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Minimax Optimization of a Unimodal Function by Variable Block Derivative Search with Time Delay

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A minimax search plan is developed for locating the maximum of a unimodal function with a sequence of blocks of simultaneous evaluations of the derivative of the function. The search strategy permits any number of blocks and any number of experiments in each block. Further, any time delay is permissible where the time delay is the number of blocks placed after the m-th block but before the results of the m-th block become known. The proposed variable block search plan is optimal in the sense that for a required final interval of uncertainty, known to contain the point where the function attains its maximum, and for any given value for time delay and the number of experiments in each block, it has the largest possible initial interval. A method of optimizing the number of experiments per block is given. An extension to non-linear programming algorithms is given.

1. INTRODUCTION

Situations often occur in which it is desired to search for the optimum of a one-dimensional unimodal function by placing sequences of simultaneous measurements (blocks) on the domain of the function. Often the physical system is such that a time delay occurs between the placement of a block of experiments and the time when the result becomes known. Let the *time delay* be τ if the $(m + \tau)$-th block is placed before the result of the m-th block becomes known, but after the result of the $(m - 1)$-st block is known. If the blocks consist of one experiment measuring the value of the function, then, if $\tau = 0$, Fibonacci search [6, 8] is the minimax technique while, if $\tau = 1$ or $\tau = 2$, [2] gives the minimax strategy. If each block consists of k function evaluation experiments, with $\tau = 0$, the minimax strategy is given in [1, 5], while [3] extends this to allow any number of experiments to be placed in each block.

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The present paper gives the minimax search strategy to determine the optimum of a unimodal function if the blocks are composed of any number of experiments evaluating the derivative of the function (or equivalently, if the zero of a monotone function is sought and experiments evaluate the function itself).

A discussion is given of how to choose the optimal number of experiments to place in each block, given certain constraints that must be satisfied.

2. Definitions

Without loss of generality, assume a maximum is being sought.

Let $f(x)$ be a unimodal function on the interval $[0, d]$ with $f(x^*) = \max\{f(x) : 0 \leqslant x \leqslant d\}$. For any $x_a < x_b$, $f(x)$ is *unimodal* if $x_b \leqslant x^*$ implies $f(x_a) < f(x_b)$ and $x^* \leqslant x_a$ implies $f(x_a) > f(x_b)$.

Define a time scale t which starts at 0 and is incremented by 1 with the passage of each time period. The start of the j-th time period is at time $t = j_0$; other times in this period are $t = j$.

A *derivative block* consists of a group of experiments, each experiment evaluating the derivative $f'(x)$, placed simultaneously on $[0, d]$ at the start of a time period. For brevity, we shall henceforth use "block" to mean "derivative block." Let $k_j \geqslant 0$ be the number of experiments in the j-th block placed at time $t = j_0$.

Let the *time delay* be τ if the $(m + \tau)$-th block is placed before the result of the m-th block is known, but after the result of the $(m - 1)$-st block is known. In other words, the result of the $(m - 1)$-st block becomes known at the time $t = m + \tau - 1$.

Let $x_{i,j}$ denote the position of the i-th experiment in the j-th block, with the ordering such that $x_{i-1,j} < x_{i,j}$. N is the total number of blocks to be placed, K the total number of experiments. Then $K = \sum_{j=1}^{N} k_j$.

Let $\mathbf{k}_N = (k_1, \ldots, k_N)$.

Let $l_n = \max\{x_{i,j} : f'(x_{i,j}) > 0,\ i = 1, \ldots, k_j;\ j = 1, \ldots, n - \tau\}$,

$$r_n = \min\{x_{i,j} : f'(x_{i,j}) < 0,\ i = 1, \ldots, k_j;\ j = 1, \ldots, n - \tau\}.$$

Let $l_0 = 0$, $r_0 = d$.

(If the search is for the zero of a monotone decreasing function, replace $f'(x_{i,j})$ by $f(x_{i,j})$.)

