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Rational and Heron tetrahedra

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Abstract

Buchholz [R.H. Buchholz, Perfect pyramids, Bull. Austral. Math. Soc. 45 (1991) 353–368] began a systematic search for tetrahedra having integer edges and volume by restricting his attention to those with two or three different edge lengths. Of the fifteen configurations identified for such tetrahedra, Buchholz leaves six unsolved. In this paper we examine these remaining cases for integer volume, completely solving all but one of them. Buchholz also considered Heron tetrahedra, which are tetrahedra with integral edges, faces and volume. Buchholz described an infinite family of Heron tetrahedra for one of the configurations. Another of the cases yields a new infinite family of Heron tetrahedra which correspond to the rational points on a two-parameter elliptic curve.

MSC: primary 11D25; secondary 11D41, 51M25

1. Introduction

Heron triangles, that is, triangles with integer edges and integer area, have been much studied. In fact a complete parametrisation has been known for centuries [9]. We note that a triangle with rational edges and area can be scaled up to one having integer edges and area so that the problem can be recast in terms of rationals instead of integers. We will call a triangle *rational* if all its edges have rational length.

In [4], Buchholz has investigated a natural generalisation of Heron triangles to three dimensions. He defines a *perfect pyramid* (we use *Heron tetrahedron*) as a tetrahedron with integer

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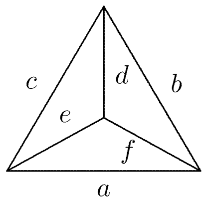

Fig. 1. The tetrahedron (a, b, c, d, e, f).

edges, face areas and volume, and he has discovered an infinite family of them. As above, the search is equivalent to searching for tetrahedra with rational edges (which we call *rational tetrahedra*) with rational face areas and rational volume.

With six edges (see Fig. 1) as independent variables in the volume formula, the problem is far too difficult to attack in general, so one begins by imposing restrictions on the edges in the hope of producing workable special cases. Buchholz took the approach of equating some of the edge lengths so that the tetrahedra have only one, two or three different edge lengths:

1-parameter	(i) $a = b = c = d = e = f$,	
2-parameter	(i) $a = b = c = d = e,\ f$,	(ii) $a = b = c = d,\ e = f$,
	(iii) $a = c = d = f,\ b = e$,	(iv) $a = b = c,\ d = e = f$,
	(v) $a = d = f,\ b = c = e$,	
3-parameter	(i) $a = b = c = d,\ e,\ f$,	(ii) $a = c = d = f,\ b,\ e$,
	(iii) $a = b = c,\ d = e,\ f$,	(iv) $a = d = f,\ b = c,\ e$,
	(v) $a = d = f,\ b = e,\ c$,	(vi) $a = d,\ b = e,\ c = f$,
	(vii) $a = e,\ b = f,\ c = d$,	(viii) $a = b,\ c,\ d = e = f$,
	(ix) $a = d,\ b = f,\ c = e$,	(x) $a = e,\ b = c,\ d = f$.

Buchholz was able to completely determine when the 1- and 2-parameter cases have rational volume, but only four of the 3-parameter cases succumbed. He showed that there are infinitely many tetrahedra with rational volume for each of the configurations 3(ii), 3(vi), 3(viii) and 3(ix). Whether there are infinitely many (or even finitely many) for the other configurations was left open, although four examples of 3(v) were given.

This paper aims to answer some of the questions left open by Buchholz. We show in the following sections that there are no tetrahedra with rational volume for cases 3(i), 3(iii), 3(iv) and 3(vii). We find an infinite family of tetrahedra with rational volume for case 3(v), give a parametric description of all such tetrahedra for cases 3(ii) and 3(viii), and show that 3(x) is not a separate case at all. We also make a more complete investigation into the Heron tetrahedra of configuration 3(vi), and discover an infinite family of Heron tetrahedra of configuration 3(ii). In each case, this requires a search for rational points on certain elliptic curves.

For completeness, we describe here the 4-, 5- and 6-parameter configurations.

4-parameter	(i) $a = b = c,\ d,\ e,\ f$,	(ii) $a = b = d,\ c,\ e,\ f$,
	(iii) $a = b = f,\ c,\ d,\ e$,	(iv) $a = d,\ b = e,\ c,\ f$,
	(v) $a = d,\ b = f,\ c,\ e$,	(vi) $a = b,\ d = f,\ c,\ e$,
	(vii) $a = b,\ d = e,\ c,\ f$,	
5-parameter	(i) $a = d,\ b,\ c,\ e,\ f$,	(ii) $a = b,\ c,\ d,\ e,\ f$,
6-parameter	(i) $a,\ b,\ c,\ d,\ e,\ f$.	

Table 1
4-, 5- and 6-parameter tetrahedra with rational volume

Case	a, b, c, d, e, f	V	Case	a, b, c, d, e, f	V
4(i)	8, 8, 8, 11, 13, 15	96	4(ii)	8, 8, 13, 8, 15, 14	21
4(iii)	13, 13, 6, 8, 10, 13	96	4(iv)	5, 7, 6, 5, 7, 8	24
4(v)	9, 15, 8, 9, 11, 15	84	4(vi)	10, 10, 13, 16, 15, 16	231
4(vii)	7, 7, 6, 9, 9, 4	24			
5(i)	8, 5, 7, 8, 14, 12	15	5(ii)	4, 4, 2, 5, 6, 7	6
6(i)	6, 5, 2, 7, 8, 4	6			

Table 2
Heron 4(iv), 4(vii), 5(ii) and 6(i) tetrahedra

Case	a, b, c, d, e, f	$A_{a,b,c}$	$A_{a,e,f}$	$A_{b,d,f}$	$A_{c,d,e}$	V
4(iv)	990, 901, 793, 308, 901, 793	338 184	338 184	120 120	120 120	27 320 832
4(vii)	680, 615, 185, 185, 208, 680	55 500	69 888	55 500	15 912	3 144 960
5(ii)	319, 221, 210, 175, 175, 318	23 100	26 796	18 564	14 700	1 034 880
6(i)	117, 80, 53, 52, 51, 84	1800	1890	2016	1170	18 144

They all can have rational volume. Examples are listed in Table 1.

By [6, Theorem 2], 4(i) is not Heron since one of the faces is equilateral. Heron tetrahedra with configurations 4(iv), 4(vii), 5(ii) and 6(i) are known. An example of each is listed in Table 2. It is an open question whether any Heron tetrahedra exist with configuration 4(ii), 4(iii), 4(v), 4(vi) or 5(i).

2. Some history

Once a formula for the volume of a tetrahedron is known (see Eq. (3.2)), a natural step is to look for solutions, i.e., for rational tetrahedra with rational volume. With the six edges as independent variables, most authors (as Buchholz did) place restrictions on the edges in the hope of finding cases with workable volume formulae. The simplest case—a regular rational tetrahedron—does not have rational volume. However, most authors quickly discover a slightly less restrictive type of tetrahedron traditionally called *isosceles*. These tetrahedra have four congruent faces, and opposite edges have equal length (Buchholz's case 3(vi)). Since the four faces are congruent, they have the same perimeter and area. Brown [2] reports the surprising converse: if the four faces of a tetrahedron have the same area or the same perimeter, then the tetrahedron has configuration 3(vi).[1]

In 1877, Hoppe [21] gave eight examples of Heron 3(vi) tetrahedra. Early in the 20th century, Güntsche [13] reduced the problem of finding these tetrahedra to a cubic in two variables. He did not solve the equation in general, but did find nine parametric families of solutions which we list in Appendix A. Haentzschel [18,19] solved the equation using the Weierstrass $\wp$-function, and looked at one particular case in detail. Using Carmichael's parametrisation of Heron triangles [8, p. 13, *Diophantine Analysis*], Buchholz [4] also found an infinite family of Heron 3(vi) tetrahedra, via an elliptic curve, and we generalise his result in Section 7.

[1] This is not true if the tetrahedron is degenerate—for example, the degenerate tetrahedron (17, 28, 25, 17, 28, 39) has all face areas equal but is not a 3(vi) tetrahedron.

A perfect cuboid is a rectangular parallelepiped with edges, face diagonals and body diagonal all integers.[2] Almost-perfect cuboids give rise to two types of rational tetrahedra: one occurring when the body diagonal is irrational (called *right tetrahedra* [28]), and the other when one of the face diagonals is irrational (which we call *right-faces* (*RF*) *tetrahedra*). Leech [23] discusses both of these types of tetrahedra in the context of perfect cuboids. Every rational right tetrahedron (of which there are infinitely many) has rational volume and (at least) three rational face areas. However, there are no known Heron right tetrahedra.

There are infinitely many Heron RF tetrahedra. The smallest two were known to Euler [16, p. 175] as the solutions $(x_1, x_2, x_3) = (104, 153, 672)$ and $(117, 520, 756)$ to equations

$$x_1^2 + x_2^2 = y_1^2, \qquad x_1^2 + x_3^2 = y_2^2, \qquad x_1^2 + x_2^2 + x_3^2 = y_3^2.$$

The rational RF tetrahedron $(x_1, x_2, y_1, y_3, x_3, y_2)$ has volume $\frac{1}{6}x_1x_2x_3$ and face areas $\frac{1}{2}x_1x_2$, $\frac{1}{2}x_1x_3$, $\frac{1}{2}x_2y_2$ and $\frac{1}{2}x_3y_1$, so every rational RF tetrahedron is Heron.

Intimately related to Heron RF tetrahedra are Heron 3(ii) tetrahedra. Four copies of the Heron RF tetrahedron $(x_1, x_2, y_1, y_3, x_3, y_2)$ gives rise to the Heron 3(ii) tetrahedron $(y_3, y_3, 2x_2, y_3, y_3, 2x_3)$, and every Heron 3(ii) tetrahedron can be decomposed into four copies of a Heron RF tetrahedron. Starke [27] described the Heron 3(ii) tetrahedron (1073, 1073, 990, 1073, 1073, 896) but gave an incorrect volume which has been copied in [16,22,25]. The correct volume was calculated in [1]. Buchholz [4] completely described all 3(ii) tetrahedra having rational volume. We give a more satisfying parametric description in Section 6 and use this to find families of Heron 3(ii) tetrahedra.

Other rational tetrahedra with rational volume include (7, 12, 15, 12, 9, 8) with $V = 96$ and $A_{c,d,e} = 54$, and (103, 152, 153, 72, 135, 112) with $V = 120\,960$ and $A_{c,d,e} = 4860$, found by Schwering [9, p. 222]. More such tetrahedra found by Schwering are listed in [25, p. 107]. Schubert [9, p. 224] found $(13, 14, 15, \frac{97}{8}, \frac{97}{8}, \frac{97}{8})$ with $V = 252$, $A_{a,b,c} = 84$ and height $h = 9$. Lietzmann [25, p. 107] found a tetrahedron with edges 6, 7, 8, 9, 10, 11 which has $V = 48$ and $A_{a,b,c} = 24$. Dove and Sumner [10] showed that if a tetrahedron has integer edges and integer volume, then the volume is divisible by 3. For every multiple of 3 up to 99 (except 87) they found a tetrahedron with that volume.

Güntsche appears to have been the first to describe an infinite family of Heron tetrahedra. As well as the nine 3(vi) families discussed above, he described two parametric families of Heron tetrahedra with six different edges in [12]. In 1985 a number of comments relating to Heron tetrahedra were made in [1]. In the early 1990s there was a small surge of interest in tetrahedra with integer volume, possibly prompted by problem D22 in Guy's book [15]. Kalyamanova [22] looked at a number of types of tetrahedra, most of which are also examined in Buchholz's work.

It does not seem to have been mentioned in the literature that the volume of a tetrahedron can be rational (non-integer) when the edges are all integers. This is a clear difference from the situation for Heron triangles, although we show in Section 3 that if a tetrahedron is Heron with integer edges then it has integer volume and face areas. Dove and Sumner [10] note that a mod 3 search gives $(12V)^2 \equiv 0$ or 1, but they want integer volume and so they just discard the cases where $(12V)^2 \equiv 1 \pmod 3$. Sierpiński [25] mentions the tetrahedron (3, 3, 4, 3, 3, 4), which has $V = \frac{8}{3}$, but does not comment on the fact that the volume is not an integer, and Buchholz [4] lists the tetrahedron shown in Fig. 2 without mentioning that its volume is not integral.

[2] Whether any perfect cuboids exist is unknown.

3. Rational/integer area and volume

Our two starting points are Heron's well-known formula for the area of a triangle with edges a, b, c,

$$(4A_{a,b,c})^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$
$$= 2\big(a^2b^2+a^2c^2+b^2c^2\big) - \big(a^4+b^4+c^4\big), \tag{3.1}$$

and the lesser known equation for the volume of a tetrahedron, which was first given by Euler ([20] gives a reference) and has been rediscovered many times since. Authors have expressed the equation in various ways, but Buchholz's form [4]

$$(12V)^2 = \big(a^2+d^2\big)\big(-a^2d^2+b^2e^2+c^2f^2\big) + \big(b^2+e^2\big)\big(a^2d^2-b^2e^2+c^2f^2\big)$$
$$+ \big(c^2+f^2\big)\big(a^2d^2+b^2e^2-c^2f^2\big) - a^2b^2c^2 - a^2e^2f^2 - b^2d^2f^2 - c^2d^2e^2 \tag{3.2}$$

seems to be the most suggestive, indicating clearly the importance of the relationship between opposite edges (a, d), (b, e) and (c, f) of the tetrahedron, as well as clearly showing (by the last four terms) that it matters how the edges combine to give the four faces.

