)^n + nk_2 (\xi_2 - \xi_1)^{n-1} = 0, \quad n \geq 2,
\]
and, as a consequence, \( \xi_1 = \xi_2 \). Finally \( k_1 = k_2 \).
**Lemma 3.** Let $B$, $\varphi$, and $\pi$ denote the polynomials defined in Corollary 2. Suppose that $B$ has two different zeros. Then at least one of them is also a zero of $\varphi$. If $B(\xi) = \varphi(\xi) = 0$, then $C_1(\xi) \neq 0$ and $\pi(\xi) = 0$.

**Proof.** It is a direct consequence of Lemma 1 and Lemma 2, that $B$ and $\varphi$ have at least one zero $\xi$ in common. Since $\xi$ is a simple zero of $B$, it follows $B'(\xi) \neq 0$. By using (2.5) $B' = C_1 C_2'$, hence $C_1(\xi) \neq 0$. Substituting $\xi$ in (3.1) with $n = 1$, we obtain $\pi(\xi) = 0$.

We now are able to treat the situation that $B$ in (2.5) has two different zeros.

**Theorem 2.** Let $\{u_0, u_1\}$ denote a coherent pair of quasi-definite linear functionals. Suppose that the polynomial $B$ in (2.5) has two different zeros. Then

(i) $u_1$ is classical with $D(\hat{\varphi} u_1) = \psi u_1$ for some polynomials $\hat{\varphi}$, $\psi$, degree $\hat{\varphi} \leq 2$, degree $\psi = 1$;

(ii) there exists a $\xi$ such that

$$\hat{\varphi} u_0 = \frac{\sigma_1 \sigma_2}{t_1 t_2} (x - \xi) u_1.$$  

**Proof.** Let $\xi_1, \xi_2$ denote the different zeros of $B$. By Lemma 3 at least one of them is also a zero of $\varphi$. Without loss of generality we may suppose $\varphi(\xi_1) = 0$. Then by Lemma 3 also $\pi(\xi_1) = 0$.

Put $B = (x - \xi_1) \hat{B}$, i.e. $\hat{B} = (\sigma_1 \sigma_2/t_1 t_2)(x - \xi_2)$, $\varphi = (x - \xi_1) \hat{\varphi}$, $\pi = (x - \xi_1) \pi_1$.

Then (3.1) reduces to

$$n P_n \hat{B} = C_n \hat{\varphi} + C_n \pi_1, \quad n \geq 1. \quad (3.5)$$

Moreover, the relations $\varphi u_0 = Bu_1$, $\pi u_0 = B Du_1$ and $\varphi Du_1 = \pi u_1$ from Corollary 1 reduce to

$$\hat{\varphi} u_0 = \hat{B} u_1 + M \hat{\varphi}(\xi_1), \quad (3.6)$$

$$\hat{B} Du_1 = \pi_1 u_0 + N \hat{\varphi}(\xi_1), \quad (3.7)$$

$$\hat{\varphi} Du_1 = \pi_1 u_1 + K \hat{\varphi}(\xi_1), \quad (3.8)$$

for some constants $M$, $N$ and $K$. 


From (3.5) and Proposition 1 we obtain for \( n \geq 1 \),

\[
(C_n \hat{\varphi} + C_n \pi_1) u_0 = n \frac{P}{P_n} \tilde{B} u_0 = \tilde{B}(C_n u_1 + C_n D u_1),
\]
or

\[
C_n(\hat{\varphi} u_0 - \tilde{B} u_1) = C_n(\tilde{B} D u_1 - \pi_1 u_0),
\]

and with (3.6) and (3.7)

\[
MC_n(\xi_1) = NC_n(\xi_1), \quad n \geq 1. \tag{3.9}
\]

Observe that \( C_1(\xi_1) \neq 0 \) and, by Lemma 3, \( C_1(\xi_1) \neq 0 \); so \( M = 0 \) if and only if \( N = 0 \).

For the second zero \( \xi_2 \) of \( B \) there are two possibilities: \( \varphi(\xi_2) \neq 0 \) or \( \varphi(\xi_2) = 0 \).

(i) Let \( \varphi(\xi_2) \neq 0 \). Then Lemma 1 implies that there exists a non-zero constant \( k \) such that

\[
C_n(\xi_2) + kC_n(\xi_2) = 0 \quad \text{for all} \quad n \geq 1.
\]

Since \( \xi_1 \neq \xi_2 \) we conclude from Lemma 2 that (3.9) only can be satisfied with \( M = N = 0 \).

(ii) Let \( \varphi(\xi_2) = 0 \). Then we may proceed with \( \xi_2 \) as with \( \xi_1 \) and conclude that there exist constants \( M_2 \) and \( N_2 \), such that

\[
M_2 C_n(\xi_2) = N_2 C_n(\xi_2) \quad \text{for all} \quad n \geq 1, \tag{3.10}
\]

where \( C_1(\xi_2) \neq 0, C_1(\xi_2) \neq 0 \).