The *interval of uncertainty* at time n is $U_n = [l_n, r_n]$. Clearly $x^* \in U_n$, $n = 0, \ldots, N + \tau$ and all further experiments should be placed in this interval. $U_1 \geqslant U_2 \geqslant \cdots \geqslant U_{n+\tau}$. $U_{N+\tau}$ is the *final interval of uncertainty*.

The fact that the locations l_n and r_n depend on the unknown function f

can be indicated by writing $l_n(f)$ and $r_n(f)$. Let the longest possible final interval, considering all unimodal functions f, be denoted by z:

$$z = \max_f(r_n(f) - l_n(f)).$$

Let the *information* at time $n + \tau$ be

$$I_{n+\tau} = \{x_{1,1},\ldots, x_{k_1,1},\ldots, x_{k_{n+\tau},n+\tau}, f'(x_{1,1}),\ldots, f'(x_{k_n,n})\}.$$

A *policy* $g_{n+\tau+1}(I_{n+1})$ specifies the placement of the $(n + \tau + 1)$-st block;

$$g_{n+\tau+1}(I_{n+\tau}) = \{x_{1,n+\tau+1},\ldots, x_{k_{n+\tau+1},n+\tau+1}\}.$$

A search strategy $S(d, z, \tau, \mathbf{k}_N)$ on $[0, d]$ is composed of N sequential policies $\{g_1,\ldots, g_N\}$ so that $r_n(f) - l_n(f)) \leqslant z$. Policy g_n is implemented at time n_0, $n_0 = 1,\ldots, N$. If $z = 1$, $S(d, 1, \tau, \mathbf{k}_N)$ is said to be *feasible.* A feasible strategy is *minimax* (or *optimal*) if d is maximized over all feasible S. If $S^*(d, 1, \tau, \mathbf{k}_N)$ is optimal, let $d = d^*$ so that $S(d, 1, \tau, \mathbf{k}_N) = S^*(d^*, 1, \tau, \mathbf{k}_N)$. An optimal strategy need not be unique.

Let $l_n' = l_n + \max\{x_{i,j} : x_{i,j} \in [l_n, r_n], j = n - \tau + 1,\ldots, n\}$. The *reduction ratio* $r = z/d$.

A sequential search strategy places all experiments in the interval of uncertainty, which is successively being reduced. A simultaneous search strategy cannot take advantage of the results of any experiments and hence requires many more experiments. The *equivalent simultaneous strategy* of $S^*(d^*, 1, \tau, \mathbf{k}_N)$ will be denoted $\mathscr{S}(E_N, d^*, 1, \tau, \mathbf{k}_N)$. $\mathscr{S}(E_N, d^*, 1, \tau, \mathbf{k}_N)$ reduces the interval of uncertainty from $[0, d^*]$ to a unit final interval using one block of experiments. E_N denotes the least number of experiments required to accomplish this.

3. The Search Strategy

Theorem. *The length of the maximum starting interval that may be searched to give a unit final interval of uncertainty, given* $\mathbf{k}_N$ *and* τ, *is* L_N, *where:*

(3.1) $L_n = 1, \qquad n \leqslant 0.$

(3.2) $L_n = k_{N-n+1}L_{n-1-\tau} + L_{n-1}, \qquad n > 0.$

A minimax search strategy on the above interval is:

(3.3) $x_{i,n} = l'_{n-1} + iL_{N-n-\tau}, \qquad i = 1,\ldots, k_n.$

Let this search strategy be denoted by $S^{\#}(L_N, 1, \tau, \mathbf{k}_N)$.

Before proving the theorem, it is convenient to prove the following lemma, which gives the longest interval that may be searched to give a unit final interval of uncertainty using simultaneous derivative evaluation experiments.

LEMMA 1. *If* $N = 1$, $k_1 = K$, *then* $d^* = L_1 = K + 1$. *The optimal search strategy* $S^*(L_1, 1, \tau, k_1)$ *is*:

(3.4) $x_{i,1} = i, \qquad i = 1,..., K.$

Proof. For notational convenience, define $x_{0,1} = 0$, $x_{K+1,1} = d^*$.