If a rational tetrahedron has any non-integer edges, we can scale the edges to form a similar tetrahedron with integer edges. If the tetrahedron had a rational face area or rational volume, after scaling the face area will be integer but the volume may still be rational.

Proposition 3.1. *If a tetrahedron has edges $a, b, c, d, e, f \in \mathbb{N}$ and a face area $A \in \mathbb{Q}^+$, then $A \in \mathbb{N}$.*

Proof. Without loss of generality we may assume that A is the rational area of the face (a, b, c). Then A, a, b, c satisfy Heron's formula (3.1). Since $a, b, c \in \mathbb{N}$, $(4A)^2$ is an integer. Since A is rational, $4A$ must be an integer. Consider Eq. (3.1) modulo 8. Examining each a, b, c possibility reveals that $(4A)^2$ may be congruent to 0, 3 or 7 (mod 8). The squares modulo 8 are 0 and 1, so $(4A)^2$ is divisible by 8 and hence by 16. Thus $A^2 \in \mathbb{N}$, which implies $A \in \mathbb{N}$. □

Proposition 3.2. *If a tetrahedron has edges $a, b, c, d, e, f \in \mathbb{N}$ and volume $V \in \mathbb{Q}^+$, then either $V \in \mathbb{N}$ or $V = \frac{v}{3}$ for some $v \in \mathbb{N} \setminus 3\mathbb{N}$.*

Proof. The volume equation (3.2) tells us that $(12V)^2$ is an integer. Since V is rational we must have $12V \in \mathbb{N}$. Examining all possibilities modulo 8 (as in the proof above) reveals that $(12V)^2 \equiv 0 \pmod 8$. Then $(12V)^2$ is divisible by 16, and $3V \in \mathbb{N}$.

Now consider Eq. (3.2) modulo 3. The squares are 0 and 1, and calculations reveal that $(12V)^2$ may be congruent to either of these. Since $3V \in \mathbb{N}$ and $12V \equiv 3V \pmod 3$, we can have $(3V)^2$ congruent to 0 or 1. If $(3V)^2$ is divisible by 3, then it is divisible by 9 and $V \in \mathbb{N}$. However, if $(3V)^2 \equiv 1 \pmod 3$, then $3V$ is not divisible by 3, and hence V must have the form $\frac{v}{3}$ for some $v \in \mathbb{N} \setminus 3\mathbb{N}$. □

The tetrahedron shown in Fig. 2 is an example with non-integer volume $V = \frac{476}{3}$. In such a case, where $V = \frac{v}{3}$, we can scale the edge lengths by 3, so that $\gcd(a, b, c, d, e, f) = 3$, to get a tetrahedron with integer volume $9v$.

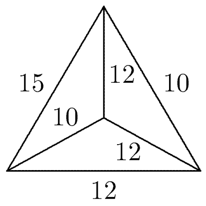

Fig. 2. A tetrahedron with edges in $\mathbb{N}$ and volume in $\mathbb{Q} \setminus \mathbb{Z}$.

Proposition 3.3. *If a tetrahedron $\mathcal{T}$ has edges $a, b, c, d, e, f \in \mathbb{N}$ and volume $V = \frac{v}{3}$, for some $v \in \mathbb{N} \setminus 3\mathbb{N}$, then $\mathcal{T}$ has no rational face area.*

Proof. The edges of a tetrahedron form three pairs (a, d), (b, e) and (c, f). We will say that a pair is (not) divisible by 3 if both of its edges are (not) divisible by three. Examining the volume equation (3.2) modulo 3 reveals that V is rational, but not integer, exactly when two of the pairs are divisible by 3 and one pair is not.

Each face contains exactly one edge from each pair, and the arrangement of edges within a face does not change its area, so the four face area equations will look like

$$(4A)^2 \equiv 2(0 \cdot 1 + 0 \cdot 0 + 0 \cdot 1) - (0 + 0 + 1) \equiv 2 \pmod 3.$$

Since 2 is not a square modulo 3, none of the faces have rational area. □

Corollary 3.4. *If a tetrahedron has integer edges, rational volume and a rational face area, then the volume and all rational face areas are integers.*

4. 3-Parameter families with no rational volume

We now examine the unsolved 3-parameter cases of [4], beginning with those cases where we are able to prove no tetrahedra with rational volume exist.

4.1. Case 3(i)

Theorem 4.1. *Rational tetrahedra with configuration* 3(i) *do not have rational volume.*

Proof. As derived in [4], the volume of a tetrahedron with configuration 3(i) is determined via the equation

$$(12V)^2 = a^2\bigl(f^2\bigl(2a^2 + e^2 - f^2\bigr) - \bigl(a^2 - e^2\bigr)^2\bigr).$$

Rearranging, we have

$$\left(\frac{12V}{a}\right)^2 + \bigl(f^2 - \bigl(a^2 - e^2\bigr)\bigr)^2 = 3e^2 f^2$$

or

$$X^2 + Y^2 = 3Z^2,$$

where $X = \frac{12V}{a}$, $Y = f^2 - (a^2 - e^2)$, $Z = ef$ are all integers. Assume that $\gcd(X, Y, Z) = 1$ and consider this equation modulo 4 to see that it has no integer solution. □

4.2. Case 3(iii)

Theorem 4.2. *Rational tetrahedra with configuration* 3(iii) *do not have rational volume.*

Proof. Buchholz expressed the volume formula as

$$(12V)^2 = a^2\bigl(16A_{a,d,f}^2 - a^2 f^2\bigr).$$

Substituting into Eq. (3.2) we can alternatively write

$$(12V)^2 = a^2\bigl(f^2\bigl(2d^2 + a^2 - f^2\bigr) - \bigl(d^2 - a^2\bigr)^2\bigr).$$

The non-square component has precisely the same form as the equation we examined in Section 4.1 and found to have no solutions. □

4.3. Case 3(iv)

Theorem 4.3. *Rational tetrahedra with configuration* 3(iv) *do not have rational volume.*

Proof. Buchholz expressed the volume formula for this case in a sum-of-squares way similar to that used to solve 3(i). We can rewrite it as

$$(12V)^2 + \bigl(be^2\bigr)^2 + \bigl(a\bigl(a^2 - b^2\bigr)\bigr)^2 = 3(abe)^2.$$

Assuming the four square terms have no common factor and considering this equation modulo 4, we see that $12V$, be^2, $a(a^2 - b^2)$ and abe must all be odd. This implies that a, b and $a^2 - b^2$ are all odd, which is clearly not possible. □

4.4. Case 3(vii)

Theorem 4.4. *Rational tetrahedra with configuration* 3(vii) *do not have rational volume.*

Proof. The volume equation which requires attention is

$$(12V)^2 = 5a^2b^2c^2 - a^4b^2 - b^4c^2 - c^4a^2.$$

We can assume that a, b and c are not all even. Suppose they are all odd. Then $(12V)^2 \equiv 2 \pmod 4$, which has no solutions. Next suppose that exactly one of the edge lengths is even, a say. Then $(12V)^2 \equiv 3 \pmod 4$, which also has no solutions.

Finally, suppose that exactly one of the edge lengths is odd, c say, and that $12V \in \mathbb{N}$. Let $a = 2^\alpha A$, $b = 2^\beta B$ and $12V = 2^\gamma v$, where A, B, v are odd and $\alpha, \beta, \gamma \geqslant 1$ (recall from Section 3 that $12V$ is even). Substituting into the volume formula gives

$$2^{2\gamma} v^2 = 2^{2\alpha+2\beta} 5A^2B^2c^2 - 2^{4\alpha+2\beta} A^4B^2 - 2^{4\beta} B^4c^2 - 2^{2\alpha} c^4A^2.$$

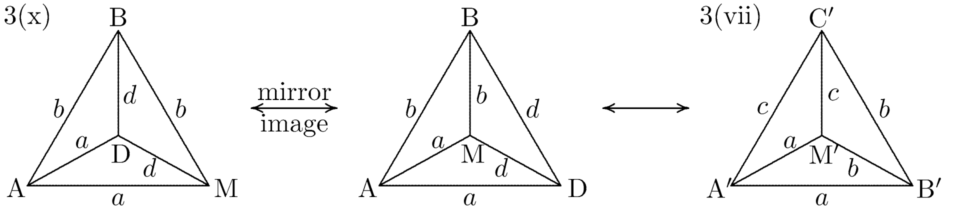

Fig. 3. The tetrahedron configurations 3(vii) and 3(x) are equivalent.

Notice that $4\alpha + 2\beta > 2\alpha + 2\beta > 2\alpha$ for all $\alpha, \beta \geqslant 1$. So the smallest power of 2 will be either 2α, 4β or $2\alpha = 4\beta$.

If $2\alpha < 4\beta$, then $2^{2\gamma-2\alpha}v^2 = 2^{2\beta}5A^2B^2c^2 - 2^{2\alpha+2\beta}A^4B^2 - 2^{4\beta-2\alpha}B^4c^2 - c^4A^2$. Consider this modulo 4 to see that $2^{2\gamma-2\alpha}v^2 \equiv 0-0-0-1 \equiv 3$ (4), and hence that there are no solutions.

If $2\alpha > 4\beta$, then $2^{2\gamma-4\beta}v^2 = 2^{2\alpha-2\beta}5A^2B^2c^2 - 2^{4\alpha-2\beta}A^4B^2 - B^4c^2 - 2^{2\alpha-4\beta}c^4A^2$. Consider this modulo 4 to see that $2^{2\gamma-2\alpha}v^2 \equiv 0-0-1-0 \equiv 3$ (4), and hence there are no solutions.

If $2\alpha = 4\beta$, then $2^{2\gamma-4\beta}v^2 = 2^{2\beta}5A^2B^2c^2 - 2^{6\beta}A^4B^2 - B^4c^2 - c^4A^2$. Consider this modulo 4 to see that $2^{2\gamma-2\alpha}v^2 \equiv 0-0-1-1 \equiv 2$ (4), and hence there are no solutions. □

4.5. Case 3(x)

Figure 3 shows that case 3(x) is simply a mirror image of case 3(vii) with the following correspondences between the edge labelings: $a_{3(\mathrm{vii})} \leftrightarrow a_{3(\mathrm{x})}, b_{3(\mathrm{vii})} \leftrightarrow d_{3(\mathrm{x})}, c_{3(\mathrm{vii})} \leftrightarrow b_{3(\mathrm{x})}$. It has the same volume, the same face areas, the same edge lengths, and hence is not a new case at all.

5. 3-Parameter families with rational volume

5.1. Case 3(v)

The volume equation in this case reduces to

$$(12V)^2 = b^4\left(3a^2 - 2b^2 + c^2\right) - a^2\left(a^2 - c^2\right)^2. \tag{5.1}$$

Buchholz conducted a search for integer solutions and found the following examples with integer volume: $(a, b, c) = (11, 15, 16), (16, 10, 15), (20, 26, 39)$. Notice that $c = \frac{3}{2}b$ for two of these solutions. This substitution yields an infinite family of rational volume tetrahedra.

Theorem 5.1. *A rational tetrahedron with configuration* 3(v) *and* $c = \frac{3}{2}b$ *has rational volume exactly when*

$$[a:b] = \left[4\left(m^2-n^2\right) : 2\left(m^2+n^2\right)\right]$$

for some $m, n \in \mathbb{N}$ *satisfying* $3m^2 > 5n^2$. *Such a tetrahedron has volume* $V = \frac{2}{3}mn(3m^2 - 5n^2)(5m^2 - 3n^2)$.

Proof. Let $b = 2B$, so that $c = 3B$, and substitute into (5.1) to get

$$(12V)^2 = \left(a^2 - B^2\right)^2\left(16B^2 - a^2\right).$$

We require

$$W^2 = 16B^2 - a^2,$$

where $W = \frac{12V}{a^2 - B^2}$ is an integer.

Using the chord method with initial solution $(a, B, W) = (-4, 1, 0)$, we find that the complete solution is given by

$$[a : B] = \left[4(m^2 - n^2) : m^2 + n^2\right].$$

Then $V = \frac{2}{3}mn(3m^2 - 5n^2)(5m^2 - 3n^2)$ is positive for $3m^2 > 5n^2$ or $3n^2 > 5m^2$. These conditions also ensure that the triangle inequalities for the faces are satisfied. Since the equations are symmetric in m and n we shall take $3m^2 > 5n^2$, which for $m, n \in \mathbb{N}$ implies $m > n$ and hence $a > 0$. $\square$

Small examples of these tetrahedra include:

m, n	a, b, c	V
2, 1	12, 10, 15	$\frac{476}{3}$
3, 2	20, 26, 39	924
4, 1	60, 34, 51	$\frac{26\,488}{3}$
4, 3	28, 50, 75	1272
5, 2	84, 58, 87	$\frac{124\,300}{3}$

m, n	a, b, c	V
3, 1	16, 10, 15	231
5, 1	48, 26, 39	$\frac{10\,675}{3}$
5, 3	32, 34, 51	3675
7, 1	96, 50, 75	$\frac{60\,137}{3}$
7, 3	80, 58, 87	38 913

The areas of the faces of these tetrahedra are given by

$$(4A_{a,a,b})^2 = 16(m^2 + n^2)^2\left(15(m^2 - n^2)^2 - 4m^2n^2\right) \quad \text{and} \tag{5.2}$$

$$(4A_{a,b,\frac{3}{2}b})^2 = \left(15(m^2 - n^2)^2 - 4m^2n^2\right)\left(9(m^2 + n^2)^2 + 64m^2n^2\right). \tag{5.3}$$

If any of the tetrahedra in this family are Heron, then Eq. (5.2) tells us that $15(m^2 - n^2)^2 - 4m^2n^2$ must be a square and hence $9(m^2 + n^2)^2 + 64m^2n^2$ must also be square. Concentrating on tetrahedra with $\gcd(a, b, \frac{3}{2}b) = 1$, we must have exactly one of m, n even. Then $9(m^2 + n^2)^2 + 64m^2n^2 \equiv 1 \pmod 4$, but $15(m^2 - n^2)^2 - 4m^2n^2 \equiv 3 \pmod 4$. So $A_{a,a,b}$ is not rational for any of these tetrahedra, and hence they are not Heron.

