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A Combinatorial Perspective on the Non-Radon Partitions

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Let E be a finite set of points in $\mathbb{R}^d$. Then $\{A, E-A\}$ is a non-Radon partition of E iff there is a hyperplane H separating A strictly from $E-A$. Or equivalently iff ${}_{\bar{A}}\mathcal{O}$ is an acyclic reorientation of $(M_{\mathrm{Aff}}(E), \mathcal{O})$, the oriented matroid canonically determined by E. If $(M(E), \mathcal{O})$ is an oriented matroid without loops then the set $NR(E, \mathcal{O}) = \{(A, E-A)\colon {}_{\bar{A}}\mathcal{O}$ is acyclic$\}$ determines $(M(E), \mathcal{O})$. In particular the matroidal properties of a finite set of points in $\mathbb{R}^d$ are precisely the properties which can be formulated in non-Radon partitions terms. The Möbius function of the poset $\mathscr{A} = \{A\colon A \subseteq E,\ {}_{\bar{A}}\mathcal{O}$ is acyclic$\}$ and in a special case its homotopy type are computed. This paper generalizes recent results of P. Edelman (A partial order on the regions of $\mathbb{R}^n$ dissected by hyperplanes, *Trans. Amer. Math. Soc.* **283** (1984), no. 2, 617–631. © 1985 Academic Press, Inc.

1. INTRODUCTION

Radon's theorem is one of the cornerstone theorems in combinatorial convexity theory. It asserts that if E is a subset of $\mathbb{R}^d$, $|E| \geqslant d+2$, it is possible to find a *Radon partition in* E, i.e., a partition $E' \cup E'' = E$ such that $\mathrm{conv}(E') \cap \mathrm{conv}(E'') \neq \varnothing$ or equivalently E' cannot be separated from E'' by any hyperplane (see [3, 7–9, 14–16, 22]).

Let $(M(E), \mathcal{O})$ be an oriented matroid on a set E [2, 12, 13, 17, 20]. We call a partition[1] $\{A, B\}$ of E a *non-Radon partition in E of the oriented matroid* $(M(E), \mathcal{O})$ if ${}_{\bar{A}}\mathcal{O} = {}_{\overline{E-A}}\mathcal{O}$ is an acyclic reorientation of $\mathcal{O}$. If E is a finite subset of $\mathbb{R}^d$ then $\{E', E''\}$ is a non-Radon partition in E iff ${}_{\bar{E'}}\mathcal{O}$ is an acyclic reorientation of the oriented matroid $(M_{\mathrm{Aff}}(E), \mathcal{O})$ of the affine dependencies of E over $\mathbb{R}$ (see [6, Theorem 2.6] for a short proof).

The familes of the non-Radon partitions of oriented matroids arises naturally in the convex theory of oriented matroids [17]. An important

[1] We call $\{\varnothing, E\}$ a partition in E.

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theorem of the theory, with a large number of applications, is the following result of Las Vergnas (see [17, Theorem 3.1]).

The number of non-Radon partitions in E of an oriented matroid $(M(E), \mathcal{O})$ *is equal to* $2^{-1}.\ t(M; 2, 0)$, *where* $t(M; x, y)$ *denotes the Tutte polynomial of M.*

All the results of the oriented matroid theory can be reformulated in non-Radon partitions terms. Indeed the family of non-Radon partitions of an oriented matroid $(M(E), \mathcal{O})$ determines canonically the oriented matroid (see Theorem 1.1).

The notation of Las Vergnas [2, 17] is followed with minor changes. We recall some definitions. If $X = (X^+, X^-)$ and $Y = (Y^+, Y^-)$ are two signed sets we say X is orthogonal to Y and we note $X \perp Y$ if $\mathbf{X} \cap \mathbf{Y} = \varnothing$, where $\mathbf{X}$ (resp. $\mathbf{Y}$) denotes the support of X (resp. Y), or $(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \varnothing$ and $(X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \varnothing$. If $(M(E), \mathcal{O})$ is an oriented matroid we note by $\mathcal{K}(\mathcal{O})$ the *signed span of* $\mathcal{O}$; i.e., if X is a signed set having support contained in E then $X \in \mathcal{K}(\mathcal{O})$ iff there are oriented circuits $X_1, ..., X_n \in \mathcal{O}$ such that $X^+ = X_1^+ \cup \cdots \cup X_n^+$, $X^- = X_1^- \cup \cdots \cup X_n^-$ and $(X_i^+ \cap X_j^-) = (X_i^- \cap X_j^+) = \varnothing$ $1 \leqslant i < j \leqslant n$. By the definitions $\mathcal{K}(\mathcal{O})$ is the set of the signed sets X of support contained in E such that $X \perp Y$ for all $Y \in \mathcal{O}^\perp$.