(a) *Optimality*

Consider any search. If the search is feasible, then it is required that $r_1 - l_1 \leqslant 1$. Hence we require that

(3.5) $x_{i+1,1} - x_{i,1} \leqslant 1, \qquad i = 0,..., K.$

Hence

(3.6) $$d^* \leqslant \max \sum_{i=0}^{K} x_{i+1,1} - x_{i,1} \leqslant K + 1 \leqslant L_1 .$$

(b) *Feasibility*

The final interval of uncertainty will be $[x_{i,1}, x_{i+1,1}]$ for some $i = 0..., K$. By (3.4) $x_{i+1,1} - x_{i,1} = 1$ for $i = 0,..., K$. Hence the search is feasible, and $d^* = L$.

Proof of the Theorem. The proof is in two parts. The first shows that, given $\mathbf{k}_N$ and τ, $[0, L_N]$ is the largest interval that can be searched to produce a unit final interval. The second shows that the proposed search is feasible, producing a unit final interval of uncertainty from the initial interval $[0, L_N]$.

(a) *Optimality*

The technique to be used is to transform a search strategy employing a sequence of blocks into an equivalent simultaneous strategy.

Let N, $\mathbf{k}_N$, and τ be given. Assume that $S^*(d^*, 1, \tau, \mathbf{k}_N)$ is known. This strategy may be transformed to the equivalent simultaneous strategy $\mathscr{S}(E_N, d^*, 1, \tau, \mathbf{k}_N)$. It will be shown that L_N is an upper bound for $E_N + 1$. Since $E_N + 1 = d^*$ by Lemma 1, L_N will also be an upper bound for d^*. The feasibility part will show that $S^{\#}(L_N, 1, \tau, \mathbf{k}_N)$ is feasible, so that $L_N = d^*$ and $S^{\#}(L_N, 1, \tau, \mathbf{k}_N)$ is the minimax search strategy.

The induction hypothesis is that, for $N > 0$, $E_N \leqslant E_N'$, where

$$E_N' = k_N(E'_{N-\tau-1} + 1) + E'_{N-1}.$$

(Define $E'_{-\tau} =,..., = E_0' = E_{-\tau} = ,..., = E_0 = 0.$)

If $N \leqslant \tau + 1$, no experimental results are available before all the blocks have been placed, so that in this case the sequential search is the same as the simultaneous search $\mathscr{S}(E_N, d^*, 1, \tau, \mathbf{k}_N)$ where

$$E_N = \sum_{n=1}^{N} k_n .$$

Let $E_N' = E_N$ for $0 < N \leqslant \tau + 1$. Thus,

$$E_N' = k_N + \sum_{n=1}^{N-1} k_n = k_N(E'_{N-\tau-1} + 1) + E'_{N-1}$$

and, since $E_N = E_N'$, the induction hypothesis is satisfied.

If $N = \tau + 2$, the results of the first block are known before the $(\tau + 2)$-nd block is placed. The length of the interval of uncertainty is reduced to $r_{\tau+1} - 1_{\tau+1} =$ either $(x_{i+1,1} - x_{i,1})$ where i is one of $0,..., k_1 - 1$ or $(r_0 - x_{k_1})$. $k_{\tau+2}$ experiments are then placed in $U_{\tau+1}$. There are $k_1 + 1$ possibilities for $U_{\tau+1}$, hence a block of $k_{\tau+2}$ experiments in the sequential search is equivalent to at most $k_{\tau+2}(k_1 + 1)$ simultaneous experiments. The total number of simultaneous experiments required to produce an equivalent reduction as sequential search is then

$$E_N \leqslant E_N' = k_{\tau+2}(k_1 + 1) + E'_{\tau+1} = k_{\tau+2}(E_1' + 1) + E'_{\tau+1}$$

and the hypothesis holds.