This family does not exhaust the integer solutions to Eq. (5.1)—the smallest example $(a, b, c) = (11, 15, 16)$ is not a member of this family—and no further progress has been made in describing the complete solution, or in determining whether Heron tetrahedra with this configuration exist. Quick searches modulo 3 and 4 show that if a Heron tetrahedron does exist, then b is even, a, c are odd, and exactly one of b, c is divisible by 3. The chord method, used to solve $4a^2 - b^2 = X^2$, where $X = \frac{4}{b}A_{a,a,b}$, gives $a = m^2 + n^2$, $b = 2(m^2 - n^2)$. This implies that $b \equiv 2 \pmod 4$. A computer search via Maple has not uncovered any Heron examples.

5.2. Case 3(viii)

In [4] Buchholz found a 2-parameter infinite family of tetrahedra with rational volume and configuration 3(viii) in the special case where $a = \frac{4d}{3}$. Here we are able to give a complete solution in terms of three parameters.

Theorem 5.2. *A rational tetrahedron with configuration* 3(viii) *has rational volume exactly when*

$$[a:c:d] = \left[2\left(4m^2+n^2+p^2\right)\left|4m^2-n^2-p^2\right| : 16mn\left|4m^2-n^2-p^2\right| : \left(4m^2+n^2+p^2\right)^2\right]$$

for some $m, n, p \in \mathbb{N}$. *The volume of such a tetrahedron is* $V = \frac{64}{3}m^2np(4m^2-n^2-p^2)^2 \times (4m^2+n^2+p^2)^2$.

Proof. The volume of a tetrahedron with configuration 3(viii) can be determined via the equation

$$(12V)^2 = c^2\left(4a^2d^2 - c^2d^2 - a^4\right).$$

Rearranging and dividing by c^2a^4, we have

$$4\left(\frac{d}{a}\right)^2 - \left(\frac{cd}{a^2}\right)^2 - \left(\frac{12V}{ca^2}\right)^2 = 1.$$

Using the chord method with initial solution $(\frac{d}{a}, \frac{cd}{a^2}, \frac{12V}{ca^2}) = (-\frac{1}{2}, 0, 0)$ we find that

$$\frac{d}{a} = \frac{4m^2+n^2+p^2}{2(4m^2-n^2-p^2)}, \qquad \frac{cd}{a^2} = \frac{4mn}{4m^2-n^2-p^2}, \qquad \frac{12V}{ca^2} = \frac{4mp}{4m^2-n^2-p^2}. \quad (5.4)$$

From the first two equations of (5.4) we have

$$d = \frac{(4m^2+n^2+p^2)a}{2(4m^2-n^2-p^2)} = \frac{4mna^2}{c(4m^2-n^2-p^2)},$$

which leads to

$$a = \frac{(4m^2+n^2+p^2)c}{8mn}.$$

Substituting this into $\frac{d}{a}$,

$$d = \frac{(4m^2+n^2+p^2)^2c}{16mn(4m^2-n^2-p^2)}.$$

Then

$$\frac{d}{c} = \frac{(4m^2+n^2+p^2)^2}{16mn(4m^2-n^2-p^2)}$$

and

$$\frac{a}{c} = \frac{(4m^2+n^2+p^2)}{8mn} = \frac{2(4m^2-n^2-p^2)(4m^2+n^2+p^2)}{16mn(4m^2-n^2-p^2)}.$$

Positive rational volume is enough to ensure that the tetrahedron exists. If $V > 0$ then $4a^2d^2 - c^2d^2 - a^4 > 0$. This can be rewritten as $a^2(4d^2 - a^2) > c^2d^2$ which implies $2d > a$, and as $d^2(4a^2 - c^2) > a^4$ which implies $2a > c$. So the triangle inequalities for faces (d, d, a) and (a, a, c) are satisfied. For the face (d, d, c) we require $2d > c$.

To see that this is satisfied, first notice that

$$\begin{aligned} 4d^4 - 4a^2d^2 + a^4 &= 4d^2\bigl(d^2 - a^2\bigr) + a^4 \\ &> a^2\bigl(d^2 - a^2\bigr) + a^4 \quad \text{since } 2d > a \\ &= a^2d^2 \\ &> 0 \end{aligned}$$

and $4d^4 - 4a^2d^2 + a^4 > 0$ is equivalent to $4d^2 > 4a^2 - \frac{a^4}{d^2}$. Now if $V > 0$ we have $d^2(4a^2 - c^2) > a^4$, which is equivalent to $4a^2 - \frac{a^4}{d^2} > c^2$. So $4d^2 > 4a^2 - \frac{a^4}{d^2} > c^2$ and hence $2d > c$. □

Now we can ask whether there are Heron tetrahedra of this configuration. So far we have found none. This configuration has two congruent faces (d, d, a) and two other isosceles faces (a, a, c) and (d, d, c). Substituting the above parametrisation into the area formulae for these faces gives

$$\begin{aligned} A^2_{a,a,c} &= 256m^2p^2\bigl(4m^2 - n^2 - p^2\bigr)^4\bigl[\bigl(4m^2 + n^2 + p^2\bigr)^2 - 16m^2p^2\bigr], \\ A^2_{d,d,a} &= 16m^2\bigl(4m^2 - n^2 - p^2\bigr)^2\bigl(4m^2 + n^2 + p^2\bigr)^4\bigl(n^2 + p^2\bigr), \\ A^2_{d,d,c} &= 64m^2p^2\bigl(4m^2 - n^2 - p^2\bigr)^2\bigl[\bigl(4m^2 + n^2 + p^2\bigr)^4 - 64m^2p^2\bigl(4m^2 - n^2 - p^2\bigr)^2\bigr]. \end{aligned}$$

In the equation for face (d, d, a) the only non-square term is $n^2 + p^2$, and we can easily make this a square. By putting $n = k^2 - l^2$, $p = 2kl$, we have an infinite family of tetrahedra with rational volume and a rational face area

$$A_{d,d,a} = 4m\bigl(k^2 + l^2\bigr)\bigl(2m - k^2 - l^2\bigr)\bigl(2m + k^2 + l^2\bigr)\bigl(4m^2 + k^4 + 2k^2l^2 + l^4\bigr)^2.$$

Then

$$\begin{aligned} A^2_{a,a,c} &= 1024m^2k^2l^2\bigl(-2m + l^2 + k^2\bigr)^4\bigl(2m + l^2 + k^2\bigr)^4 \\ &\quad \times \bigl[\bigl(k^4 + 2k^2l^2 + l^4 + 4m^2\bigr)^2 - 64k^2l^2m^2\bigr], \\ A^2_{d,d,c} &= 256m^2k^2l^2\bigl(-2m + l^2 + k^2\bigr)^2\bigl(2m + l^2 + k^2\bigr)^2 \\ &\quad \times \bigl[\bigl(k^4 + 2k^2l^2 + l^4 + 4m^2\bigr)^4 - 256k^2l^2m^2\bigl(k^4 + 2k^2l^2 + l^4 - 4m^2\bigr)^2\bigr]. \end{aligned}$$

There are tetrahedra with rational $A_{a,a,c}$. The following example occurs when $k=7$, $l=6$, $m=34$: $a=12\,546$, $c=3744$, $d=28\,577$, which has $A_{d,d,a}=43\,722\,810$, $A_{a,a,c}=10\,069\,920$, $A_{d,d,c}=3024\sqrt{670\,331\,713}$ and $V=215\,696\,452\,608$.

If we approach the 3(viii) tetrahedra by examining the faces first, and worry about the volume later, then we need solutions to

$$A^2_{a,a,c}=c^2\bigl(4a^2-c^2\bigr),\qquad A^2_{d,d,a}=a^2\bigl(4d^2-a^2\bigr),\qquad A^2_{d,d,c}=c^2\bigl(4d^2-c^2\bigr).$$

Looking at the last two of these equations, we use the chord method to see that

$$a=4mn\quad\text{or}\quad 2\bigl(m^2-n^2\bigr),\qquad c=4pq\quad\text{or}\quad 2\bigl(p^2-q^2\bigr),\qquad d=m^2+n^2=p^2+q^2.$$

Thus we have $m=r^2-s^2+u^2$, $n=2rs$, $p=r^2+s^2-u^2$, $q=2ru$. A quick search with $a=4mn$, $c=4pq$ reveals that $(r,s,u)=(3,28,35)$, $(4,21,35)$, $(18,35,38)$, $(19,36,35)$, $(25,24,23)$ gives non-degenerate tetrahedra with all faces rational, but irrational volume.

There are many degenerate Heron tetrahedra with configuration 3(viii), which unfortunately only makes it impossible to use modular arguments to show that there are not any non-degenerate Heron tetrahedra. However, we make the following conjecture.

Conjecture 5.3. *There are no non-degenerate Heron tetrahedra with configuration* 3(viii).

5.3. Case 3(ix)

This configuration has two copies of the face (a,b,c) as well as two different isosceles faces, and the volume is given by

$$\begin{aligned}(12V)^2&=a^2\bigl(2a^2b^2+2a^2c^2+2b^2c^2-2a^4-b^4-c^4\bigr)\\&=a^2\bigl(2a^2\bigl(b^2+c^2-a^2\bigr)-\bigl(b^2-c^2\bigr)^2\bigr).\end{aligned}$$

It is easy to show that a 3(ix) tetrahedron with rational volume has a even and b, c odd. Buchholz lists the tetrahedra with $(a,b,c)=(12,7,11)$, $(28,15,27)$, $(36,19,35)$, all of which have rational volume.[3] A computer search reveals many small examples.

Working from his first example, Buchholz put $a=3(c-b)$ and rearranged the volume formula into the homogeneous quadratic

$$\left(\frac{12V}{c-b}\right)^2=(63b-39c)^2-34(11b-7c)^2.$$

Solving using the chord method gave the infinite family

$$[a:b:c]=\bigl[12p^2-144pq+408q^2:7p^2-78pq+238q^2:11p^2-126pq+374q^2\bigr]$$

with $V=48|p^2-34q^2|(p^2-12pq+34q^2)^2$.

[3] These three examples all satisfy $b=\frac{a}{2}+1$, $c=a-1$, but they are the only examples that do.

This procedure can be easily applied to any known 3(ix) tetrahedron with rational volume to find other infinite families. For example, $(a,b,c)=(12,19,23)$ leads to

$$[a:b:c]=\left[12p^2-336pq+408q^2:19p^2-46pq+646q^2:23p^2-158pq+782q^2\right],$$

which has $V=432|p^2-34q^2|(p^2-28pq+34q^2)^2$, and $(a,b,c)=(28,15,27)$ leads to

$$[a:b:c]$$
$$=\left[28p^2-1680pq+24\,920q^2:15p^2-886pq+13\,350q^2:27p^2-1606pq+24\,030q^2\right],$$

which has $V=\frac{784}{3}|p^2-890q^2|(p^2-60pq+890q^2)^2$. Note that putting $p=4, q=-1$ in the parametrisation from $(a,b,c)=(12,19,23)$ gives $(12,7,11)$. However, no single parametrisation gives many small tetrahedra with rational volume. Unfortunately, we have been unable to find a general solution or a solution which does not rely on a known example.

Leaving the volume, we move our focus to the face areas. The relevant equations are $A^2_{a,b,c}=2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$, $A^2_{a,b,b}=a^2(4b^2-a^2)$ and $A^2_{a,c,c}=a^2(4c^2-a^2)$. A quick search reveals tetrahedra with all face areas rational only for $b=c$. Case 3(ix) then reduces to Buchholz's case 2(iii) which is not Heron. Searching the infinite families above has not revealed any Heron tetrahedra, although we have not proved that there cannot be any.

6. Heron tetrahedra in family 3(ii)

In this case we have two pairs of congruent faces, both isosceles. Therefore it would seem likely that there is a better chance of finding Heron tetrahedra since we have only three quantities to make rational simultaneously. This is indeed the case.

6.1. Rational volume

In [4] Buchholz shows that there are infinitely many tetrahedra with rational volume and configuration 3(ii). As with case 3(viii) we are able to use a sum-of-squares approach to arrive at a complete parametric description of those tetrahedra. We then go on to look for Heron tetrahedra of this configuration.

Theorem 6.1. *A rational tetrahedron with configuration* 3(ii) *has rational volume exactly when*

$$[a:b:e]=\left[4m^2+n^2+p^2:8mn:2\left|4m^2-n^2-p^2\right|\right]$$

for some $m,n,p\in\mathbb{N}$ *such that* $4m^2-n^2-p^2\neq 0$. *Such a tetrahedron has volume* $V=\frac{32}{3}m^2np|4m^2-n^2-p^2|$.

Proof. The volume of a tetrahedron with configuration 3(ii) can be determined via the equation

$$(12V)^2=b^2e^2\left(4a^2-b^2-e^2\right).$$

Rearranging, we have

$$4A^2-B^2-W^2=1,$$

where $A=\frac{a}{e}$, $B=\frac{b}{e}$, $W=\frac{12V}{be^2}$.