The following interesting theorem of Arnaldo Mandel generalizes a result of a previous version of this paper.

THEOREM 1.1 ([21]). *Let* $(M(E), \mathcal{O})$ *be an acyclic oriented matroid. Then the set of its non-Radon partitions determines canonically the oriented matroid.*

Proof. Let $NR(E, \mathcal{O}) = \{(A, E - A)\colon {}_{\bar{A}}\mathcal{O} \text{ acyclic}\}$ be considered as a family of signed sets. Then $NR(E, \mathcal{O}) \subseteq \mathcal{K}(\mathcal{O}^\perp)$ because if $X \in NR(E, \mathcal{O})$ then $X \perp Y$ for all $Y \in \mathcal{O}$. $NR(E, \mathcal{O})$ is the set of the maximal elements of $\mathcal{K}(\mathcal{O}^\perp)$. Let $\mathcal{B}(\mathcal{O}) = \{X\colon \mathbf{X} \subseteq E,\ \ X \perp Y,\ \ Y \in NR(E, \mathcal{O})\}$. From Proposition 1.2 $\mathcal{K}(\mathcal{O}) = \mathcal{B}(\mathcal{O})$ and as $\mathcal{O}$ is the set of the minimal nonempty elements of $\mathcal{K}(\mathcal{O})$ the theorem follows. ∎

PROPOSITION 1.2 ([21]). *Let* $(M(E), \mathcal{O})$ *be an oriented matroid. Let* $T(\mathcal{O}^\perp)$ *be the set of maximal elements of* $\mathcal{K}(\mathcal{O}^\perp)$. *Then* $\mathcal{K}(\mathcal{O})$ *is the family of the signed sets of support contained in E and orthogonal to the elements of* $T(\mathcal{O}^\perp)$.

Proof. Let $\mathcal{B}(\mathcal{O}) = \{X\colon \mathbf{X} \subseteq E,\ \ X \perp Y \text{ for all } Y \in T(\mathcal{O}^\perp)\}$. By the definitions $\mathcal{K}(\mathcal{O}) \subseteq \mathcal{B}(\mathcal{O})$. We prove the equality $\mathcal{K}(\mathcal{O}) = \mathcal{B}(\mathcal{O})$ by induction on $|E|$.

If $E = \{e\}$ the equality is trivial. Suppose that the equality holds for $|E'| < n$ and let $|E| = n$. It is easy to prove that $\mathcal{K}((\mathcal{O}/A)\backslash B) = \{X - A\colon$