Again Lemma 2 implies that at least one of the relations (3.9) and (3.10) has to be a trivial one. Without loss of generality we may suppose that (3.9) is trivial, i.e. \( M = N = 0 \).

In both cases (3.6) reduces to

\[
\hat{\varphi} u_0 = \tilde{B} u_1 = \frac{\sigma_1 \sigma_2}{t_1 t_2} (x - \xi_2) u_1 \tag{3.11}
\]

This proves assertion (ii) of the theorem.
To prove the first assertion we use (2.6) and (3.5) with \( n = 1 \):

\[
\frac{\partial}{\partial t} P_1 u_0 = \frac{\partial}{\partial t} C_1 u_1 + \frac{\partial}{\partial t} C_1 D u_1 = \left( \frac{P_1}{P_1} B - C_1 \pi_1 \right) u_1 + \frac{\partial}{\partial t} C_1 D u_1,
\]
or

\[
\frac{P_1}{P_1} (\partial u_0 - B u_1) = C_1 (\partial D u_1 - \pi_1 u_1).
\]

With (3.11) and (3.8) we obtain \( KC_1(\xi) = 0 \). Since, by Lemma 3, \( C_1(\xi) \neq 0 \), we have \( K = 0 \) and (3.8) reduces to

\[
\phi D u_1 = \pi_1 u_1.
\]

Finally \( D(\phi u_1) = \phi' u_1 + \phi D u_1 = (\phi' + \pi_1) u_1 = \psi u_1 \), where \( \psi \) is a polynomial of degree \( \leq 1 \). Since \( u_1 \) is quasi-definite the degree of \( \psi \) has to be 1; thus \( u_1 \) is classical.

**Examples.** A linear functional is positive-definite if and only if it can be represented by a distribution function \( \Psi \) as (see [3], Ch. II)

\[
\langle u, p \rangle = \int_a^b p(x) d\Psi(x), \quad p \in \mathcal{P}.
\]

Then a coherent pair of positive-definite linear functionals \( \{u_0, u_1\} \) corresponds to a coherent pair of distribution functions \( \{d\Psi_0, d\Psi_1\} \). We mention all coherent pairs of distribution functions which follow from Theorem 1 and 2. The classical polynomials are given in their usual notation (see e.g. Szego [12]) and not in their monic version; a linear change in the variable gives again a coherent pair.

A. Laguerre Case. The distribution function \( d\Psi(x) = x^\alpha e^{-x} dx \) with \( \alpha > -1 \) on \( (0, \infty) \) defines a positive-definite classical functional \( u \). The functional \( u \) satisfies \( D(\phi u) = \psi u \) with \( \phi(x) = x \).

From Theorem 1 and Theorem 2 we obtain the following coherent pairs.

\[
d\Psi_0(x) = x^\alpha e^{-x} dx, \quad d\Psi_1(x) = \frac{1}{x - \xi} x^{\alpha + 1} e^{-x} dx + M\delta(\xi), \tag{3.12}
\]

where we have to take \( \alpha > -1, \xi \leq 0, M \geq 0 \).

\[
d\Psi_0(x) = (x - \xi) x^{\alpha - 1} e^{-x} dx, \quad d\Psi_1(x) = x^\alpha e^{-x} dx, \tag{3.13}
\]
where $\zeta < 0$, $\alpha > 0$.

\[ d\Psi_0(x) = e^{-x} \, dx + M\delta(0), \quad d\Psi_1(x) = e^{-x} \, dx, \quad (3.14) \]

with $M \geq 0$. In (3.12) the $d\Psi_1$ has to be interpreted as

\[ \int_{-\infty}^{\infty} f(x) \, d\Psi_1(x) = \int_{0}^{\infty} f(x) \frac{1}{x - \zeta} x^{\alpha + 1} e^{-x} \, dx + M f(\zeta), \]

so the spectrum of $\Psi_1$ is $[0, \infty) \cup \{\zeta\}$. The spectrum of all other distribution functions is $[0, \infty)$. It is not difficult to check that (3.12), (3.13) and (3.14) indeed define coherent pairs. For (3.12) and (3.13) compare [7]. Since (3.14) has not been mentioned in [7] we give a proof of it.

Let $\{P_n\}$ denote an orthogonal polynomial sequence with respect to $d\Psi_0$. Since $L_n(0) = 1$ for all $n \geq 0$ (see [12], 5.1.7) we have

\[ \int_{-\infty}^{\infty} \{L_n^{(0)} - L_{n-1}^{(0)}\} P_k \, d\Psi_0 = \int_{0}^{\infty} \{L_n^{(0)} - L_{n-1}^{(0)}\} P_k e^{-x} \, dx = 0 \]

if $k \leq n - 2$. This implies

\[ L_n^{(0)} - L_{n-1}^{(0)} = c_n P_n + c_{n-1} P_{n-1}, \]

for some constants $c_n$ and $c_{n-1}$. Then differentiation gives (compare [12], p. 102)

\[ L_n^{(0)} - L_{n-1}^{(0)} = -c_n P'_n - c_{n-1} P'_{n-1}. \]

Remark. If $\alpha \neq 0$, then (3.7) and (3.8) with $N = K = 0$ imply that $d\Psi_0$ in (3.13) cannot have a term $M\delta(0)$.