Using the chord method with the initial solution $(A, B, W) = (-\frac{1}{2}, 0, 0)$ we find that the complete solution is given by

$$[a:b:e] = \left[4m^2 + n^2 + p^2 : 8mn : 2\left|4m^2 - n^2 - p^2\right|\right].$$

Substituting this into the two face area equations, we have

$$(A_{a,a,b})^2 = 16m^2n^2\left[\left(4m^2 + n^2 + p^2\right)^2 - 16m^2n^2\right]$$

and

$$(A_{a,a,e})^2 = 16m^2\left(4m^2 - n^2 - p^2\right)^2\left(n^2 + p^2\right).$$

Both $(A_{a,a,b})^2$ and $(A_{a,a,e})^2$ are non-negative for all $m, n, p \in \mathbb{N}$, so the triangle inequalities are satisfied for the faces. □

Assuming rational volume, we now want faces with rational area to get Heron tetrahedra.

6.2. *Heron 3(ii) tetrahedra—known families*

As mentioned in Section 2, Heron 3(ii) tetrahedra are intimately linked to Heron RF tetrahedra. Historically, at least two parametric families of Heron RF tetrahedra are known. The first, from [1], gives large Heron RF tetrahedra such as

$$(386\,678\,175, 332\,273\,368, 379\,083\,360, 509\,828\,993, 504\,093\,032, 635\,318\,657).$$

Putting $x = 1$, $y = 2$ in the parametrisation below gives the corresponding Heron 3(ii) tetrahedron with edges $a = 635\,318\,657$, $b = 773\,356\,350$, $e = 758\,166\,720$.

The corresponding Heron 3(ii) tetrahedra have edges

$$\begin{aligned}
a &= \left(y^4 + 6y^2x^2 + x^4\right)\left(y^8 - y^4x^4 + x^8\right)\left(y^8 + 2y^6x^2 + 11y^4x^4 + 2y^2x^6 + x^8\right) \\
&\quad \times \left(y^8 - 4y^6x^2 + 8y^4x^4 - 4y^2x^6 + x^8\right), \\
b &= 2\left(x^8 - y^8\right)\left(y^4 + 6y^2x^2 + x^4\right)\left(y^8 - 4y^6x^2 + 5y^4x^4 - 4y^2x^6 + x^8\right) \\
&\quad \times \left(y^8 + 2y^6x^2 + 11y^4x^4 + 2y^2x^6 + x^8\right), \\
e &= 24x^3y^3\left(y^2 + x^2\right)\left(y^4 + 3y^2x^2 + x^4\right)\left(y^8 - y^4x^4 + x^8\right) \\
&\quad \times \left(y^8 - 4y^6x^2 + 8y^4x^4 - 4y^2x^6 + x^8\right),
\end{aligned}$$

with face areas

$$\begin{aligned}
A_{a,a,b} &= 2x^2y^2\left(y^4 - 3y^2x^2 + x^4\right)\left(y^4 - y^2x^2 + x^4\right)\left(y^6 - 2x^2y^4 + x^4y^2 + x^6\right) \\
&\quad \times \left(y^6 + x^2y^4 - 2x^4y^2 + x^6\right)\left(x^8 - y^8\right)\left(y^4 + 6y^2x^2 + x^4\right)^2 \\
&\quad \times \left(y^8 + 2y^6x^2 + 11y^4x^4 + 2y^2x^6 + x^8\right)^2,
\end{aligned}$$

$$\begin{aligned}A_{a,a,e} = {} & 12x^3y^3\big(x^2+y^2\big)\big(x^4+3x^2y^2+y^4\big)\big(x^6+4x^4y^2-6x^3y^3+4x^2y^4+y^6\big)\\ & \times \big(x^6+4x^4y^2+6x^3y^3+4x^2y^4+y^6\big)\big(x^8-x^4y^4+y^8\big)^2\\ & \times \big(x^8-4x^6y^2+8x^4y^4-4x^2y^6+y^8\big)^2\end{aligned}$$

and volume

$$\begin{aligned}V = {} & 16x^5y^5\big(x^2+y^2\big)\big(x^8-y^8\big)\big(x^4+6x^2y^2+y^4\big)\big(x^4-x^2y^2+y^4\big)\\ & \times \big(x^4-3x^2y^2+y^4\big)\big(x^4+3x^2y^2+y^4\big)\big(x^6+x^4y^2-2x^2y^4-3xy^5+y^6\big)\\ & \times \big(x^6+x^4y^2-2x^2y^4+3xy^5+y^6\big)\big(x^6-3yx^5-2x^4y^2+x^2y^4+y^6\big)\\ & \times \big(x^6+3yx^5-2x^4y^2+x^2y^4+y^6\big)\big(x^8-4x^6y^2+8x^4y^4-4x^2y^6+y^8\big)\\ & \times \big(x^8-x^4y^4+y^8\big)\big(x^8+2x^6y^2+11x^4y^4+2x^2y^6+y^8\big).\end{aligned}$$

The second parametrisation of Heron RF tetrahedra, derived from [23, p. 525] by putting $n=-(m+l)$ in Leech's formulae for r and t, leads to the 2-parameter infinite family of Heron 3(ii) tetrahedra with edges

$$\begin{aligned}a &= \big(2m^4+2m^3l+l^2m^2+2l^3m+2l^4\big)\big(m^4+2m^3l+7l^2m^2+6l^3m+2l^4\big),\\ b &= 2l(l+2m)(m-l)(l+m)\big(m^2+2l^2\big)\big(3m^2+4lm+2l^2\big),\\ e &= 4\big(l^2+lm+m^2\big)m^2(m+2l)^2\big(m^2-2lm-2l^2\big),\end{aligned}$$

with face areas

$$\begin{aligned}A_{a,a,b} &= 2m(m+2l)\big(l^2+lm+m^2\big)l(l+2m)(l-m)(l+m)\big(m^2+2l^2\big)\\ &\quad \times \big(3m^2+4lm+2l^2\big)\big(m^4+2l^2m^2+4l^3m+2l^4\big),\\ A_{a,a,e} &= 2(l-m)(l+2m)\big(m^2+2lm+2l^2\big)\big(l^2+lm+m^2\big)m^2(m+2l)^2\\ &\quad \times \big(-m^2+2lm+2l^2\big)l(l+m)\big(5m^2+2lm+2l^2\big)\end{aligned}$$

and volume

$$\begin{aligned}V &= \frac{16}{3}\big(l^2+lm+m^2\big)^2m^3(m+2l)^3l^2(l+2m)^2(l-m)^2(l+m)^2\\ &\quad \times \big(-m^2+2lm+2l^2\big)\big(3m^2+4lm+2l^2\big)\big(m^2+2l^2\big).\end{aligned}$$

Putting $m=2$, $l=1$ leads to the Heron 3(ii) tetrahedron found by Starke [27], with edges $a=1073$, $b=990$, $e=896$.

So there are infinitely many Heron 3(ii) tetrahedra. It would be nice to be able to describe all Heron 3(ii) tetrahedra, and so we move on to examining their face areas.

6.3. Rational face areas—using elliptic curves

Returning to the parametrisation we found for 3(ii) tetrahedra with rational volume, we want to determine the conditions on m, n, p that will ensure that the face areas are rational. For the face (a, a, e) this is easy. If we put $n = 2qr$, $p = q^2 - r^2$, for $(q, r) = 1$, then we have $A_{a,a,e} = 4m(q^2 + r^2)|4m^2 - (q^2 + r^2)^2| \in \mathbb{N}$. In this case the other face satisfies

$$(A_{a,a,b})^2 = 64m^2q^2r^2\big(q^8 + 4q^6r^2 + 6q^4r^4 + 4q^2r^6 + r^8 + 16m^4 + 8m^2q^4 + 8m^2r^4 - 48m^2q^2r^2\big). \tag{6.1}$$

The choice of $m = 14$, $q = 4$, $r = 1$, for example, makes face area $A_{a,a,b}$ rational and we arrive at Starke's Heron tetrahedron (1073, 896, 1073, 1073, 990, 1073), with $A_{a,a,b} = 436\,800$, $A_{a,a,e} = 471\,240$ and $V = 124\,185\,600$. To find Heron 3(ii) tetrahedra in general we require the octic factor of (6.1) to be a square:

$$q^8 + 4q^6r^2 + 6q^4r^4 + 4q^2r^6 + r^8 + 16m^4 + 8m^2q^4 + 8m^2r^4 - 48m^2q^2r^2 = Z^2.$$

Notice that this is a quartic in m:

$$\begin{aligned} Z^2 &= 16m^4 + 8\big((q^2 - r^2)^2 - 4q^2r^2\big)m^2 + \big(q^2 + r^2\big)^4 \\ &= M^4 + 2\big((q^2 - r^2)^2 - 4q^2r^2\big)M^2 + \big(q^2 + r^2\big)^4, \end{aligned}$$

where $M = 2m$.[4] Using Mordell's transformation [24, Chapter 10, Theorem 2] with $c = \frac{1}{3}(4q^2r^2 - (q^2 - r^2)^2)$, $d = 0$, $e = (q^2 + r^2)^4$ we can birationally transform it into the cubic

$$\begin{aligned} Y^2 &= 4X^3 - \frac{4}{3}\big(q^8 + 14q^4r^4 + r^8\big)X - \frac{8}{27}\big(q^4 + r^4\big)\big(q^8 - 34q^4r^4 + r^8\big) \\ &= \frac{4}{27}\big(3X + q^4 + 6r^2q^2 + r^4\big)\big(3X + q^4 - 6r^2q^2 + r^4\big)\big(3X - 2q^4 - 2r^4\big). \end{aligned}$$

Putting $X = x + \frac{2}{3}(q^4 + r^4)$, $Y = 2y$ and dividing by 4, we have the 2-parameter family of elliptic curves

$$E_{q,r}:\quad y^2 = x^3 + 2\big(q^4 + r^4\big)x^2 + \big(q^4 - r^4\big)^2x = x\big(x + \big(q^2 - r^2\big)^2\big)\big(x + \big(q^2 + r^2\big)^2\big). \tag{6.2}$$

Following the transformations back, we can recover m via the formula

$$m = \frac{y}{2(x + (q^2 - r^2)^2)}. \tag{6.3}$$

Since the curve and transformation are symmetrical in q, r we will take $q > r$. We now examine this family of elliptic curves in detail.

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[4] Note that $[n : p] = [2qr : q^2 - r^2]$ provides all possible ways to make $n^2 + p^2$ a square. If we had used $n = \frac{2qr}{g}$, $p = \frac{q^2 - r^2}{g}$, we would now have the same equation, $Z^2 = M^4 + 2((q^2 - r^2)^2 - 4q^2r^2)M^2 + (q^2 + r^2)^4$, but with $M = 2gm$.

Theorem 6.2. *For all q, r, the torsion subgroup of $E_{q,r}(\mathbb{Q})$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.*

Proof. In fact, we will show that

$$E_{q,r}(\mathbb{Q})_{\mathrm{t}} = \big\{\mathcal{O}, (0,0), \big(-\big(q^2-r^2\big)^2, 0\big), \big(-\big(q^2+r^2\big)^2, 0\big), \\ \big(\big(q^4-r^4\big), \pm 2q^2\big(q^4-r^4\big)\big), \big(-\big(q^4-r^4\big), \pm 2r^2\big(q^4-r^4\big)\big)\big\}.$$

Examining (6.2) reveals that the points of order 2 are $(0,0)$, $(-(q^2-r^2)^2, 0)$ and $(-(q^2+r^2)^2, 0)$. Suppose $P_4 = (x_4, y_4)$ is a point of order 4. Then the x-coordinate of $2P_4$ is given by

$$x(2P_4) = \frac{(x_4-(q^4-r^4))^2(x_4+(q^4-r^4))^2}{4y_4^2}.$$

But $2P_4 = (0,0)$, so the points of order 4 have $x_4 = -(q^4-r^4)$ or (q^4-r^4). Substituting into (6.2) gives the four values of y_4. By Mazur's theorem [26, p. 58], either we have found all of the torsion points or $E_{q,r}(\mathbb{Q})_{\mathrm{t}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_8$.

Suppose $P_8 = (x_8, y_8)$ is a point of order 8. Then the x-coordinate of $2P_8$ is given by

$$x(2P_8) = \frac{(x_8-(q^4-r^4))^2(x_8+(q^4-r^4))^2}{4y_8^2}.$$

But $x(2P_8) = (q^4-r^4)$, so we require q, r such that q^4-r^4 is a square. This occurs only when $q^2 = r^2 = 1$ or $q^2 = 1, r = 0$ [24, p. 17]—that is, only when $E_{q,r}$ is singular. Hence there are no points of order 8. □

Theorem 6.3. *The torsion points in $E_{q,r}(\mathbb{Q})$ correspond to degenerate tetrahedra.*

Proof. Points of order 2 have $y = 0$ which in turns implies $m = 0$ and $b = 0$. For the points of order 4 we have

$$m = \frac{\pm 2q^2(q^4-r^4)}{2((q^4-r^4)+(q^2-r^2)^2)} = \pm\frac{1}{2}\big(q^2+r^2\big)$$

or

$$m = \frac{\pm 2r^2(q^4-r^4)}{2(-(q^4-r^4)+(q^2-r^2)^2)} = \mp\frac{1}{2}\big(q^2+r^2\big),$$

which both give $e = 0$. □

To ensure that m is positive we are interested only in points on $E_{q,r}$ which satisfy $x > -(q^2-r^2)^2$, $y > 0$ or $x < -(q^2-r^2)^2$, $y < 0$. Two different infinite-order points $P = (x_0, y_0)$ and $P' = (x_1, y_1)$ give the same $m = m_0$ when

$$x_1 = \frac{-(q^2-r^2)^2(x_0+(q^2+r^2)^2)}{x_0+(q^2-r^2)^2}, \qquad y_1 = \frac{-4y_0q^2r^2(q^2-r^2)^2}{(x_0+(q^2-r^2)^2)^2}. \tag{6.4}$$

So each m is given by exactly one point in each of the regions, and we can restrict the points of interest to those satisfying just $x > -(q^2 - r^2)^2$, $y > 0$ (on the unbounded component) or just $x < -(q^2 - r^2)^2$, $y < 0$ (on the bounded component). Calculations show that $P' = -P + P_2 + 2P_4$, where $P_2 = (-(q^2 + r^2)^2, 0)$ and $P_4 = (-(q^4 - r^4), -2r^2(q^4 - r^4))$. Note that $(x_0, -y_0)$ and $(x_1, -y_1)$ give $-m_0$, and hence the same tetrahedron as P if we take $b = |16mqr|$.