$X \in \mathscr{K}(\mathcal{O})$ and $\mathbf{X} \cap B = \varnothing\}$ (see [2] Proposition 5.5). If $M(E)$ has a loop e then for all $Y \in T(\mathcal{O}^{\perp})$, $e \notin \mathbf{Y}$. By inductive hypothesis $\mathscr{B}(\mathcal{O}/e) = \mathscr{K}(\mathcal{O}/e)$ and then necessarily $\mathscr{B}(\mathcal{O}) = \mathscr{K}(\mathcal{O})$. Suppose $M(E)$ has no loops. Let $X \in \mathscr{B}(\mathcal{O})$ and let e be an element of $E - \mathbf{X}$ or if $\mathbf{X} = E$ an element such that $(X-e) \perp Y$ for all $Y \in T(\mathcal{O}^{\perp}/e)$. Then by the inductive hypothesis $(X-e) \in \mathscr{K}(\mathcal{O} \backslash e)$. In this case $(X-e) \perp Y$ for all $Y \in \mathcal{O}^{\perp}$ because $(Y-e) \in \mathscr{K}(\mathcal{O}^{\perp}/e)$ and necessarily $X \in \mathscr{K}(\mathcal{O})$ because $\mathbf{X} \cap \mathbf{Y} \neq \{e\}$. Suppose now $X \in \mathscr{B}(\mathcal{O})$, $\mathbf{X} = E$ and for every e there is $Y_e \in T(\mathcal{O}^{\perp})$ such that $(X-e)$ is not orthogonal to $(Y-e)$. We can suppose $X^+ = E$ by reorientation if necessary of $\mathcal{O}$. Then for every e we can suppose $Y_e^+ = E - \{e\}$, $Y_e^- = \{e\}$. Suppose $\mathcal{O}$ has a positive circuit Z. Then necessarily $\mathbf{Z} = E$, $Z = X$ and $X \in \mathscr{K}(\mathcal{O})$. If $\mathcal{O}$ has not a positive circuit then $(E, \varnothing) \in T(\mathcal{O}^{\perp})$. But this hypothesis is not possible because $(E, \varnothing)$ is not orthogonal to X and the proposition holds. ∎

This paper was suggested by the work of Paul H. Edelman concerning a partial order on the regions of $\mathbb{R}^n$ dissected by hyperplanes [10]. It is not difficult to attach Edelman's study to the more general topics proposed above. Indeed let $\mathscr{H} = \{H_1, H_2, ..., H_m\}$ be a set of hyperplanes in $\mathbb{R}^n$ such that $\bigcap_{i=1}^m H_i = \{0\}$. Let $E = \{h_1, h_2, ..., h_m\}$ be a set of vectors of $\mathbb{R}^n$ normal to the hyperplanes $\mathscr{H}$ (i.e., $H_i = \{x : \langle x, h_i \rangle = 0\}$, $1 \leqslant i \leqslant m$). Let $M(E)$ be the matroid on E determined by linear dependence over $\mathbb{R}$ (i.e., $\{h_{i_1}, ..., h_{i_k}\}$ is an independent set of M iff $h_{i_1}, ..., h_{i_k}$ are linearly independent vectors of $\mathbb{R}^n$). The matroid $M(E)$ has a canonical orientation $\mathcal{O}$: If $\mathbf{C} = \{h_{i_1}, ..., h_{i_k}\}$ is a circuit of $M(E)$ and $\sum_{j=1}^k \lambda_j h_{i_j} = 0$, then $C = (C^+, C^-)$ with $C^+ = \{h_{i_j} : \lambda_j > 0\}$, $C^- = \{h_{i_j} : \lambda_j < 0\}$ is a signed circuit of $\mathcal{O}$; or equivalently if $H = \{h'_{i_1}, ..., h'_{i_k}\}$ is a hyperplane of M then h and h', $h, h' \in E - H$, have the same sign in the signed cocircuit C of support $E - H$ if and only if they are on the same side of the vector subspace of $\mathbb{R}^n$, of dimension $n-1$, spanned by $\{h'_{i_1}, ..., h'_{i_k}\}$.

It is a standard result of matroid theory that *the number of connected n-cells (or regions) of* $\mathscr{R} = \mathbb{R}^n - \bigcup_{i=1} H_i$ *is* $t(M; 2, 0)$, *where* $t(M; x, y)$ *is the Tutte polynomial of the matroid M* (see Zaslavsky [25]). More precisely let R_0 be a fixed region of $\mathscr{R}$ which, without loss of generality, we assume to have the property that for all $x \in R_0$ and h_i, $1 \leqslant i \leqslant m$, $\langle h_i, x \rangle < 0$. By a classical result on linear inequalities (see [23, Theorem 22.1]) *for any set* A, $A \subseteq E$, *exactly one of the following alternatives holds*:

(i) *there is a vector* $x \in \mathbb{R}^n$ *such that* $\langle x, h_i \rangle > 0$ *if* $h_i \in A$ *and* $\langle x, h_i \rangle < 0$ *if* $h_i \in E - A$;

(ii) *there are nonnegative real numbers* $\lambda_1, ..., \lambda_m$, *not all zero, such that* $\sum_{h_i \in E-A} \lambda_i h_i - \sum_{h_j \in A} \lambda_j h_j = 0$ (i.e., *the orientation* ${}_{\bar{A}}\mathcal{O}$ *has a positive circuit*).