B. Jacobi Case. The distribution function $d\Psi(x) = (1 - x)^\alpha (1 + x)^\beta$ with $\alpha > -1$, $\beta > -1$ on $(-1, 1)$ represents a positive-definite classical functional $u$ with $D(u) = \psi u$, where $\psi(x) = 1 - x^2$.

Theorem 1 and Theorem 2 give the coherent pairs

\[ d\Psi_0(x) = (1 - x)^\alpha (1 + x)^\beta \, dx, \]

\[ d\Psi_1(x) = \frac{1}{|x - \zeta|} (1 - x)^{\alpha + 1} (1 + x)^{\beta + 1} \, dx + M\delta(\zeta), \quad (3.15) \]
with $x > -1$, $\beta > -1$, $|\xi| \geq 1$, $M \geq 0$,
\[
\begin{align*}
d\mathcal{P}_0(x) &= |x - \xi| (1 - x)^{s-1} (1 - x)^{\beta - 1} \, dx, \\
d\mathcal{P}_1(x) &= (1 - x)^s (1 + x)\beta \, dx
\end{align*}
\]  \hspace{1cm} (3.16)

with $|\xi| > 1$, $x > 0$, $\beta > 0$,
\[
\begin{align*}
d\mathcal{P}_0(x) &= (1 + x)^{\beta - 1} \, dx + M\delta(1), \\
d\mathcal{P}_1(x) &= (1 + x)\beta \, dx,
\end{align*}
\]  \hspace{1cm} (3.17)

with $\beta > 0$, $M \geq 0$ and
\[
\begin{align*}
d\mathcal{P}_0(x) &= (1 - x)^{s-1} \, dx + M\delta(-1), \\
d\mathcal{P}_1(x) &= (1 - x)^s \, dx,
\end{align*}
\]  \hspace{1cm} (3.18)

with $x > 0$, $M \geq 0$.

The spectrum of $\mathcal{P}_1$ in (3.15) is $[-1, 1]$; the spectrum of the other distribution functions is $[-1, 1]$.

Again it is easy to check that this indeed are coherent pairs (for (3.15) and (3.16) compare [7]). The coherence of (3.17) follows with $P^{(0, \beta-1)}_\alpha(1) = 1$ for all $\alpha \geq 0$ (see [12], (4.1.1)) and
\[
\frac{d}{dx}(P^{(0, \beta-1)}_\alpha - P^{(0, \beta-1)}_{\alpha-1}) = \frac{1}{2} (2\alpha + \beta - 1) P^{(0, \beta)}_{\alpha-1},
\]
(see [1], p. 782). The coherence of (3.18) follows in a similar way.

C. Hermite Case. In the Hermite case the distribution function is
\[
d\mathcal{F}(x) = e^{-x^2} \, dx \text{ on } (-\infty, \infty) \text{ with } \varphi(x) \equiv 1.
\]

Theorem 1 and 2 imply that there cannot exist coherent pairs.

4. SYMMETRICALLY COHERENT PAIRS

In this section $u_0$ and $u_1$ denote symmetric quasi-definite linear functionals and $\{P_n\}$ and $\{T_n\}$ the corresponding MOPS. The polynomials of even degree are even functions and the polynomials of odd degree odd ones. In this situation (2.1) only can be satisfied with $\sigma_n = 0$ for all $n \geq 1$.

Therefore Iserles et al. [4] introduced the concept of symmetrically coherent pair. The pair $\{u_0, u_1\}$ of symmetric functionals is called a symmetrically coherent pair if there exist non-zero constants $\sigma_n$ such that
\[
T_n = \frac{P_{n+1}^{\sigma_{n+1}}}{n+1} - \sigma_{n-1} \frac{P_{n-1}^{\sigma_{n-1}}}{n-1} \text{ for } n \geq 2.
\]
In this section we assume \( \{u_0, u_1\} \) to be a symmetrically coherent pair and we will prove that again at least one of the functionals has to be classical. Therefore we will use the polynomials
\[
C_n = \sigma_{n-1} \frac{T_n}{t_n} - \frac{T_{n-2}}{t_{n-2}}, \quad n \geq 1.
\]

Proposition 1 is replaced by Proposition 2 which can be proved in the same way.