Now suppose that two different m values give the same tetrahedron. Then

$$g_0 a = 4m_0^2 + \left(q^2 + r^2\right)^2, \qquad g_0 b = 16 m_0 q r, \qquad g_0 e = 2\left|4m_0^2 - \left(q^2 + r^2\right)^2\right|$$

and

$$g_1 a = 4m_1^2 + \left(q^2 + r^2\right)^2, \qquad g_1 b = 16 m_1 q r, \qquad g_1 e = 2\left|4m_1^2 - \left(q^2 + r^2\right)^2\right|.$$

Beginning with b, we have $g_0 = \frac{g_1 m_0}{m_1}$. Substituting into the equations for a, equating and simplifying, we have

$$(m_0 - m_1)\left(4m_0 m_1 - \left(q^2 + r^2\right)^2\right) = 0.$$

So

$$m_1 = \frac{(q^2 + r^2)^2}{4m_0}$$

will give us the same tetrahedron as m_0 (it is easily checked that this satisfies the equations for e, but only because we allow for $4m^2 - (q^2 + r^2)^2 < 0$). In terms of points on the curve, if

$$m_0 = \frac{y_0}{2(x_0 + (q^2 - r^2)^2)}$$

comes from a point $P = (x_0, y_0)$ (and hence from $-P + P_2 + 2P_4$) of infinite order, then

$$\begin{aligned} m_1 &= \frac{2(q^2 + r^2)^2 (x_0 + (q^2 - r^2)^2)}{4y_0} \\ &= \frac{(q^2 + r^2)^2 (x_0 + (q^2 - r^2)^2) y_0}{2x_0 (x_0 + (q^2 - r^2)^2)(x_0 + (q^2 + r^2)^2)} \\ &= \frac{(q^2 + r^2)^2 y_0}{2x_0 (x_0 + (q^2 + r^2)^2)} \end{aligned}$$

comes from the points $-P + 2P_4 = (\frac{(q^4 - r^4)^2}{x_0}, \frac{y_0 (q^4 - r^4)^2}{x_0^2})$ and $-(-P + 2P_4) + P_2 + 2P_4 = P + P_2$.

Pulling all of this together we have the following result.

Theorem 6.4. *Let P be a point of infinite order on $E_{q,r}$, and P_2, P_4 be generators of the torsion subgroup $E_{q,r}(\mathbb{Q})_{\mathrm{t}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Let $\mathcal{T}_P$ be the Heron* 3(ii) *tetrahedron to which P corresponds. Then exactly four points on $E_{q,r}$ correspond to $\mathcal{T}_P$, namely P, $-P + P_2 + 2P_4$, $P + P_2$ and $-P + 2P_4$.*

Note that if we allow $b = |16mqr|$, then there are exactly eight points which correspond to $\mathcal{T}_P$ (the four listed above, and their negatives). In some ways this is nicer since it results in every point of infinite order corresponding to a Heron 3(ii) tetrahedron. If we put $P' = P + P_4$, then we get a new tetrahedron corresponding to the (eight) remaining linear combinations of P, P_2 and P_4. Note also that if we do not allow $4m^2 - (q^2 + r^2)^2 < 0$ then only two points correspond to a single tetrahedron.

Corollary 6.5. *If $E_{q,r}$ has rank greater than zero, then $E_{q,r}(\mathbb{Q})$ generates infinitely many Heron* 3(ii) *tetrahedra.*

We would very much like to know for which q, r the rank of $E_{q,r}$ is greater than zero, and also whether we can find the same tetrahedron through points on different curves. Without answers to these questions, the best we can do to describe all Heron 3(ii) tetrahedra is to give the following definition and theorem.

Definition 6.6. Define the set of Heron 3(ii) tetrahedra generated by q, r to be $\mathcal{S}_{q,r} := \{\mathcal{T}_P\colon P \text{ is a point of infinite order on } E_{q,r}\}$. Note that $\mathcal{S}_{q,r} = \emptyset$ when the rank of $E_{q,r}$ is zero.

Theorem 6.7. $\bigcup_{q>r>0,\ (q,r)=1} \mathcal{S}_{q,r}$ *is the set of all Heron* 3(ii) *tetrahedra.*

6.4. Examples

If we put $q = 4$, $r = 1$ then we have the curve

$$E_{4,1}\colon \quad y^2 = x^3 + 514x^2 + 65\,025x \tag{6.5}$$

which has rank 1. Magma gives $P_\infty = (735, -26\,880)$ as the infinite generator, and $P_2 = (-289, 0)$, $P_4 = (-255, -510)$ as the generators of the torsion subgroup $E_{4,1}(\mathbb{Q})_{\mathrm{t}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$.

Linear combinations of these generators reveal, from points on the unbounded component of the curve with $y > 0$, the Heron tetrahedra shown in Table 3.

The results of a search for other $0 < r < q < 10$ which give $E_{q,r}$ with positive rank are listed in Table 4, along with two of the Heron IF tetrahedra from each curve. Notice that $(925, 1040, 925, 925, 1512, 925)$ comes from both $q = 5$, $r = 4$ and $q = 7$, $r = 6$.

7. Heron tetrahedra in family 3(vi)

The main positive result of [4] was the discovery of an infinite family of Heron tetrahedra. Since we require that all face areas be rational, the search becomes easier if two or more faces are congruent. Case 3(vi) yielded the fortunate situation where all four faces are congruent, and it was in this case that Buchholz found his infinite family.

Buchholz was not the first to find infinite families of Heron 3(vi) tetrahedra. In [14], Güntsche describes (without giving the derivation) two parametric families of Heron 3(vi) tetrahedra:

$$a = 10(p^2 + 1)(p^2 - 1)(p^4 + 3p^2 + 1),$$

$$b = (2p^2 + 3)(4p^2 + 1)(p^4 + 2p^2 + 2),$$

$$c = (p^2+4)(3p^2+2)(2p^4+2p^2+1),$$

and

$$a = (p^2+25)(2p^2+1)(16p^4+9p^2+25),$$
$$b = 2(p^2-3)(p^2+4)(65p^4+58p^2+25),$$
$$c = 7(p^2+1)(9p^2+1)(2p^4+2p^2+25).$$

The derivations of these two parametrisations, along with seven other parametric solutions, described in [13], are elementary but clever. All nine families are listed in Appendix A.

Table 3
Heron IF tetrahedra from $E_{4,1}$

Point	m	a, b, e
$-P_\infty$	14	1073, 896, 990
$P_\infty + 2P_4$	$\frac{289}{56}$	
$P_\infty + P_2 + P_4$	$\frac{187}{6}$	1105, 528, 1904
$-P_\infty + P_2 + P_4$	$\frac{51}{22}$	
$-2P_\infty$	$\frac{35\,409}{7280}$	5 082 931 681, 4 124 440 320, 5 150 674 238
$2P_\infty + 2P_4$	$\frac{525\,980}{35\,409}$	
$-2P_\infty - (P_2 + P_4)$	$\frac{1\,568\,471}{55\,826}$	78 978 572 273, 41 205 393 904, 131 466 535 200
$2P_\infty - (P_2 + P_4)$	$\frac{474\,521}{184\,526}$	
$3P_\infty$	$\frac{2\,207\,906\,666}{9\,133\,889\,119}$	24 130 171 304 018 995 232 753, 1 290 673 579 030 077 904 256, 48 182 344 978 505 714 696 610
$-3P_\infty + 2P_4$	$\frac{2\,639\,693\,955\,391}{8\,831\,626\,664}$	
$3P_\infty + P_2 + P_4$	$\frac{664\,401\,147\,939}{82\,189\,288\,522}$	27 337 745 700 228 926 240 785, 25 697 250 655 203 047 838 192, 2 742 681 355 438 915 546 544
$-3P_\infty + P_2 + P_4$	$\frac{698\,608\,952\,437}{78\,164\,840\,934}$	

Table 4
Other $E_{q,r}$ with positive rank for $0 < r < q < 10$

q, r	a, b, e	a, b, e
5, 4	3485, 6720, 1066	925, 1040, 1512
6, 1	31 705, 19 008, 24 206	159 877, 41 496, 293 040
6, 5	7085, 13 920, 714	80 825, 42 840, 155 672
7, 2	79 913, 83 776, 19 950	240 461, 37 240, 475 728
7, 3	4453, 5544, 4550	20 213, 19 110, 30 624
7, 6	925, 1512, 1040	697, 1344, 306
8, 3	618 759 457, 812 657 664, 62 948 930	619 368 865, 54 937 248, 1 235 916 864
9, 4	24 466 780 848 673, 22 774 534 763 904, 38 119 368 909 310	32 316 422 059 585, 42 224 531 714 928, 30 682 359 334 704
9, 8	28 209 025, 55 752 192, 5 600 770	4 321 153, 5 562 144, 6 581 856

7.1. Buchholz's theorem

A more sophisticated approach using elliptic curves was taken by Buchholz [4] to find an infinite family of Heron 3(vi) tetrahedra. We provide a slightly more straight-forward proof of his theorem, which can then be generalised so that we can analyse this case in more detail.

Theorem 7.1. [4, Theorem 1] *There exists an infinite number of Heron* 3(vi) *tetrahedra.*

Proof. The volume of a 3(vi) tetrahedron is given by

$$(12V)^2 = 2\left(a^2+b^2-c^2\right)\left(a^2-b^2+c^2\right)\left(-a^2+b^2+c^2\right). \tag{7.1}$$

Using Carmichael's parametrisation of Heron triangles [8, p. 13] to ensure the face area is integral, we substitute $a = n(m^2+k^2)$, $b = m(n^2+k^2)$ and $c = (m+n)(mn-k^2)$ to get

$$(12V)^2 = \left(4mn(m+n)\left(mn-k^2\right)\right)^2\left(k^2(m+n)^2-\left(mn-k^2\right)^2\right)\left(m^2-k^2\right)\left(n^2-k^2\right).$$

We require positive integers m, n, k, v such that

$$v^2 = \left(k^2(m+n)^2-\left(mn-k^2\right)^2\right)\left(m^2-k^2\right)\left(n^2-k^2\right). \tag{7.2}$$

The conditions $\gcd(m,n,k) = 1$, $m > n > 0$ and $mn > k^2 > \frac{m^2n}{2m+n}$ ensure we have exactly one Heron triangle from each similarity class [3, Theorem 1.5].

Dividing by k^8 and putting $x = \frac{m}{k}$, $y = \frac{n}{k}$, and $z = \frac{v}{k^4}$ we arrive at the following quartic in rational x and y:

$$z^2 = \left((x+y)^2-(xy-1)^2\right)\left(x^2-1\right)\left(y^2-1\right). \tag{7.3}$$

Notice that $(x,z) = (1,0)$ is a solution of (7.3) for all y. It is also worth noting that Eq. (7.3) is symmetrical in x and y.

Any Heron 3(vi) tetrahedron (a,b,c,a,b,c) will satisfy Eq. (7.3). We shall use the same example Buchholz chose (the first example in his Table 1) and put $y = \frac{4}{3}$ to get

$$(9z)^2 = 7(-x+7)(7x+1)(x-1)(x+1). \tag{7.4}$$

In Section 7.2 we use other examples to get different curves giving different infinite families.

We now depart from Buchholz's original proof. If we expand the right-hand side and put $x = w-1$ we have

$$(9z)^2 = -49w^4+532w^3-1204w^2+672w.$$

Divide by w^4 and put $u = \frac{1}{w}$, $Z = \frac{9z}{w^2}$ to get

$$Z^2 = 672u^3-1204u^2+532u-49.$$

Finally, we multiply by 84^2 and put $X = 168u$, $Y = 84Z$ to arrive at the elliptic curve

$$E_{\frac{4}{3}}:\quad Y^2 = X^3-301X^2+22\,344X-345\,744.$$

Now, as Buchholz noted, $x = \frac{7}{5}$ gives us a Heron tetrahedron. Following the transformations through, we find that the tetrahedron corresponds to the point $P = (70, 294)$ on the elliptic curve $E_{\frac{4}{3}}$.

Using the formula given in [26] we can calculate the x-coordinate of $2P$ and substitute into $E_{\frac{4}{3}}$ to see that $2P = (\frac{2125}{9}, \frac{30\,932}{27})$. Since this is not an integral point we use the Nagell–Lutz theorem [26, p. 56] to conclude that it is a point of infinite order, and hence that P is a point of infinite order.