Assume the hypothesis holds for $N = M - 1 \geqslant \tau + 2$. When the M-th block is to be placed the results of the first $M - \tau - 1$ blocks are known. By the induction hypothesis, a search with $M - \tau - 1$ blocks is equivalent to at most $E'_{M-\tau-1}$ experiments and hence there-are at most $E'_{M-\tau-1} + 1$ possibilities for U_{M-1}, in which the M-th block could be placed. The placement of the M-th block in the sequential search is then equivalent to at most $k_M(E'_{M-\tau-1} + 1)$ experiments in the simultaneous search. Since the previous $(M - 1)$ blocks are equivalent to at most E'_{M-1} experiments by the hypothesis, we have

$$E_M \leqslant E_M' = k_M(E'_{M-\tau-1} + 1) + E'_{M-1},$$

proving the hypothesis.

By Lemma 1, $d^* = E_N + 1 \leqslant E_N' + 1$. Using the substitution $\mathscr{L}_n = E_n' + 1$, $n = 1,..., N$ it follows that $\mathscr{L}_N \geqslant d^*$ where $\mathscr{L}_N$ is determined by

$$\begin{aligned} \mathscr{L}_n &= 1, & n &\leqslant 0, \\ \mathscr{L}_n &= k_n \mathscr{L}_{n-\tau-1} + \mathscr{L}_{n-1}, & n &> 0. \end{aligned}$$

It is shown below that $\mathscr{L}_N = L_N$ and hence $L_N \geqslant d^*$, which completes the optimality part of the proof. The next part will show that $L_N = d^*$ and that $S^{\#}(L_N, 1, \tau, \mathbf{k}_N)$ is the minimax strategy.

To show that $\mathscr{L}_N = L_N$, an inductive proof can be used to obtain generalized formulas for $\mathscr{L}_n$ and L_n. It can then be shown that the formulas for L_N and $\mathscr{L}_N$ are identical and hence $\mathscr{L}_N = L_N$.

The induction hypothesis is that

$$\begin{aligned} \mathscr{L}_n = 1 &+ \sum_{i_1=1}^{n} k_{i_1} + \sum_{i_1=1}^{n} \sum_{i_2=i_1+\tau+1}^{n} k_{i_1} k_{i_2} + \cdots \\ &+ \sum_{i_1=1}^{n} \cdots \sum_{i_{A_n}=i_{A_n-1}+\tau+1}^{n} k_{i_1} \cdots k_{i_{A_n}} \end{aligned}$$

and that

$$\begin{aligned} L_n = 1 &+ \sum_{i_1=N-n+1}^{N} k_{i_1} + \sum_{i_1=N-n+1}^{N} \sum_{i_2=i_1+\tau+1}^{N} k_{i_1} k_{i_2} + \cdots \\ &+ \sum_{i_1=N-n+1}^{N} \cdots \sum_{i_{A_n}=i_{A_n-1}+\tau+1}^{N} k_{i_1} \cdots k_{i_{A_n}}, \end{aligned}$$

where

$$A_n = \left[\frac{n+1}{\tau+1}\right].$$

These formulas are easily verified by induction. For $n = N$, it is clear that $\mathscr{L}_N = L_N$.

(b) *Feasibility*

By induction. The hypothesis is that the search strategy $S_N^{\#}(L_N, 1, \tau, \mathbf{k}_N)$ is feasible on $[0, L_N]$.

For $N = 1,..., \tau + 1$ it is easily verified by (3.1), (3.2), and (3.3) that the length of the final interval of uncertainty, $r_{N+\tau} - l_{N+\tau}$, will be unity. Hence the search is feasible.

Now assume $S_N^{\#}(L_N, 1, \tau, \mathbf{k}_N)$ is feasible for $N \leqslant M - 1$ on $[0, L_N]$. For $N = M$, by (3.3), the first block of experiments is placed at $x_{i,1} = l_0' + iL_{M-\tau-1} = iL_{M-\tau-1}$, $i = 1,..., k_1$. The second block is placed at $x_{i,2} = l_1' + iL_{M-\tau-2} = k_1L_{M-\tau-1} + iL_{M-\tau-2}$, $i = 1,..., k_2$. The $(\tau + 1)$-st block is placed at $x_{i,\tau+1} = l_\tau' + iL_{M-2\tau-1} = \sum_{j=1}^{\tau} k_jL_{M-\tau-j} + iL_{M-2\tau-1}$, $i = 1,..., k_{\tau+1}$. At this point the result of the first block of experiments is received. If $f'(x_{k_{1,1}}) < 0$, then

$$U_{\tau+1} = [l_{\tau+1}, r_{\tau+1}] = [x_{i,1}, x_{i+1,1}], \qquad \text{for some } i = 0,..., k_1 - 1.$$