**Proposition 2.** Let \( \{u_0, u_1\} \) denote a symmetrically coherent pair, then
\[
n \frac{P_n}{p_n} u_0 = D(C_{n+1} u_1) \quad \text{for} \quad n \geq 1.
\]

**Corollary 2.** Let \( \{u_0, u_1\} \) denote a symmetrically coherent pair, then
\[
\varphi D u_1 = \pi u_1, \quad \varphi \varphi u_0 = x B u_1, \quad \pi u_0 = B D u_1
\]

with
\[
\varphi = \frac{3}{x p_3} C_2 + \frac{P_1}{x p_1} C_4, \quad (4.1)
\]
\[
\pi = -3 \frac{p_3}{x p_3} C_2 + \frac{P_1}{x p_1} C_4, \quad (4.2)
\]
\[
B = \frac{1}{x} \{C_2 C_4 - C_4 C_2\}, \quad (4.3)
\]
where degree \( \varphi \leq 4 \), degree \( \pi \leq 3 \) and degree \( B = 4 \).

**Proof.** Proposition 2 with \( n = 1 \) and \( n = 3 \) reads
\[
\frac{P_1}{p_1} u_0 = C_2 u_1 + C_2 D u_1, \quad (4.4)
\]
\[
3 \frac{P_3}{p_3} u_0 = C_4 u_1 + C_4 D u_1, \quad (4.5)
\]
where \( P_1, P_3, C_2 \) and \( C_4 \) are odd polynomials. Elimination of \( u_0 \) gives the first identity of Corollary 2. Elimination of \( D u_1 \) gives the second and elimination of \( u_1 \) gives the last relation. The leading coefficient of \( B \) is
\[2(\sigma_1 \sigma_3 \tau_2 \ell_4) \neq 0.\]
All above mentioned polynomials are either even or odd. Then all zeros, apart from \(x = 0\) in the odd polynomials, appear in pairs \([-\xi, \xi]\). A result similar to Corollary 2 has been given in [8] based on Proposition 2 with \(n = 1\) and \(n = 2\). We have chosen the definition of \(B\) in such a way that we have the next lemma.

**Lemma 4.** (i) If \(B\) in (4.3) is of the form \(B = 2(\sigma_1 \sigma_3/t_2 t_4)(x^2 - \xi^2)^2\), then \(C_2 = (\sigma_1/t_2)(x^2 - \xi^2)\) and \((x^2 - \xi^2) | C_4\).

(ii) If \(C_2 | B\), then \(B\) is of the form \(B = (2\sigma_1 \sigma_3/t_2 t_4)(x^2 - \xi^2)^2\).

**Proof.** Put \(C_2 = (\sigma_1/t_2)(x^2 - x^2)\) and \(C_4 = (\sigma_3/t_4)(x^4 + \beta^2 x^2 + \gamma^2)\). Then (4.3) gives

\[
B = \frac{2\sigma_1 \sigma_3}{t_2 t_4} (x^4 - 2x^2 x^2 - x^2 \beta^2 - \gamma^2).
\]

(i) If \(B = 2(\sigma_1 \sigma_3/t_2 t_4)(x^2 - \xi^2)^2\), then \(x^2 = \xi^2\) and \(-x^2 \beta^2 - \gamma^2 = \xi^4\). This implies \(C_3 = (\sigma_1/t_2)(x^2 - \xi^2)\) and \(C_4(\xi) = 0\), i.e. \((x^2 - \xi^2) | C_4\).

(ii) If \(C_2 | B\), then \(B(x) = 0\), i.e. \(-x^4 - x^2 \beta^2 - \gamma^2 = 0\) and \(B = (2\sigma_1 \sigma_3/t_2 t_4)(x^2 - 2x^2 x^2 + x^4)\) has the desired form. \(\square\)

Lemma 4 enables us to characterize \(\{u_0, u_1\}\) in the case that \(B\) is a pure square.

**Theorem 3.** Let \(\{u_0, u_1\}\) denote a symmetrically coherent pair of quasi-definite linear functionals. Let \(B\) in (4.3) be of the form \(B = (2\sigma_1 \sigma_3/t_2 t_4)(x^2 - \xi^2)^2\). Then

(i) \(u_0\) is classical with \(D(u_0) = \psi u_0\) for some polynomials \(\psi\), degree \(\psi \leq 2\), degree \(\psi = 1\);

(ii) \(\psi u_0 = 2(\sigma_1 \sigma_3/t_2 t_4)(x^2 - \xi^2) u_1\).

**Proof.** It follows from Lemma 4(i) and (4.1) that we can write \(\phi = (x^2 - \xi^2) \phi\), for a polynomial \(\phi\) with degree \(\phi \leq 2\). The elimination of \(Du_1\) from (4.4) and (4.5) can be done in such a way that we obtain

\[
x \phi u_0 = x \frac{2\sigma_1 \sigma_3}{t_2 t_4} (x^2 - \xi^2) u_1,
\]

i.e.

\[
\phi u_0 = 2 \frac{\sigma_1 \sigma_3}{t_2 t_4} (x^2 - \xi^2) u_1 + M \delta(0),
\]
for some constant $M$. We will show that $M = 0$. Then $u_0$ is classical, since by (4.4),

$$D(\hat{\varphi} u_0) = 2 \frac{\sigma_3}{t_4} D(C_2 u_1) = 2 \frac{\sigma_3}{t_4} p_1 u_0.$$ 

In order to prove that $M = 0$ in (4.7) we use Proposition 2 with $n = 2$:

$$2 \frac{p_2}{p_2} u_0 = C_3 u_1 + C_3 D u_1. \quad (4.8)$$

Elimination of $Du_1$ from (4.4) and (4.8) gives

$$\left(2 \frac{p_2}{p_2} C_2 - \frac{P_1}{p_1} C_3 \right) u_0 = (C_2 C_3 - C_3 C_2) u_1,$$

which will be abbreviated as

$$q_4 u_0 = b_4 u_1, \quad (4.9)$$

where $q_4$ and $b_4$ are even polynomials, degree $q_4 \leq 4$, degree $b_4 = 4$.