Buchholz showed that a Heron tetrahedron corresponding to a point on $E_{\frac{4}{3}}$ must satisfy $\frac{4}{3} < x < 2$. Transforming these inequalities we require $56 < X < 72$, and assume $Y \geqslant 0$ since we are only concerned with the X-coordinates of the points $(X, \pm Y)$. We have one point in this range, which we will refer to as the *strict range* due to the strict conditions placed on m, n, k. The density of the rational points on an elliptic curve with rank $\geqslant 1$ and with three rational points of order two[5] implies that there are infinitely many rational points satisfying $56 < X < 72$.

Since the conditions on x come directly from conditions on m, n, k which limit solutions to a single triangle from each similarity class, we know that each of the rational points satisfying $56 < X < 72$ will give a distinct tetrahedron. Hence there are infinitely many Heron 3(vi) tetrahedra. □

We can extract the primitive triangle of each similarity class from the representative given by the strict conditions by dividing the edge lengths by $g = \gcd(a, b, c)$. Since tetrahedra of this type are formed from four copies of one triangle, this procedure also gives us the primitive Heron 3(vi) tetrahedron.

To make it easier to find examples of tetrahedra in this infinite family, we disregard the requirement $k^2 > \frac{m^2 n}{2m+n}$ (and $m > n$ if we like since x and y are interchangeable in Eq. (7.3)). Thus we need only find rational points on $E_{\frac{4}{3}}$ satisfying $0 < X < 96$, which we refer to as the *general range*. Four points in this general range will correspond to the same tetrahedron (see Section 7.3), and this makes it easier to find tetrahedra. Calculations show that $P, 3P, 5P, \ldots, 19P$ all correspond to distinct tetrahedra (see Theorem 7.10), although only $P, 7P, 13P$ lie in the strict range $56 < X < 72$.

Note. The tetrahedron given by $3P$, for example, does not correspond to a point in the strict range for $E_{\frac{4}{3}}$. This tetrahedron corresponds to a rational point in the strict range on the curve $E_{\frac{83\,841\,161}{83\,619\,373}}$.

7.2. More infinite families

Since $(-1, 0)$ is a solution to (7.3) for all y, we can find other infinite families of tetrahedra by following the same process and using the value for y from a different solution of (7.3). We could also consider E_x for each initial solution by using the value for x instead of y, although the strict range would change since the condition $1 > \frac{x^2 y}{2x+y}$ is not symmetrical in x and y.

[5] In fact, if the rank of E is positive then $E(\mathbb{Q})$ is dense on the component(s) of $E(\mathbb{R})$ which contain at least one rational point. A published proof of this accepted fact has eluded us, but [5] and [17] give two different explanations—with a bit of hand waving. Much of the content of [5] can be found in the proof of [7, Lemma 5].

Table 5
E_y from eight known Heron 3(vi) tetrahedra

a, b, c	m, n, k	x, y	$E_y : Y^2 =$	r
203, 195, 148	39, 35, 25	$\frac{39}{25}, \frac{7}{5}$	$X^3 - 774X^2 + 148\,680X - 6\,350\,400$	2
888, 875, 533	21, 20, 15	$\frac{7}{5}, \frac{4}{3}$	$X^3 - 301X^2 + 22\,344X - 345\,744$	1
1804, 1479, 1183	58, 33, 30	$\frac{29}{15}, \frac{11}{10}$	$X^3 - 7371X^2 + 12\,709\,620X - 2\,353\,220\,100$	1
2431, 2296, 2175	77, 68, 44	$\frac{7}{4}, \frac{17}{11}$	$X^3 - 30\,618X^2 + 234\,206\,280X - 435\,251\,589\,696$	2
2873, 2748, 1825	39, 34, 26	$\frac{3}{2}, \frac{17}{13}$	$X^3 - 23\,490X^2 + 135\,649\,800X - 158\,244\,840\,000$	1
3111, 2639, 2180	1989, 1281, 1071	$\frac{13}{7}, \frac{61}{51}$	$X^3 - 2\,926\,840X^2 + 2\,063\,902\,108\,800X - 237\,953\,522\,903\,040\,000$	1
5512, 5215, 1887	364, 265, 260	$\frac{7}{5}, \frac{53}{52}$	$X^3 - 879\,165X^2 + 173\,862\,397\,800X - 923\,242\,148\,010\,000$	2
8484, 6625, 6409	318, 175, 150	$\frac{53}{25}, \frac{7}{6}$	$X^3 - 1807X^2 + 780\,780X - 50\,381\,604$	2

Since we begin with a tetrahedron, we have at least one rational point on the curve. There exists m, n, k which satisfy the strict conditions and determine this tetrahedron. So there is a rational point in the strict range for X. In all the examples considered so far, that point has infinite order and there are infinitely many rational points in the strict range. We conjecture in Section 7.4, where we look at the general case instead of starting with a known example, that all Heron tetrahedra with edges (a, b, c, a, b, c) correspond to a point of infinite order on some E_y.

Table 5 lists the curves we get when we start with each of the eight examples given by Buchholz [4] and Fricke [11]. The values of m, n, k listed are those which satisfy the strict conditions. All of the curves have $E_y(\mathbb{Q})_t \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The infinite families are not disjoint if we allow tetrahedra from points outside the strict range. For example, the tetrahedra with x equal to $\frac{7}{5}$ in the above table will correspond to points on either $E_{\frac{4}{3}}$ or $E_{\frac{53}{52}}$, as well as to points on $E_{\frac{7}{5}}$.

Note. There is no infinite family which contains all of the eight tetrahedra listed above because they only have three possible x- and y-values each (see Section 7.3) and they do not all have a common one.

7.3. m, n, k as functions of a, b, c

Buchholz showed in [3, p. 13] how to express m, n, k as functions of a, b, c:

$$m = (a + b + c)(a - b + c), \tag{7.5}$$

$$n = (a + b + c)(-a + b + c), \tag{7.6}$$

$$k = 4A = \sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}, \tag{7.7}$$

where A is the area of the triangle. Since we can permute these edges in six ways (without changing the triangle) there are six m, n, k combinations which all give the Heron triangle with edges a, b, c.

Suppose that $a_0 > b_0 > c_0$ are the edges of a Heron triangle with area A_0. The possible m, n, k combinations are listed in Table 6.

Notice that n_1 and n_2 are the same, although m_1 and m_2 are different. Similarly, $n_3 = n_4$ and $n_5 = n_6$. The m_i also form three pairs, but the subscripts of these pairs are not the same as the subscripts of the n_i pairs. The three values that m_i can take are the same as the three values n_i can take.

Now suppose that T_0 is the Heron tetrahedron $(a_0, b_0, c_0, a_0, b_0, c_0)$. The possible values for $x_i = \frac{m_i}{k_i}$ and $y_i = \frac{n_i}{k_i}$ are

$$\frac{(a_0+b_0+c_0)(-a_0+b_0+c_0)}{4A_0}, \qquad \frac{(a_0+b_0+c_0)(a_0-b_0+c_0)}{4A_0},$$
$$\frac{(a_0+b_0+c_0)(a_0+b_0-c_0)}{4A_0}.$$

For each y_i equal to one of these, x_i can take both of the other values. This means that if we construct the curve E_{y_i}, there will be two X-values corresponding to the two values of x_i which give rational points on the curve. These X-values are non-zero since the tetrahedron is not degenerate, and hence each X gives two Y-values. In other words, each Heron tetrahedron T_0 corresponds to exactly four points on each of the three curves E_{y_1}, E_{y_3} and E_{y_5}.

Examples 7.2. The tetrahedron (203, 195, 148, 203, 195, 148) corresponds to points on $E_{\frac{5}{2}}$, $E_{\frac{7}{5}}$ and $E_{\frac{39}{25}}$. The tetrahedron (2431, 2296, 2175, 2431, 2296, 2175) corresponds to points on $E_{\frac{7}{4}}$, $E_{\frac{17}{11}}$ and $E_{\frac{29}{15}}$. Note that these tetrahedra do not share a common curve, so no single curve can generate all Heron 3(vi) tetrahedra.

We can now argue that there are infinitely many tetrahedra corresponding to points on a curve E_{y_0}, with positive rank, without needing the strict range. Using the general range instead, if there is a rational point in that range (and we show in the next section that two points of order 2 lie in the general range) then the rational points are dense in that range. Taking all the rational points in that range, we can partition them into four subsets such that each contains exactly

Table 6
Six ways to get m, n, k from a_0, b_0, c_0

i	m_i	n_i	k_i
1	$(a_0+b_0+c_0)(a_0-b_0+c_0)$	$(a_0+b_0+c_0)(-a_0+b_0+c_0)$	$4A_0$
2	$(a_0+b_0+c_0)(a_0-c_0+b_0)$	$(a_0+b_0+c_0)(-a_0+c_0+b_0)$	$4A_0$
3	$(a_0+b_0+c_0)(b_0-a_0+c_0)$	$(a_0+b_0+c_0)(-b_0+a_0+c_0)$	$4A_0$
4	$(a_0+b_0+c_0)(b_0-c_0+a_0)$	$(a_0+b_0+c_0)(-b_0+c_0+a_0)$	$4A_0$
5	$(a_0+b_0+c_0)(c_0-a_0+b_0)$	$(a_0+b_0+c_0)(-c_0+a_0+b_0)$	$4A_0$
6	$(a_0+b_0+c_0)(c_0-b_0+a_0)$	$(a_0+b_0+c_0)(-c_0+b_0+a_0)$	$4A_0$

one point corresponding to each tetrahedron. The four subsets are infinite and hence there are infinitely many non-similar Heron 3(vi) tetrahedra generated by E_{y_0}.

However, it is still useful to maintain the strict range since each Heron 3(vi) tetrahedron corresponds to a unique point in the strict range of a unique curve. As a practical matter, if we have m, n, k which do not satisfy the strict conditions then we can easily construct the m, n, k which do satisfy them.

7.4. The general case

As noted in the proof of Theorem 7.1, $(x, z) = (1, 0)$ is a solution to Eq. (7.3) for all y. So we can make similar transformations in the general case to those made for the specific example already studied.

Begin by putting $y = \frac{N}{K}$, where $N, K \in \mathbb{N}$ are relatively prime. Substitute $x = \frac{1}{w} - 1$ in Eq. (7.3) and multiply throughout by w^4 to arrive at

$$\begin{aligned}(K^2w^2z)^2 = {}& 8NK(N^2 - K^2)w^3 - 4(N^2 - K^2)(N^2 + 3NK - K^2)w^2 \\ &+ 4(N^2 - K^2)(N^2 + NK - K^2)w - (N^2 - K^2)^2.\end{aligned}$$

To eliminate the leading coefficient, multiply by $(NK(N^2 - K^2))^2$ and put $X = 2NK \times (N^2 - K^2)w$, $Y = NK^3(N^2 - K^2)w^2z$. Then

$$\begin{aligned}E_y\colon\quad Y^2 = {}& (X - NK(N^2 - K^2)) \\ &\times (X - K(N - K)(N^2 - K^2))(X - N(N + K)(N^2 - K^2)).\end{aligned} \tag{7.8}$$

To simplify calculations, substitute $X = W + NK(N^2 - K^2)$ into E_y to get

$$\begin{aligned}E'_y\colon\quad Y^2 &= W^3 - (N^2 - K^2)^2W^2 - K^2N^2(N^2 - K^2)^2W \\ &= W(W - K^2(K^2 - N^2))(W - N^2(N^2 - K^2)).\end{aligned}$$

Theorem 7.3. *If a Heron* 3(vi) *tetrahedron corresponds to a rational point on* E'_y *then* $y > 1$.

Proof. Since $n, k > 0$ we have $y > 0$. If $y = 1$ then $n = k$ in Eq. (7.1) implies that the volume is zero, and hence the tetrahedron is degenerate.

The transformation from the quartic (7.3) to E'_y is described by

$$W = \frac{2NK(N^2 - K^2)}{x + 1} - NK(N^2 - K^2), \qquad Y = \frac{NK^3(N^2 - K^2)z}{(x + 1)^2}.$$

For a rational point on the quartic to correspond to a Heron tetrahedron with $a, b, c > 0$, we require $x > 0$, $x > \frac{1}{y}$. If we translate these conditions, we find that

- if $N > K$ then $x > 0 \Leftrightarrow NK(K^2 - N^2) < W < NK(N^2 - K^2)$ and $x > \frac{1}{y} \Leftrightarrow W < NK(N - K)^2$. So rational points on E'_y, $y > 1$, which correspond to Heron tetrahedra, satisfy the general range $NK(K^2 - N^2) < W < NK(N - K)^2$;

- if $N < K$ then $x > 0 \Leftrightarrow NK(N^2 - K^2) < W < NK(K^2 - N^2)$ and $x > \frac{1}{y} \Leftrightarrow NK \times (N-K)^2 < W$. So rational points on E'_y, $0 < y < 1$, which correspond to Heron tetrahedra, satisfy the general range $NK(N-K)^2 < W < NK(K^2 - N^2)$.

We can locate the points of order two on E'_y, $(0,0)$, $(K^2(K^2-N^2),0)$ and $(N^2(N^2-K^2),0)$, in relation to these ranges. If $N > K$ then $NK(K^2-N^2) < K^2(K^2-N^2) < 0 < NK(N^2-K^2)$ and $NK(N-K)^2 < N^2(N^2-K^2)$, so the points $(K^2(K^2-N^2),0)$ and $(0,0)$ lie within the general range and the point $(N^2(N^2-K^2),0)$ is outside the general range. Hence all of the rational points which correspond to Heron tetrahedra lie on the bounded component of E'_y when $y > 1$. If $N < K$ then $N^2(N^2-K^2) < 0 < NK(N-K)^2$ and $NK(K^2-N^2) < K^2(K^2-N^2)$. So the general range lies between the two components of the curve E'_y for $0 < y < 1$. We therefore require $N > K$ for the curve to produce Heron tetrahedra. □

Not all curves E'_y with $y > 1$ will produce Heron tetrahedra. For example, $E'_{\frac{5}{4}}(\mathbb{Q}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and it is easy to show that points of order 2 give degenerate tetrahedra.