By (3.3)

$$| x_{i+1,1} - x_{i,1} | = L_{M-\tau-1}, \qquad i = 0,..., k_1 - 1.$$

Since $M - \tau - 1$ blocks remain, by the induction assumption this region can be feasibly searched.

If $f'(x_{k_{1,1}}) > 0$, then

$$U_{\tau+1} = [k_1L_{M-\tau-1}, L_M].$$

By (3.2), $| L_M - k_1L_{M-\tau-1} | = L_{M-1}$. τ blocks have been placed in this region already, positioned at

$$(3.7)\quad l_1' + iL_{M-\tau-2}, i = 1,..., k_2 ;...; l_\tau' + iL_{M-2\tau-1}, i = 1,..., k_{\tau+1}.$$

$M - \tau - 1$ blocks remain to be placed. But, given $(k_2,..., k_N)$ and an interval of length L_{M-1} to search, (3.7) gives the precise placement of the first τ blocks of $S_{M-1}^{\#}(L_{M-1}, 1, \tau, \mathbf{k}_{M-1})$. By the induction hypothesis this search is feasible and $L_N = d^*$. This proves the hypothesis.

This completes the proof of the theorem.

4. Example

Suppose $N = 5$, $K = 8$, and $\tau = 1$ and that it is required that $k_i \leqslant 2$, $i = 1,..., 5$. It will be shown in Section 5 that the optimal $\mathbf{k}_5 = (2, 1, 2, 1, 2)$. In this case,

$$\begin{aligned} L_0 &= 1, \\ L_1 &= k_5 + 1 = 3, \\ L_2 &= k_4 + L_1 = 4, \\ L_3 &= k_3L_1 + L_2 = 10, \\ L_4 &= k_2L_2 + L_3 = 14, \\ L_5 &= k_1L_3 + L_4 = 34. \end{aligned}$$

Thus the interval [0, 34] may be searched to give a unit final interval of uncertainty. The placements in the first two blocks are given by:

$$\begin{aligned} x_{11} &= L_3 = 10, \\ x_{21} &= 2L_3 = 20 = l_1', \\ x_{12} &= l_1' + L_2 = 24. \end{aligned}$$

Before placing the third block, the results of the first block become known and the interval of uncertainty, U_3, is reduced to one of [0, 10], [10, 20] or [20, 34]. Suppose $f'(x_{11}) < 0$ so $U_3 = [0, 10]$. Then

$$\begin{aligned} l_3' &= 0, \\ x_{13} &= l_3' + L_1 = 3, \\ x_{23} &= l_3' + 2L_1 = 6 = l_4', \\ x_{14} &= l_4' + L_0 = 7. \end{aligned}$$

At this point, the result of the third block becomes known so that $U_5 = [0, 3], [3, 6]$, or $[6,10]$. Suppose $f'(x_{23}) > 0$. Then

$$\begin{aligned} l_5' &= 7, \\ x_{15} &= l_5' + L_{-1} = 8, \\ x_{25} &= l_5' + 2L_{-1} = 9. \end{aligned}$$

After the results of these experiments are obtained, the final interval of uncertainty is reduced to one of [6, 7], [7, 8], [8, 9], or [9, 10].

Figure 1 depicts the placement of the first four blocks of experiments. Block 3 has been placed three times, to cover the three cases $U_3 = [0, 10]$, $U_3 = [10, 20]$, and $U_3 = [20, 34]$. Similarly, block 4 has been placed four times. In practice, of course, these blocks are placed only once in the appropriate interval, as above. Block 5, consisting of two experiments, has yet to be placed in Figure 1. It is clear, however, that after it is placed, the final interval of uncertainty is unity. Figure 1 thus shows the power of sequential search in that the experiments in block 3 are equivalent to three times as many experiments than if all experiments had been placed simultaneously, while block 4 is equal to four times as many, and block 5 is equal to ten times as many.