These considerations prove the following proposition of Las Vergnas:

PROPOSITION 1.3 ([18]). *Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a set of hyperplanes in $\mathbb{R}^n$ such that $\bigcap_{i=1}^m H_i = \{0\}$. Let $E = \{h_1, \ldots, h_m\}$ be a set of vectors of $\mathbb{R}^n$ such that $H_i = \{x: \langle x, h_i \rangle = 0\}$, $1 \leqslant i \leqslant m$, and suppose there is a region R_0 of $\mathcal{R} = \mathbb{R}^n - \bigcup_{i=1}^m H_i$ such that for $x \in R_0$ and $h_i \in E$ $\langle x, h_i \rangle < 0$. Let $(M_{\mathrm{Lin}}(E), \mathcal{O})$ be the oriented matroid on E determined by linear dependence over $\mathbb{R}$. Then the map $\phi: \mathcal{R} \to \mathcal{A}$, where $\phi(R) = \{h_i: \langle h_i, x \rangle > 0, x \in R\}$ and $\mathcal{A}$ is the set $\{A: A \subseteq E, {}_{\bar{A}}\mathcal{O}$ is an acyclic reorientation of $\mathcal{O}\}$ is an one-to-one map.*

We remark that Paul Edelman considers also noncentral arrangements of hyperplanes (i.e., such that $\bigcap_{i \in I} H_i = \varnothing$). But this point of view is only apparently more general from the matroidal viewpoint. Indeed let $\{H_i \equiv \sum_{j=1}^n a_{ij} x_j = b_i\}_{1 \leqslant i \leqslant m}$ be a noncentral arrangement of hyperplanes of $\mathbb{R}^n$ and let $H'_i \equiv \sum_{j=1}^n a_{ij} x_j - b_i x_{n+1} = 0$, $1 \leqslant i \leqslant m$, $H'_{m+1} \equiv x_{n+1} = 0$. Let $(M_{\mathrm{Lin}}(E), \mathcal{O})$ be the oriented matroid on $E = \{h_1 = (a_{11}, \ldots, a_{1m}, -b_1), \ldots, h_m = (a_{m1}, \ldots, a_{mm}, -b_m), h_{m+1} = (0, 0, \ldots, 0, -1)\}$ determined by linear dependence over $\mathbb{R}$. Let R_0 be a fixed (not necessarily bounded) region of $\mathcal{R} = \mathbb{R}^n - \bigcup_{i=1}^m H_i$ which, without loss of generality, we assume to have the property that for all $x \in R_0$ and $h_i \in E$ $\langle (x, 1), h_i \rangle < 0$. Then ${}_{\bar{A}}\mathcal{O}$ is acyclic and $h_{m+1} \notin A$ iff there is a vector $y \in \mathbb{R}^n$ such that $\langle (y, 1), h_i \rangle > 0$ if $h_i \in A$ and $\langle (y, 1), h_i \rangle < 0$ if $h_i \in E - A$ (see [23, Theorem 22.1]); i.e., iff the vector $y \in \mathbb{R}^n$ satisfies the inequalities $\langle y, (a_{i1}, \ldots, a_{in}) \rangle > b_i$ if $h_i \in A$ and $\langle y, (a_{i1}, \ldots, a_{in}) \rangle < b_i$ if $h_i \in E - (A \cup \{h_{m+1}\})$. These considerations are incorporated into the following version of Proposition 1.3:

PROPOSITION 1.3'. *Let $\mathcal{H} = \{H_i \equiv \sum a_{ij} x_j = b_i\}_{1 \leqslant i \leqslant m}$ be a set of hyperplanes in $\mathbb{R}^n$ such that $\bigcap_{1 \leqslant i \leqslant m} H_i = \varnothing$. Let $E = \{h_1 = (a_{11}, \ldots, a_{1m}, -b_1), \ldots, h_m, h_{m+1} = (0, \ldots, 0, -1)\}$ and suppose there is a region R_0 of $\mathcal{R} = \mathbb{R}^n - \bigcup_{i=1}^m H_i$ such that for $x \in R_0$, $\langle h_i, x \rangle < 0$, $1 \leqslant i \leqslant m$. Let $(M_{\mathrm{Lin}}(E), \mathcal{O})$ be the oriented matroid on E determined by linear dependence over $\mathbb{R}$. Then the map $\phi: \mathcal{R} \to \mathcal{A}'$, where $\phi(R) = \{h_i: \langle h_i, x \rangle > 0, x \in R\}$ and $\mathcal{A}' = \{A: A \subseteq E - \{h_{m+1}\}$ and ${}_{\bar{A}}\mathcal{O}$ is an acyclic reorientation of $\mathcal{O}\}$ is an one-to-one map.*

Finally we remark that if the generalizations to oriented matroids of Edelman's results tend to become quite technical our proofs are certainly more straightforward.

2. THE MAIN THEOREMS

The set of faces of an acyclic oriented matroid $(M(E), \mathcal{O})$, also called matroid polytope, ordered by inclusion constitutes a lattice denoted here by $P(M)$. The lattice $P(M)$ has the Jordan–Hölder chain property with height function ρ_M (see [17, Theorem 1.1]). We remark that we consider

$\varnothing$ and E faces of $P(M)$, and given $F, G \in P(M)$, $F \leqslant G$, then the interval $[F, G]$ of $P(M)$ is isomorphic to $P(M(G)/F)$.

Theorem 2.1 is a fundamental result of oriented matroid theory because it is equivalent to the Euler relation to matroid polytopes (see Theorem 2.1′). We remark that the equivalence of Theorems 2.1 and 2.1′ actually holds in any lattice with the Jordan–Hölder property as pointed out by Lindström (see [19, Theorem 2]). This result is also implicit in Rota [24].

THEOREM 2.1 ([5]). *Let $P(M)$ be the lattice of the faces of a matroid polytope $(M(E), \mathcal{O})$. Let μ be the Möbius function of the lattice $P(M)$. Then for all $F \in P(M)$ $\mu(\varnothing, F) = (-1)^{\rho(F)}$.*

THEOREM 2.1′ ([5, 11, 13, 20]). *Let $P(M)$ be the lattice of the faces of a matroid polytope, $f_i(M)$ be the number of faces of $P(M)$ with rank i. Then the lattice $P(M)$ satisfies the Euler relation, i.e.,*

$$\sum_{i=0}^{\operatorname{rank} M} (-1)^i f_i(M) = 0.$$

The following variant of Euler's relation is technically the most important result of this paper.

THEOREM 2.2. *Let $(M(E), \mathcal{O})$ be an acyclic oriented matroid and ${}_{\bar{A}}\mathcal{O}$ be an acyclic reorientation of $\mathcal{O}$ with $A \neq \varnothing, E$. Then the following equivalent identities hold:*

$$\sum_{\substack{F \in P(M) \\ F \cap A = \varnothing}} (-1)^{\operatorname{rank} F} = 0, \tag{2.3}$$

$$\sum_{\substack{F \in P(M) \\ F \cap A \neq \varnothing}} (-1)^{\operatorname{rank} F} = 0, \tag{2.4}$$

where $P(M)$ denotes the lattice of the faces of the matroid polytope $(M(E), \mathcal{O})$.

LEMMA 2.5 ([17]). *Let $(M(E), \mathcal{O})$ be an acyclic oriented matroid and p be an extreme point of $\mathcal{O}$. Then F is a face of $(M, \mathcal{O})/\{p\}$ iff $F \cup \{p\}$ is a face of $(M, \mathcal{O})$.*

LEMMA 2.6 ([4, 20]). *Let $(M(E), \mathcal{O})$ be an acyclic oriented matroid and ${}_{\bar{A}}\mathcal{O}$ be an acyclic reorientation of $\mathcal{O}$. Then there is an order $a_1, a_2, \ldots, a_n$ of the points of A such that for every i, $1 \leqslant i \leqslant n$, ${}_{\overline{\{a_1, \ldots, a_i\}}}\mathcal{O}$ is an acyclic reorientation of $\mathcal{O}$.*