Elimination of $u_0$ from (4.6) and (4.9) gives

$$\hat{\varphi} b_4 - 2 \frac{\sigma_1 \sigma_3}{t_2 t_4} q_4 (x^2 - \xi^2) = 0, \quad (4.10)$$

and then elimination of $u_0$ from (4.7) and (4.9) leads to $M q_4(0) = 0$.

If $M = 0$ we are ready. Therefore suppose $M \neq 0$. Then $q_4(0) = 0$.

Since $P_2(0) \neq 0$, we obtain $C_3(0) = 0$, i.e. by Lemma 4(i) $\xi = 0$. Then (4.7) reduces to

$$\hat{\varphi} u_0 = 2 \frac{\sigma_1 \sigma_3}{t_2 t_4} x^2 u_1 + M \delta(0). \quad (4.11)$$

Putting $q_4 = x^2 q_2$, $b_4 = x^2 b_2$, we obtain from (4.9) and (4.10)

$$x^2 q_2 u_0 = x^2 b_2 u_1, \quad (4.12)$$

$$\hat{\varphi} b_2 - 2 \frac{\sigma_1 \sigma_3}{t_2 t_4} q_2 x^2 = 0. \quad (4.13)$$
Then elimination of $u_1$ from (4.11) and (4.12) gives $M b_2(0) = 0$. Since we had assumed $M \neq 0$ we obtain $b_2(0) = 0$, i.e.

$$C_2 C_3 - C_1 C_4 = b_4 = x^2 b_2 = \frac{\sigma_1 \sigma_2}{t_2 t_3} x^4.$$  

It is easy to see that then $C_4 = (\sigma_3 / t_3) x^3$.

We have found that $M \neq 0$ implies $C_2 = (\sigma_1 / t_2) x^2$ and $C_3 = (\sigma_2 / t_1) x^3$. Then elimination of $D u_1$ from (4.4) and (4.8) can be done in such a way that one arrives at

$$q_2 u_0 = \frac{\sigma_1 \sigma_2}{t_2 t_3} x^3 u_1.$$  \hspace{1cm} (4.14)

Relation (4.13) reduces to

$$\frac{\sigma_2}{t_3} - \frac{2 \sigma_3}{t_4} q_2 = 0.$$  \hspace{1cm} (4.15)

Finally (4.11), (4.14) and (4.15) imply $M = 0$, a contradiction. This completes the proof of the theorem.  

In order to treat the situation where $B$ in (4.3) has two different pairs of zeros $\{-\xi_1, \xi_1\}$ and $\{-\xi_2, \xi_2\}$ we derive a basic relation similar to relation (3.1).

By Proposition 2 and Corollary 2 we have

$$(2n + 1) \frac{P_{2n+1}}{P_{2n+1}} B u_1$$

$$= (2n + 1) \frac{P_{2n+1}}{x P_{2n+1}} x B u_1$$

$$= (2n + 1) \frac{P_{2n+1}}{x P_{2n+1}} x \varphi u_0 = \varphi D(C_{2n+2} u_1)$$

$$= \varphi C_{2n+2} u_1 + \varphi C_{2n+2} Du_1 = (\varphi C_{2n+2} + C_{2n+2} \pi) u_1.$$  

Hence

$$(2n + 1) \frac{P_{2n+1}}{x P_{2n+1}} B = \varphi \frac{C_{2n+2}}{x} + C_{2n+2} \frac{\pi}{x}, \quad n \geq 0.$$  \hspace{1cm} (4.16)

We have used the fact that $P_{2n+1}$, $C_{2n+2}$ and $\pi$ are odd polynomials.
Lemma 5. Let \( \zeta \) be such that \( B(\zeta) = 0, \varphi(\zeta) \neq 0 \), where \( B \) and \( \varphi \) denote the polynomials defined in (4.3) and (4.1). Then there exists a \( k \) independent of \( n, k \neq 0 \), such that

\[
C_{2n+2}(\zeta) + k \frac{C_{2n+2}^{}(\zeta)}{\zeta} = 0 \quad \text{for all } n \neq 0.
\]

Proof. Substitution of \( \zeta \) in (4.16) gives

\[
\varphi(\zeta) \frac{C_{2n+2}^{}(\zeta)}{\zeta} + C_{2n+2}(\zeta) \frac{\pi(\zeta)}{\zeta} = 0.
\]

(If \( \zeta = 0 \), then \( \pi(\zeta)/\zeta \) has to be read as \( \pi(\zeta)/\zeta = \lim_{x \to 0} \pi(x)/x \); the same should be done with \( C_{2n+2}^{}(\zeta) \).)