Theorem 7.4. *Let H be a Heron* 3(vi) *tetrahedron corresponding to a point P_H of infinite order in the strict range of E'_y. Then H generates an infinite family*

$$S_H = \{G\colon\ G \text{ corresponds to a point } P_G \text{ in the strict range of } E'_y\}$$

of Heron 3(vi) *tetrahedra.*

Proof. P_H is a point of infinite order in the strict range of E'_y which corresponds to H. Since the rational points are dense on the curve, there are infinitely many rational points in this range. Since our conditions on m, n, k ensure that the rational points in the strict range correspond to distinct similarity classes of faces of the tetrahedra, we have infinitely many distinct similarity classes of Heron tetrahedra. □

It is worth noting here that we do not need an initial tetrahedron H to know whether an elliptic curve E'_y will give us an infinite family S_H. If the rank is positive, and since there are three rational points of order 2, the rational points are dense everywhere on the curve, i.e., they are dense in the strict range without us needing to find a point in that range.

Theorem 7.5. *An elliptic curve $E'_{N/K}$ produces an infinite family*

$$S_{N/K} = \{G\colon\ G \text{ corresponds to a point } P_G \text{ in the strict range of } E'_{N/K}\}$$

of Heron 3(vi) *tetrahedra if and only if $E'_{N/K}$ has rank $r \geqslant 1$ and $K < N < \sqrt{3}K$.*

Proof. If the rank of $E'_{N/K}$ is zero, there can be at most finitely many tetrahedra produced by the curve, and hence we do not have an infinite family.

On the other hand, if the rank is greater than zero, then the rational points are dense on both components of the curve. Exactly four of the infinitely many rational points on the bounded component correspond to any one tetrahedron (see Section 7.3). Hence the curve produces infinitely many Heron 3(vi) tetrahedra. So $S_{N/K}$ may be non-empty. In fact, $S_{N/K} \neq \emptyset$ exactly when there is a strict range on the curve $E'_{N/K}$.

The strict conditions on x and y are $x > y$ and $1 > \frac{x^2 y}{2x+y}$. Substituting $\frac{N}{K}$ for y we find that we require $x > \frac{N}{K}$ and $\frac{N}{K}x^2 - 2x - \frac{N}{K} < 0 \Leftrightarrow \frac{1}{N}(K - \sqrt{N^2+K^2}) < x < \frac{1}{N}(K + \sqrt{N^2+K^2})$, which together give us $\frac{N}{K} < x < \frac{1}{N}(K + \sqrt{N^2+K^2})$. In particular, we require $\frac{N}{K} < \frac{1}{N}(K + \sqrt{N^2+K^2}) \Leftrightarrow N^2 - K^2 < K\sqrt{N^2+K^2} \Leftrightarrow N^4 - 2N^2K^2 + K^4 < K^2N^2 + K^4$ since $0 < K < N$. This leads to $N < \sqrt{3}K$. □

It is possible that P_G is a point of finite order. If the rank of $E'_{N/K}$ is zero, then we may have $S_{N/K} = \emptyset$ or

$$S_{N/K} = \{G\colon\ G \text{ corresponds to a point } P_G \text{ of order 6 in the strict range of } E'_{N/K}\}.$$

Lemma 7.6. *The torsion subgroup $E'_y(\mathbb{Q})_t$ of $E'_y(\mathbb{Q})$ has the form $\mathbb{Z}_2 \oplus \mathbb{Z}_{2i}$ with i equal to either* 1 *or* 3.

Proof. The torsion subgroup $E'_y(\mathbb{Q})_t$ contains the point at infinity, $\mathcal{O}$, and the three points of order two, $(0,0)$, $(K^2(K^2-N^2),0)$ and $(N^2(N^2-K^2),0)$. By Mazur's theorem [26, p. 58] $E'_y(\mathbb{Q})_t$ must be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{2i}$, where $i = 1,2,3$ or 4.

Clearly, $i = 1$ is a possibility since we always have three points of order two.

If $i = 2$ or 4 then there must be a point of order four on E'_y, (u,v) say. Using the duplication formula [26, p. 31] we can calculate the W-coordinate w of $2(u,v)$:

$$w = \frac{u^4 + 2N^2K^2(N^2-K^2)^2u^2 + N^4K^4(N^2-K^2)^4}{4u(u - K^2(K^2-N^2))(u - N^2(N^2-K^2))} = \frac{(u^2 + N^2K^2(N^2-K^2)^2)^2}{4v^2}. \quad (7.9)$$

But $2(u,v) = (N^2(N^2-K^2),0)$ so

$$N^2(N^2-K^2) = \frac{(u^2 + N^2K^2(N^2-K^2)^2)^2}{4u(u - K^2(K^2-N^2))(u - N^2(N^2-K^2))}. \quad (7.10)$$

Rearranging Eq. (7.10) we have

$$\begin{aligned} &\left(u^2 - 2N^2(N^2-K^2)u - N^2K^2(N^2-K^2)^2\right)^2 = 0 \\ \iff\quad & u^2 - 2N^2(N^2-K^2)u - N^2K^2(N^2-K^2)^2 = 0, \end{aligned}$$

and hence $u = N(N^2-K^2)(N \pm \sqrt{N^2+K^2})$. Note that we now require $N^2 + K^2 = Y^2$ for some $Y \in \mathbb{N}$, as well as $N^2 - K^2 = X^2$, where $X = \frac{u^2 + N^2K^2(N^2-K^2)^2}{2vN} \in \mathbb{N}$.

Multiplying, we have $N^4 - K^4 = (XY)^2$. Mordell shows in [24, p. 17] that the only solutions are $N^2 = K^2 = 1$ and $N^2 = 1$, $K = 0$, none of which give appropriate $y = \frac{N}{K}$. So there are no points of order four on E'_y. □

We note that all of the examples of Heron 3(vi) tetrahedra we have examined have torsion subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

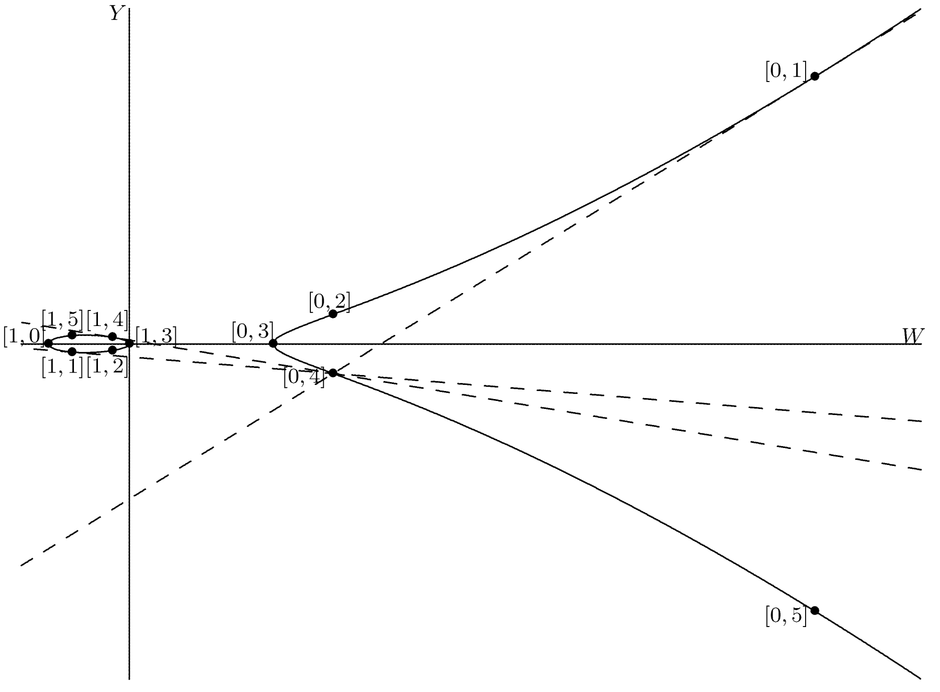

Fig. 4. Labeling torsion points as elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_6$.

Theorem 7.7. *Let H be a Heron* 3(vi) *tetrahedron corresponding to a point P_H on E'_y. Then P_H is a point of order six, or a point of infinite order.*

Proof. By Lemma 7.6, P_H is either a point of infinite order or a point of order two, three or six. The points of order two give degenerate tetrahedra, so P_H does not have order two. The points of order three are inflection points of E'_y and hence lie on the unbounded component of E'_y. So P_H is not a point of order three. The remaining possibility is that P_H is one of the four points of order six on the bounded component (see Fig. 4). □

Conjecture 7.8. *Every Heron* 3(vi) *tetrahedron H generates an infinite family S_H of Heron* 3(vi) *tetrahedra.*

Proof of some cases. By Theorem 7.7 we must show that $E'_y(\mathbb{Q})$ contains no points of order six, or that the points of order six do not lie in the strict range.

Suppose that $E'_y(\mathbb{Q})_t \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$, where $y = \frac{N}{K}$ and $(N, K) = 1$. We can label the points of finite order as the elements of $\mathbb{Z}_2 \oplus \mathbb{Z}_6$, as in Fig. 4.

If P is a point of order six, then it must correspond to one of the elements $[1, \pm 1], [1, \pm 2]$ since $[0, \pm 1]$ do not give tetrahedra. Let $Q = (u, v)$ be the point corresponding to $[1, 5]$. Then $K^2(K^2 - N^2) < u < 0$ and $3Q$ corresponds to the element $3[1, 5] = [1, 3]$, i.e., the W- and Y-coordinates of $3Q$, denoted $W(3Q)$ and $Y(3Q)$, respectively, must be zero.

Using the duplication and addition formulae [26, p. 31] we find that

$$W(3Q)=\frac{u(u^4+6N^2K^2(K^2-N^2)^2u^2-4N^2K^2(K^2-N^2)^4u-3N^4K^4(K^2-N^2)^4)^2}{(3u^4-4(K^2-N^2)^2u^3-6N^2K^2(K^2-N^2)^2u^2-N^4K^4(K^2-N^2)^4)^2}. \tag{7.11}$$

The point Q does not have order two, so $u \neq 0$ and hence $W(3Q)=0$ exactly when

$$u^4+6N^2K^2\left(K^2-N^2\right)^2u^2-4N^2K^2\left(K^2-N^2\right)^4u-3N^4K^4\left(K^2-N^2\right)^4=0.$$

This is equivalent to

$$u^4=N^2K^2\left(K^2-N^2\right)^2\left[3N^2K^2\left(K^2-N^2\right)^2+4\left(K^2-N^2\right)^2u-6u^2\right]. \tag{7.12}$$

Case 1(a). Suppose that N, K, K^2-N^2 are square-free. Since $(N,K)=1$, we have $u=NK(K^2-N^2)\mathcal{U}$ for some non-zero $\mathcal{U}\in\mathbb{Z}$. This contradicts $K^2(K^2-N^2)<u<0$ since $N>K$. Hence we have shown that $E_y'(\mathbb{Q})_{\mathrm{t}}\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$ if N, K, K^2-N^2 are square-free. An example with N, K, K^2-N^2 all square-free is $N=3$, $K=2$.

Case 1(b). Suppose that N and K are square-free, and that $K^2-N^2=-n_0^{2^\alpha}n_1^{2^{\alpha-1}}\cdots n_{\alpha-1}^2n_\alpha$, where the n_i are square-free products of not necessarily distinct primes. Then $K^2-N^2|u^2$ implies that $u=n_0^{2^{\alpha-1}}n_1^{2^{\alpha-2}}\cdots n_{\alpha-2}^2n_{\alpha-1}n_\alpha\mathcal{U}_0$ for some integer $\mathcal{U}_0<0$.

Substituting into (7.12) and dividing by $\nu_0=n_0^{2^{\alpha+1}}n_1^{2^\alpha}\cdots n_{\alpha-2}^{2^3}n_{\alpha-1}^{2^2}n_\alpha^{2^2}$ gives

$$\begin{aligned}\mathcal{U}_0^4={}&N^2K^2n_0^{2^\alpha}n_1^{2^{\alpha-1}}\cdots n_{\alpha-2}^{2^2}n_{\alpha-1}^2\big[3N^2K^2n_0^{2^\alpha}n_1^{2^{\alpha-1}}\cdots n_{\alpha-2}^{2^2}n_{\alpha-1}^2\\&+4n_0^{2^\alpha+2^{\alpha-1}}n_1^{2^{\alpha-1}+2^{\alpha-2}}\cdots n_{\alpha-2}^{2^2+2}n_{\alpha-1}^{2+1}n_\alpha\mathcal{U}_0-6\mathcal{U}_0^2\big]\end{aligned} \tag{7.13}$$

which in turn implies $\mathcal{U}_0=n_0^{2^{\alpha-2}}n_1^{2^{\alpha-3}}\cdots n_{\alpha-3}^2n_{\alpha-2}n_{\alpha-1}\mathcal{U}_1$.