Proof of Theorem 2.2. We remark that the equivalence of the conditions (2.3) and (2.4) is a clear consequence of the Euler relation (see Theorem 2.1′). We now prove (2.3). For every X, $X \subseteq E$, let $X' = X \cap \{p\colon p$ extreme point of $(M(E), \mathcal{O})\}$ and let $L = \{F\colon F$ is a face of $M(E)$ and $F \cap A = \emptyset\}$, $L' = \{F'\colon F'$ is a face of $M(E')$ and $F' \cap A' = \emptyset\}$. Then $L \simeq L'$, $F \rightsquigarrow F'$ because $\rho_{M(E)}(F) = \rho_{M(E')}(F')$. Indeed the extreme points of a face F of $M(E)$ are also extreme points of $M(E)$ and there is a base of F constituted by extreme points (see [17]). Then, without loss of generality we assume that all the points of $(M(E), \mathcal{O})$ are extreme points. We proceed by induction on $|A|$. Suppose $A = \{p\}$. From Theorem 2.1′ and Lemma 2.5 we have

$$\sum_{F \in P(M)} (-1)^{\operatorname{rank} F} = 0, \qquad \sum_{\substack{F \in P(M)\\ p \in F}} (-1)^{\operatorname{rank} F} = -\sum_{F \in P(M/p)} (-1)^{\operatorname{rank} F} = 0$$

and the identity (2.3) is true. Suppose Theorem 2.2 holds for all B, $|B| < n$ and let $|A| = n$. From Lemma 2.6 there is an order $a_1, \ldots, a_n$ of the points of A such that $_{\overline{\{a_1, \ldots, a_{n-1}\}}}\mathcal{O} = \mathcal{O}'$ is an acyclic orientation.

As $_{\bar{A}}\mathcal{O}$ and $\mathcal{O}'$ are acyclic orientations a_n is an extreme point of $\mathcal{O}'$ (see [17]). Moreover, from the induction hypothesis we have

$$\sum_{\substack{F \in P(M)\\ F \cap \{a_1, \ldots, a_{n-1}\} = \emptyset}} (-1)^{\operatorname{rank} F} = 0 \qquad \text{and} \qquad \sum_{\substack{F \in P(M/a_n)\\ F \cap \{a_1, \ldots, a_{n-1}\} = \emptyset}} (-1)^{\operatorname{rank} F} = 0.$$

From Lemma 2.5 the last identity is equivalent to

$$\sum_{\substack{F \in P(M)\\ a_n \in F\\ F \cap \{a_1, \ldots, a_{n-1}\} = \emptyset}} (-1)^{\operatorname{rank} F} = 0.$$

But, as

$$\sum_{\substack{F \in P(M)\\ F \cap A = \emptyset}} (-1)^{\operatorname{rank} F} = \sum_{\substack{F \in P(M)\\ F \cap \{a_1, \ldots, a_{n-1}\} = \emptyset}} (-1)^{\operatorname{rank} F} - \sum_{\substack{F \in P(M)\\ a_n \in F\\ F \cap \{a_1, \ldots, a_{n-1}\} = \emptyset}} (-1)^{\operatorname{rank} F} = 0$$

we have concluded the proof of (2.3). ∎

Remark 2.7. The converse of Theorem 2.2 is not true. Indeed let $(M_{\mathrm{Aff}}(\{1, \ldots, 4\}), \mathcal{O})$ be the oriented matroid of the affine dependencies on $\{1, \ldots, 4\}$ over $\mathbb{R}$ determined by Fig. 1. Then for $A = \{2\}$ the equalities (2.3) and (2.4) hold but $_{\overline{\{2\}}}\mathcal{O}$ is not an acyclic reorientation of $\mathcal{O}$. ∎

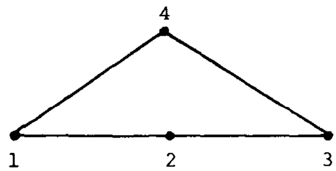

FIGURE 1

Corollary 2.9 is an application of Theorem 2.2 to convex polytopes. Theorem 2.8 is a necessary auxiliary result implicit in [17] (see [6] for a short proof).