The relation with \( n = 0 \) reads

\[
\varphi(\zeta) \frac{C_{1}^{*}(\zeta)}{\zeta} + C_{1}(\zeta) \frac{\pi(\zeta)}{\zeta} = 0.
\]

Since \( C_{1}^{*}(\zeta)/\zeta = 2(\sigma_{1}/t_{2}) \neq 0 \) and \( \varphi(\zeta) \neq 0 \) it follows \( \pi(\zeta)/\zeta \neq 0 \). Then the lemma is satisfied with \( k = \varphi(\zeta)(\zeta/\pi(\zeta)) \neq 0 \).

Lemma 6. Suppose that there exist \( \zeta_1, \zeta_2, k_1 \neq 0 \) and \( k_2 \neq 0 \) such that

\[
C_{2n+2}(\zeta_1) + k_1 \frac{C_{2n+2}^{}(\zeta_1)}{\zeta_1} = 0 \quad \text{and} \quad C_{2n+2}(\zeta_2) + k_2 \frac{C_{2n+2}^{}(\zeta_2)}{\zeta_2} = 0
\]

(4.17)

for all \( n \geq 0 \). Then \( \zeta_1 = \pm \zeta_2 \) and \( k_1 = k_2 \).

Proof. The polynomials \( T_{2n} \) and \( C_{2n} \) are even polynomials. Write \( T_{2n}^*(x^2) = T_{2n}(x) \) and \( C_{2n}^*(x^2) = C_{2n}(x) \), then \( \{T_{2n}^*\} \) are orthogonal with respect to the functional \( u_1^* \) defined by the moments \( (u_1^*, x^n) = (u_1, x^{2n}) \), \( n = 0, 1, 2, \ldots \). The relations (4.17) become

\[
C_{n+1}^*(\zeta_j^2) + 2k_{j} C_{n+1}^{*}^{}(\zeta_j^2) = 0, \quad j = 1, 2, \quad n \geq 0.
\]

Proceeding as in the proof of Lemma 2 we obtain \( \zeta_1^2 = \zeta_2^2 \) and \( k_1 = k_2 \).

Lemma 7. Suppose \( B \) in (4.3) has two different pairs of zeros. Then

(i) at least one pair of zeros of \( B \) is also a pair of zeros of \( \varphi \);

(ii) if \( (x^2 - \zeta_1^2) \mid B \) and \( (x^2 - \zeta_2^2) \mid \varphi \), then \( C_{2}(\zeta) \neq 0 \) and \( (x^2 - \zeta^2) \mid \pi \).
Proof. Assertion (i) of the lemma is a direct consequence of Lemma 5 and Lemma 6. If \([-\xi, \xi]\) is a common pair of zeros of \(B\) and \(\varphi\), then (4.16) with \(n=0\) implies \(x^2-\xi^2\mid C_2(\pi/x)\). Since \(B\) has two different pairs of zeros Lemma 4(ii) implies \(C_2(\xi) \neq 0\). Hence \(x^2-\xi^2\mid \pi\). 

**Theorem 4.** Let \(\{u_0, u_1\}\) denote a symmetrically coherent pair of quasi-definite linear functionals. Let \(B\) in (4.3) be of the form

\[
B = \frac{2}{t_2 t_4} (x^2 - \xi_1^2)(x^2 - \xi_2^2) \quad \text{with} \quad \xi_1^2 \neq \xi_2^2.
\]

Then

(i) \(u_1\) is classical with \(D(\tilde{\varphi} u_1) = \psi u_1\) for some polynomials \(\tilde{\varphi}, \psi\),

degree \(\tilde{\varphi} \leq 2\), degree \(\psi = 1\);

(ii) there exists \(\xi \in \{\xi_1, \xi_2\}\) such that

\[
\tilde{\varphi} u_0 = \frac{2}{t_2 t_4} (x^2 - \xi^2) u_1.
\]

Proof. According to Lemma 7(i) we may suppose that \([-\xi_1, \xi_1]\) is also a pair of zeros of \(\varphi\). Then, by Lemma 7(ii), \(C_2(\xi_1) \neq 0\) and \([-\xi_1, \xi_1]\) is also a pair of zeros of \(\pi\). Put

\[
B = (x^2 - \xi_1^2) \tilde{B}, \quad \varphi = (x^2 - \xi_1^2) \tilde{\varphi}, \quad \pi = (x^2 - \xi_1^2) \pi_1.
\]

Then (4.16) becomes

\[
(2n+1) \frac{P_{2n+1}}{P_{2n+1}} \tilde{B} = \tilde{\varphi} C_{2n+2} + C_{2n+2} \pi_1, \quad n \geq 0.
\] (4.18)