Substituting into (7.13) and dividing by $\nu_1=n_0^{2^\alpha}n_1^{2^{\alpha-1}}\cdots n_{\alpha-3}^{2^3}n_{\alpha-2}^{2^2}n_{\alpha-1}^{2^2}$ gives

$$\begin{aligned}\mathcal{U}_1^4={}&N^2K^2n_0^{2^{\alpha-1}}n_1^{2^{\alpha-2}}\cdots n_{\alpha-3}^{2^2}n_{\alpha-2}^2\big[3N^2K^2n_0^{2^{\alpha-1}}n_1^{2^{\alpha-2}}\cdots n_{\alpha-3}^{2^2}n_{\alpha-2}^2\\&+4n_0^{2^\alpha+2^{\alpha-2}}n_1^{2^{\alpha-1}+2^{\alpha-3}}\cdots n_{\alpha-3}^{2^3+2}n_{\alpha-2}^{2^2+1}n_{\alpha-1}^2n_\alpha\mathcal{U}_1-6\mathcal{U}_1^2\big]\end{aligned} \tag{7.14}$$

which in turn implies $\mathcal{U}_1=n_0^{2^{\alpha-3}}n_1^{2^{\alpha-4}}\cdots n_{\alpha-4}^2n_{\alpha-3}n_{\alpha-2}\mathcal{U}_2$.

Continuing in this manner we eventually reach the last few steps.

$$\begin{aligned}\mathcal{U}_{\alpha-1}^4={}&N^2K^2n_0^2\\&\times\big[3N^2K^2n_0^2+4n_0^{2^\alpha+1}n_1^{2^{\alpha-1}}\cdots n_{\alpha-3}^{2^3}n_{\alpha-2}^{2^2}n_{\alpha-1}^2n_\alpha\mathcal{U}_{\alpha-1}-6\mathcal{U}_{\alpha-1}^2\big]\end{aligned} \tag{7.15}$$

implies that $\mathcal{U}_{\alpha-1}=n_0\mathcal{U}_\alpha$.

Substituting into (7.15) and dividing by $v_\alpha = n_0^{2^2}$ gives

$$\begin{aligned}\mathcal{U}_\alpha^4 &= N^2K^2\big[3N^2K^2 + 4n_0^{2^\alpha}n_1^{2^{\alpha-1}}\cdots n_{\alpha-1}^2 n_\alpha \mathcal{U}_\alpha - 6\mathcal{U}_\alpha^2\big]\\ &= N^2K^2\big[3N^2K^2 + 4\big(N^2-K^2\big)\mathcal{U}_\alpha - 6\mathcal{U}_\alpha^2\big].\end{aligned}$$

Back substitution shows that $u = n_0^{2^\alpha}n_1^{2^{\alpha-1}}\cdots n_{\alpha-1}^2 n_\alpha \mathcal{U}_\alpha = (N^2-K^2)\mathcal{U}_\alpha$. Since $(N,K)=1$ and N, K are square-free we have $u = NK(K^2-N^2)\mathcal{U}$ for some non-zero $\mathcal{U}\in\mathbb{Z}$. This contradicts $K^2(K^2-N^2) < u < 0$ since $N > K$. Hence we have shown that $E_y'(\mathbb{Q})_{\mathrm{t}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if N, K are square-free. An example covered by this case is $N = 7$, $K = 2$.

Case 2. The reduction modulo p theorem [26, p. 123] can be used to show that $E_y'(\mathbb{Q})_{\mathrm{t}} \cong \mathbb{Z}_2\oplus\mathbb{Z}_2$ for $y = \frac{N}{K}$ such that $N, K, N^2-K^2 \not\equiv 0 \pmod 7$.

The discriminant of E_y' is

$$d_y' = K^4N^4\big(K^2+N^2\big)^2\big(K^2-N^2\big)^6$$

and is congruent to 1 (mod 7) for all $N, K, N^2-K^2 \not\equiv 0 \pmod 7$. So we can consider E_y' modulo 7 for those N, K. Calculations show that

$$Y^2 \equiv W^3 - 2W^2 - W(7) \quad \text{or} \quad Y^2 \equiv W^3 - W^2 - 2W(7) \quad \text{or} \quad Y^2 \equiv W^3 - 4W^2 - 4W(7),$$

and that $|E_y'(\mathbb{F}_7)| = 4$ in each of these cases. Since $E_y'(\mathbb{Q})_{\mathrm{t}}$ is a subgroup of $E_y'(\mathbb{F}_7)$ there cannot be points of order three or six. This case applies to $N = 8$, $K = 5$, for example, which is not covered by Case 1(a) or 1(b).

It may or may not be useful to observe that if we let the point which behaves like $[1,5]$ be $Q = (u, v)$, then $[1,5] + [0,3] = -[1,4]$ tells us that the point which behaves like $[1,4]$ is

$$Q' = \left(\frac{N^2(N^2-K^2)(u-K^2(K^2-N^2))}{u-N^2(N^2-K^2)}, \frac{N^2(N^2+K^2)(N^2-K^2)^2 v}{(u-N^2(N^2-K^2))^2}\right). \qquad \square$$

While it would be preferable to know if/when $E_{N/K}'(\mathbb{Q})$ contains points of order six, and for which N/K the rank of $E_{N/K}'$ is positive, we can still make the following statement.

Theorem 7.9. *The disjoint union* $\bigcup_{K<N<\sqrt{3}K} S_{N/K}$ *contains all Heron* 3(vi) *tetrahedra.*

Proof. Every Heron 3(vi) tetrahedron has exactly one strict y and therefore corresponds to a point in the strict range of exactly one E_y'. That is, every Heron 3(vi) tetrahedron is an element of exactly one $S_{N/K}$. $\square$

We can also define infinite families of Heron 3(vi) tetrahedra as in Theorem 7.10. We have not looked closely at these families since they do not necessarily include all Heron 3(vi) tetrahedra, but they are useful when it comes to finding examples (as seen at the end of Section 7.1).

Theorem 7.10. *Let H be a Heron* 3(vi) *tetrahedron corresponding to a point P_H of infinite order in the strict range of E'_y. The points iP_H correspond to Heron* 3(vi) *tetrahedra exactly when i is odd, and*

$$O_H = \{G\colon\ G \text{ corresponds to a point } iP_H \text{ for some odd } i \in \mathbb{N}\}$$

is an infinite family of tetrahedra generated by H.

Proof. First, note that we get H from both P_H and $-P_H$ since the tetrahedron is determined by $x = \frac{m}{k}$, i.e., we are only concerned with the W-coordinate of points on E'_y, so we can restrict i to positive integers in the definition of O_H without any consequence.

Now P_H corresponds to a tetrahedron and must therefore be on the bounded component of the curve. The tangent at P_H intersects E'_y a third time on the unbounded component of the curve. To find $3P_H$, and all other odd multiples of P_H, we can join P_H, on the bounded component of the curve, to an even multiple of P_H, on the unbounded component. Then the third point of intersection will always be on the bounded component. Similarly, to find all even multiples of P_H we join two points on the bounded component and the third point of intersection is on the unbounded component.

We have previously shown that the points corresponding to Heron tetrahedra lie on the bounded component of the curve, so iP_H corresponds to a Heron tetrahedron if and only if i is odd. Since at most four of the iP_H can correspond to a single tetrahedron, and there are infinitely many odd $i \in \mathbb{N}$, O_H must be infinite. □

7.5. Examples

In Table 7, a point in the strict range is given along with the corresponding values of m, n, k. The points of order 2 are denoted e_i with $X(e_1) < X(e_2) < X(e_3)$, where $X(e_i)$ denotes the X-coordinate of e_i. The edges a, b, c of the tetrahedra resulting from m, n, k are not given due to space limitations. However, they are easily calculated using Carmichael's parametrisation: $a = n(m^2 + k^2)$, $b = m(n^2 + k^2)$, $c = (m + n)(mn - k^2)$.

Appendix A. Parametric families of Heron 3(vi) tetrahedra

Güntsche's derivations of the following nine parametric families of Heron 3(vi) tetrahedra can be found in [13].

(1)
$$\begin{aligned} a &= 10(p^2+1)(p^2-1)(p^4+3p^2+1),\\ b &= (2p^2+3)(4p^2+1)(p^4+2p^2+2),\\ c &= (p^2+4)(3p^2+2)(2p^4+2p^2+1). \end{aligned}$$

(2)
$$\begin{aligned} a &= (p^2+25)(2p^2+1)(16p^4+9p^2+25),\\ b &= 2(p^2-3)(p^2+4)(65p^4+58p^2+25),\\ c &= 7(p^2+1)(9p^2+1)(2p^4+2p^2+25). \end{aligned}$$

(3)
$$\begin{aligned} a &= (p+2)(p-2)(p^2+1)(p^2+6p+18)(p^2-6p+18),\\ b &= (p+3)(p-3)(p^2+36)(p^2+2p+2)(p^2-2p+2),\\ c &= 10p(p^2+6)(p^4-8p^2+36). \end{aligned}$$

Table 7
Points in the strict range of various E_y with positive rank

y	Strict range	Point in strict range	m, n, k
$\frac{3}{2}$	$75-15\sqrt{13}<X<24$	$5P_\infty$	(Too long to reproduce here)
$\frac{4}{3}$	$56<X<72$	e_1+P_∞	21, 20, 15
$\frac{6}{5}$	$726-66\sqrt{61}<X<300$	e_1+3P_∞	839 014 791 813 714 203 293 331, 798 226 451 545 670 539 646 262, 665 188 709 621 392 116 371 885
$\frac{7}{5}$	$504-42\sqrt{74}<X<175$	$e_1-P_{\infty 1}+P_{\infty 2}$	39, 35, 25
$\frac{7}{6}$	$1183-91\sqrt{85}<X<504$	$P_{\infty 2}$	318, 175, 150
$\frac{8}{5}$	$4056-312\sqrt{89}<X<1200$	e_1+8P_∞	(Too long to reproduce here)
$\frac{9}{8}$	$2601-153\sqrt{145}<X<1152$	$e_1+P_{\infty 1}$	6376, 5949, 5288
$\frac{10}{7}$	$8670-510\sqrt{149}<X<2940$	e_1+5P_∞	(Too long to reproduce here)
$\frac{10}{9}$	$3610-190\sqrt{181}<X<1620$	e_1+P_∞	32 852 691, 29 751 770, 26 776 593
$\frac{11}{7}$	$396-22\sqrt{170}<X<\frac{1078}{9}$	e_2+4P_∞	11 589 632 675 766 589 835 335, 10 064 717 126 964 768 113 483, 6 404 819 989 886 670 617 671
$\frac{11}{10}$	$4851-231\sqrt{221}<X<2200$	e_2+P_∞	58, 33, 30
$\frac{12}{11}$	$6348-276\sqrt{265}<X<2904$	e_2+P_∞	16 159, 14 316, 13 123

(4)
$$\begin{aligned} a &= 4(4p^3+16p^2+5p+1)(16p^3+20p^2+16p+1),\\ b &= (2p+1)(2p-1)(4p^2+12p+1)(20p^2+12p+5),\\ c &= (2p+3)(6p+1)(16p^4+48p^3+72p^2+12p+1). \end{aligned}$$

(5)
$$\begin{aligned} a &= (p-2)(p+4)(p^2+1)(p^2+2p+2)(p^4+4p^3+16p^2+36p+16),\\ b &= 2(p^2+4p+8)(p^3+7p^2+5p+4)(p^4+2p^3+p^2+5p+4),\\ c &= p(p+3)(p^2+p+4)(p^6+4p^5+6p^4+24p^3+82p^2+80p+32). \end{aligned}$$

(6)
$$\begin{aligned} a &= (p^{10}+9p^8+16p^7+58p^6-416p^5+290p^4+400p^3+1125p^2+3125)\\ &\quad\times(p^{10}+9p^8-16p^7+58p^6+416p^5+290p^4-400p^3+1125p^2+3125),\\ b &= (p+1)(p-1)(p+5)(p-5)(p^2+2p+5)(p^2-2p+5)\\ &\quad\times(p^2+4p+5)(p^2-4p+5)(p^8+12p^6+214p^4+300p^2+625),\\ c &= 8p(p^2+5)(p^4-2p^2+25)\\ &\quad\times(p^{12}+6p^{10}-89p^8+1364p^6-2225p^4+3750p^2+15\,625). \end{aligned}$$

(7)
$$\begin{aligned} a &= (p^3+1139p^2+3331p+1681)(49p^3-269p^2+851p+1681),\\ b &= 16(3p+5)(25p+9)(p^4-12p^3+630p^2+1812p+1681),\\ c &= (p+1)(p-31)(p^2+18p+49)(1201p^2+2370p+1681). \end{aligned}$$

(8)
$$\begin{aligned} a &= 2(2p+3)(p^2+4)(p^2+3p+1)(p^3+14p^2+9p+1),\\ b &= (p+4)(p^2-4p-2)(5p^2+4p+1)(p^3+12p^2+8p+4),\\ c &= (p-1)(3p+2)(p^2+7p+2)(p^4+12p^3+58p^2+44p+10). \end{aligned}$$

(9) $$\begin{aligned} a &= (p+4)\bigl(8p^2+12p+5\bigr)\bigl(8p^2+13p+4\bigr)\bigl(16p^3+49p^2+44p+16\bigr),\\ b &= (4p+1)\bigl(4p^2+13p+8\bigr)\bigl(5p^2+12p+8\bigr)\bigl(16p^3+44p^2+49p+16\bigr),\\ c &= 4(p+1)(p-1)(2p+3)(3p+2)\bigl(32p^4+152p^3+257p^2+152p+32\bigr). \end{aligned}$$

Interestingly, substituting appropriate p up to 20 (those p for which the edges are positive) in each of the nine parametrisations gives only one repeated tetrahedron—all the other tetrahedra obtained are distinct.

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