THEOREM 2.8. *Let E be a finite set of $\mathbb{R}^d$ and $(M_{Aff}(E), \mathcal{O})$ be the oriented matroid of affine dependencies of E over $\mathbb{R}$. Then the following two statements are equivalent:*

(i) ${}_{\bar{A}}\mathcal{O}$ *is an acyclic reorientation of* $\mathcal{O}$;

(ii) *there is a hyperplane H separating A strictly from $E-A$.*

COROLLARY 2.9 ([15, Chap. 8, Sect. 5, Exercise 2). *Let P be a d-dimensional convex polytope and F be a facet of P. Denote by $f_i(P; F)$ the number of i-faces of P that are disjoint from F. Then $\sum_{i=0}^{d-1}(-1)^i f_i(P; F) = 1$.*

COROLLARY 2.10. *Let $(M(E), \mathcal{O})$ be an acyclic oriented matroid, let $\mathcal{A} = \{A\colon A \subseteq E,\ {}_{\bar{A}}\mathcal{O}$ is acyclic$\}$ and suppose $\mathcal{A}$ ordered by inclusion. Then*

$$\mu(\varnothing, A) = \begin{cases} (-1)^{\operatorname{rank} A} & \text{if } A \text{ is a face of } (M(E), \mathcal{O}), \\ 0 & \text{otherwise,} \end{cases}$$

where μ denotes the Möbius function of $\mathcal{A}$.

Proof. We proceed by induction on $|A|$. If $A=\varnothing$ then $\mu(\varnothing, \varnothing)=1$. Suppose Corollary 2.10 is true for all A', $|A'|<n$ and let $|A|=n$. If $A=E$ then by the induction hypothesis $\mu_{P(M)}(\varnothing, E)=\mu_{\mathcal{A}}(\varnothing, E)$ and by Theorem 2.1 $\mu_{P(M)}(\varnothing, E)=(-1)^{\operatorname{rank} E}$. Suppose now $A \neq E$. From the definition of Möbius function and the induction hypothesis $\mu_{\mathcal{A}}(\varnothing, A)= -\sum_{F\in P(M), F\subsetneq A}(-1)^{\operatorname{rank} F}$. As $E-A\neq\varnothing, E$, by the identity 2.3 relative to $E-A$ we have $\sum_{F\in P(M), F\subseteq A}(-1)^{\operatorname{rank} F}=0$. Then if A is a face $\mu_{\mathcal{A}}(\varnothing, A)= \mu_{P(M)}(\varnothing, A)=(-1)^{\operatorname{rank} A}$ and if it is not then we have necessarily $\mu_{\mathcal{A}}(\varnothing, A)=0$. ∎

COROLLARY 2.11. *Let $(M(E), \mathcal{O})$ be an acyclic oriented matroid and p an extreme point of $(M, \mathcal{O})$. Let $\mathcal{A}' = \{A\colon A\subseteq E-\{p\}$ and ${}_{\bar{A}}\mathcal{O}$ is acyclic$\}$ and suppose $\mathcal{A}'$ ordered by inclusion. Then*

$$\mu(\varnothing, A)=\begin{cases}(-1)^{\operatorname{rank} A} & \text{if } A \text{ is a face of } (M(E), \mathcal{O}),\\ 0 & \text{otherwise,}\end{cases}$$

where μ denotes the Möbius function of $\mathscr{A}'$.

Proof. Since $\mathscr{A}'$ is an ideal of the poset $\mathscr{A}=\{A\colon A\subseteq E$ and ${}_{\bar{A}}\mathcal{O}$ is acyclic$\}$ we conclude that $\mu_{\mathscr{A}'}(\varnothing, A)=\mu_{\mathscr{A}}(\varnothing, A)$ and Corollary 2.11 is a result of Corollary 2.10. ∎

We remark that from Proposition 1.3′ and Corollary 2.11 we can deduce, by simple interpretation of the definitions, Theorem 1.8 and Corollary 1.10 of Edelman [10].