Moreover, the relations \(x\varphi u_0 = xBu_1, \ B Du_1 = \pi u_0\) and \(\varphi Du_1 = \pi u_1\) from Corollary 2 give

\[
x^2 \tilde{\varphi} u_0 = x^2 \tilde{B} u_1 + M\tilde{\varphi}(\xi_1) + M\tilde{\varphi}(-\xi_1),
\] (4.19)

\[
x \tilde{B} Du_1 = x\pi_1 u_0 + N\tilde{\varphi}(\xi_1) + N\tilde{\varphi}(-\xi_1),
\] (4.20)

\[
x \varphi Du_1 = x\pi_1 u_1 + K\tilde{\varphi}(\xi_1) + K\tilde{\varphi}(-\xi_1),
\] (4.21)

where we have used the fact that the functionals applied on polynomials of odd degree have to give zero.

We will show \(M = N = K = 0\).
It follows from (4.18) and Proposition 2 that

\[
(\tilde{C} C_{2n+2} + C_{2n+2} \pi_1) u_0 = (2n + 1) \frac{P_{2n+1}}{P_{2n+1}} \tilde{B} u_0 = \tilde{B} (C_{2n+2} u_1 + C_{2n+2} D u_1).
\]

Hence

\[
\frac{C_{2n+2}}{x} \{ x^2 \phi u_0 - x^2 \tilde{B} u_1 \} = C_{2n+2} \{ x^2 \tilde{B} u_1 - x \pi_1 u_0 \}, \quad n \geq 0.
\]

Then (4.19) and (4.20) imply

\[
2 \frac{C_{2n+2}(\xi_1)}{\xi_1} M = 2C_{2n+2}(\xi_1) N, \quad n \geq 0. \tag{4.22}
\]

Observe that \( C_{2}(\zeta_1) \zeta_1 = 2(\zeta_1/\zeta_2) \neq 0, C_2(\zeta_1) \neq 0 \); then \( M = 0 \) if and only if \( N = 0 \). Consider the second pair of zeros \( \{ -\zeta_2, \zeta_2 \} \) of \( B \). There are two possibilities: \( \phi(\zeta_2) \neq 0 \) and \( \phi(\zeta_2) = 0 \). If \( \phi(\zeta_2) \neq 0 \), then Lemma 5 and Lemma 6 imply that relation (4.22) has to be trivial, i.e. \( M = N = 0 \). If \( \phi(\zeta_2) = 0 \), then we can proceed with \( \zeta_2 \) as with \( \zeta_1 \) and arrive at a relation for \( \zeta_2 \) similar to relation (4.22) for \( \zeta_1 \). Again Lemma 6 implies that at least one of the relations has to be a trivial one, and without loss of generality we may suppose that the relation (4.22) for \( \zeta_1 \) is trivial. Hence in both cases we obtain \( M = N = 0 \).

In order to prove that \( K = 0 \) we proceed as follows. With (4.4) and (4.18) for \( n = 0 \) we obtain

\[
\phi \frac{P_1}{P_1} u_0 = \phi (C_2 u_1 + C_2 D u_1) = \left( \frac{P_1}{P_1} \tilde{B} - C_2 \pi_1 \right) u_1 + \phi C_2 D u_1,
\]

or

\[
\frac{P_1}{x P_1} \{ x^2 \phi u_0 - x^2 \tilde{B} u_1 \} = C_2 \{ x \phi D u_1 - x \pi_1 u_1 \}.
\]

Then (4.19) with \( M = 0 \) and (4.21) imply

\[
2K C_2(\zeta_1) = 0.
\]

Since \( C_2(\zeta_1) \neq 0 \), we have \( K = 0 \).

Now we are able to prove the assertions of the theorem. Relation (4.21) with \( K = 0 \) reads \( x \phi D u_1 = x \pi_1 u_1 \). Since \( u_1 \) is symmetric and \( \phi \) and \( \pi_1 \) are
odd polynomials we have \( \langle \hat{\phi} D u_1, 1 \rangle = \langle \pi_1 u_1, 1 \rangle = 0 \). Then the relation can be reduced to

\[
\hat{\phi} Du_1 = \pi_1 u_1.
\] (4.23)

Then \( D(\hat{\phi} u_1) = \hat{\phi}' u_1 + \hat{\phi} Du_1 = (\hat{\phi} + \pi_1) u_1 = \psi u_1 \), with degree \( \hat{\phi} \leq 2 \), degree \( \psi \leq 1 \). However, \( \psi \) is an odd polynomial and \( \psi \equiv 0 \) is impossible for a quasi-definite functional \( u_1 \). Then degree \( \psi = 1 \) and \( u_1 \) is classical. This proves assertion (i) of the theorem.

In the same way (4.19) with \( M = 0 \) reduces to

\[
x \hat{\phi} u_0 = x \tilde{B} u_1
\] (4.24)
or

\[
\hat{\phi} u_0 = \tilde{B} u_1 + L \delta(0).
\] (4.25)

Observe that \( \tilde{B} = (2 \pi_1 \sigma_3 / t_2 t_4)(x^2 - \xi^2) \).