COROLLARY 2.12. *Let $(M(E), \mathcal{O})$ be an acyclic oriented matroid, let $\mathscr{A}=\{A\colon A\subseteq E$ and ${}_{\bar{A}}\mathcal{O}$ is acyclic$\}$ and suppose $\mathscr{A}$ is ordered by inclusion. Then for $A, B\in\mathscr{A}$*

$$\mu_{\mathscr{A}}(A, B)=\begin{cases}(-1)^{\operatorname{rank}(B-A)} & \text{if } A\subseteq B \text{ and } B-A \text{ is a face of } (M, {}_{\bar{A}}\mathcal{O}),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let $\mathcal{O}'={}_{\bar{A}}\mathcal{O}$, $\mathscr{A}'=\{C\colon {}_{\bar{C}}\mathcal{O}$ is acyclic$\}$. From the definitions $\mathscr{A}'=\{X\bigtriangleup A\colon X\in\mathscr{A}\}$. Moreover if $A\subseteq B$ the interval $[A, B]$ of the poset $\mathscr{A}$ is isomorphic to the interval $[\varnothing, B-A]$ of the poset $\mathscr{A}'$ where $X\rightsquigarrow X-A$. Then $\mu_{\mathscr{A}}(A, B)=\mu_{\mathscr{A}'}(\varnothing, B-A)$ and Corollary 2.12 is a consequence of Corollary 2.10. ∎

More information concerning the poset $\mathscr{A}$ can be obtained considering the *order complex* $\Delta(\mathscr{A})$; i.e., the abstract simplicial complex whose vertices are the elements of $\mathscr{A}-\{\varnothing, E\}$ and whose simplicies are the chains $A_0< A_1<\cdots<A_k$ in $\mathscr{A}-\{\varnothing, E\}$. Let $|\Delta(\mathscr{A})|$ be the geometric realization of $\Delta(\mathscr{A})$. The following theorem generalizes a result of Edelman (see [10, Theorem 2.7]).

THEOREM 2.13. *Let $(M(E), \mathcal{O})$ be an acyclic oriented matroid of rank r and suppose that $\mathcal{O}$ has r extreme points. Then $|\Delta(\mathscr{A})|$ has the homotopy type of the $(r-2)$-dimensional sphere.*

Lemma 2.14 is attributed by Björner [1] to Quillen (see [1] for an elementary proof).

LEMMA 2.14 ([1]). *Let Δ be a geometric simplicial complex covered by a family of subcomplexes $(\Delta_i)_{i\in I}$. Assume that every finite intersection $\Delta_{i_1}\cap\Delta_{i_2}\cap\cdots\cap\Delta_{i_n}$ is either empty or contractible. Then Δ has the same homotopy type as the nerve N of the covering.*

Proof of Theorem 2.13. Let $a_1,\ldots, a_r$ be the extreme points of $\mathcal{O}$. For every i, $1 \leqslant i \leqslant r$, let Δ_i be the abstract simplicial complex whose vertices are the elements of $\mathscr{A}_i = \{A\colon A \in \mathscr{A},\ a_i \in A,\ A \neq E\}$ and whose simplicies are the chains $A_0 < A_1 < \cdots < A_k$ in $\mathscr{A}_i$. Then Δ is covered by the family of subcomplexes $(\Delta_i)_{1 \leqslant i \leqslant r}$. For every I, $I \subsetneq \{1,..., r\}$ the flat F_i of M spanned by the elements $\{a_i\}_{i \in I}$ is a face of $\mathcal{O}$ and $F_i \in \mathscr{A}$. Then $\bigcap_{i \in I} \Delta_i = \{A\colon A \in \mathscr{A},\ F_i \subseteq A\} = St\{F_i\}$; i.e., $\bigcap_{i \in I} |\Delta_i|$ is a cone with peak F_i and thus it is contractible. Then from Lemma 2.14 $|\Delta|$ has the same homotopy type of the nerve N of the covering $(|\Delta_i|)_{1 \leqslant i \leqslant r}$. As it is clear that N is the simplicial complex of the faces of a $(r-1)$-simplex our theorem follows. ∎

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