We will prove that \( L = 0 \) in (4.25), which completes the proof of assertion (ii) of the theorem. Elimination of \( u_0 \) from (4.8) and (4.24) gives, using (4.23),

\[
C_3 \hat{\phi} + C_3 \pi_1 - 2 \frac{P_2}{P_2} \tilde{B} = 0.
\]

Then elimination of \( u_0 \) from (4.8) and (4.25) gives

\[
2 \frac{P_2(0)}{P_2} L = 0.
\]

Since \( P_2(0) \neq 0 \), we obtain \( L = 0 \). 

Theorem 3 and Theorem 4 enables us to give all symmetrically coherent pairs which can be represented by distribution functions. In Theorem 3 and Theorem 4 the \( \xi \) may be complex, in the distribution functions below we always assume the \( \xi \) to be real.

D. Hermite Case. The classical distribution function is \( d \mathcal{P}(x) = e^{-x^2} dx \) on \( (-\infty, \infty) \) with \( \mathcal{P}(x) = 1 \). Theorem 3 and Theorem 4 give the symmetrically coherent pairs of distribution functions on \( (-\infty, \infty) \)

\[
\begin{align*}
&\left\{ e^{-x^2} dx, \frac{e^{-x^2}}{x^2 + \xi^2} dx \right\} \quad \text{with} \quad \xi \neq 0, \\
&\left\{ (x^2 + \xi^2) e^{-x^2} dx, e^{-x^2} dx \right\}.
\end{align*}
\]

It is easy to prove that these pairs are indeed symmetrically coherent pairs.
E. Gegenbauer Case. The classical distribution function is \( dP(x) = (1 - x^2)^\alpha \) on \((-1, 1)\) with \( \alpha > -1 \); the corresponding functional \( u \) satisfies \( D(\psi u) = \psi u \) with \( \phi(x) = 1 - x^2 \).

We obtain the following symmetrically coherent pairs of distribution functions with obvious definition of the spectra

\[
\left\{ (1 - x^2)^{\alpha - 1} dx, \frac{(1 - x^2)^\alpha}{x^2 + \zeta^2} dx \right\}, \quad \alpha > 0, \quad |\zeta| \neq 0,
\]

and

\[
\left\{ (1 - x^2)^{\alpha - 1} dx, \frac{(1 - x^2)^\alpha}{\zeta^2 - x^2} dx + M\delta(\zeta) + M\delta(-\zeta) \right\},
\]

with \( \alpha > 0, \ |\zeta| > 1, \ M > 0, \)

\[
\left\{ x(1 - x^2)^{\alpha - 1} dx, (1 - x^2)^\alpha dx \right\}, \quad \alpha > 0
\]

\[
\left\{ (x^2 - x^2)(1 - x^2)^{\alpha - 1} dx, (1 - x^2)^\alpha dx \right\}
\]

with \( |\zeta| > 1, \ \alpha > 0 \) and

\[
\left\{ dx + M\delta(1) + M\delta(-1), dx \right\}, \quad M \geq 0.
\]

Again one can prove that the mentioned pairs are coherent pairs.

Remark. In \([2]\) the concept of generalized coherent pairs has been introduced. It reads for linear functionals: let \( u_0 \) and \( u_1 \) denote quasi-definite linear functionals and let \( \{P_n\} \) and \( \{T_n\} \) denote their MOPS, then \( \{u_0, u_1\} \) is called a generalized coherent pair if there exist constants \( \sigma_n, \tau_n \) such that

\[
T_n = P_{n+1}^\sigma - \frac{\sigma_n}{n+1} - \sigma_n \frac{P_{n+1}^\tau - \tau_n}{n+1} \quad \text{for} \quad n \geq 2.
\]

Let \( \alpha > -1, \ \zeta_1 < 0, \ \zeta_2 < 0, \ M > 0, \) then

\[
\left\{ x^\alpha e^{-x} dx, \frac{1}{x - \zeta_2} x^{\alpha+1} e^{-x} dx + M\delta(\zeta_2) \right\}
\]

and

\[
\left\{ (x - \zeta_1) x^\alpha e^{-x} dx, x^{\alpha+1} e^{-x} dx \right\}
\]
are coherent pairs. From this observation it easily follows that
\[
\left\{ (x - \xi_1) x^s e^{-x} \mathrm{d}x, \frac{1}{x - \xi_2} x^{s+1} e^{-x} \mathrm{d}x + M\delta(\xi_2) \right\}
\]
is a generalized coherent pair. (Obviously the \(dx\)-terms are distribution functions on \([0, \infty)\) and if \(M \neq 0\) the last term gives a contribution from \(\xi_2\) outside \((0, \infty)\)). Here none of the distribution functions is a classical one, so the results of this paper cannot be generalized to generalized coherent pairs.

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REFERENCES