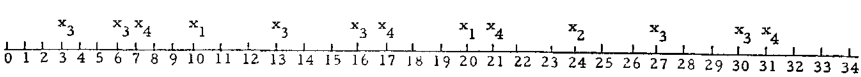

FIG. 1. Placement of experiments for $\mathbf{k} = (2, 1, 2, 1, 2)$.

5. Arrangement of Experiments

The selection of $\mathbf{k}_N$ depends on N (the time allotted to the search), τ, the maximum experiments allowable per block and the desired reduction ratio. For given values of τ and the reduction ratio, decreasing N requires increasing K.

The optimal $\mathbf{k}_N$ can be determined as follows. From (3.2):

$$(5.1)\quad L_N = k_1 L_{N-\tau-1} + k_2 L_{N-\tau-2} +,..., + k_{\tau+1} L_{N-2\tau+1} + L_{N-\tau-1},$$

where $L_{N-\tau-1},..., L_{N-2\tau-1}$ are independent of $k_1,..., k_{\tau+1}$. Now $L_n \geqslant L_{n-1}$ for all n. If there are no constraints on the number of experiments per block, it is clear that L_N is maximized by setting $k_2 = ,..., = k_{\tau+1} = 0$ and placing the available experiments in block 1, say $k_1 = k_1'$. Similarly $L_{N-\tau-1}$ could be expanded as L_N was, and this would show that $L_{N-\tau-1}$ is maximized by setting $k_{\tau+2} = k'_{\tau+2}$, $k_{\tau+3} = ,..., = k_{2\tau+3} = 0$, if there are no constraints. Continuing in this manner, we find that

$$(5.2)\quad L_N = k_1' k'_{\tau+2},..., k_b' + 1,$$

where $b = [N/(\tau+1)](\tau+1) + 1$. Let $h = (b+\tau)/(\tau+1)$, $a = [K/h]$ and $P = K - ah$. Then L_N in (5.2) is maximized by setting

$$k_i = \begin{Bmatrix} a+1, & i = 1, \tau+2,..., (P-1)\tau + P, \\ a, & i = P\tau + P + 1,..., b, \\ 0, & \text{otherwise.} \end{Bmatrix}$$

Now consider the case in which constraints on the k_i exist, $k_i \leqslant c_i$. Suppose that $c_i = c$, $i = 1,..., N$. Set $k_i = c$, $i = 1, \tau+2,..., b$ and determine if the required reduction ratio is attained. If not, then it is clear that, to optimize (5.1), experiments should be added, one at a time to blocks $2, \tau+3,...$. If these blocks each has one experiment, and the reduction ratio is not obtained, continue adding experiments in the prescribed manner until either the desired reduction ratio is attained or until each of these blocks has c experiments in it. If the latter holds, start adding experiments to blocks $3, \tau+4,...$. Continue in this manner until the desired reduction ratio is obtained or until all blocks have c experiments in them, so that no greater reduction ratio is possible, given the constraints.

Very similar reasoning holds if we are given K, N, τ, $z = 1$, and $k_i \leqslant c_i$ and wish to minimize the reduction ratio. This was the case in the example of Section 4, where $N = 5$, $K = 8$, $\tau = 1$, and $c_i = 2$, $i = 1,..., 5$. Using the above reasoning, experiments are first placed in blocks 1, 3, and 5. Since the maximum per block is 2, set $k_1 = k_3 = k_5 = 2$ as this does not violate $K = 8$. Two experiments remain to be placed, and following the above discussion set $k_2 = k_4 = 1$. Thus $\mathbf{k}_5 = (2, 1, 2, 1, 2)$. This gives the optimum reduction ratio subject to the constraints.

7. Asymptotic Results

In practice, it is frequently desired to use a search procedure in which N is not specified initially. In this case, the asymptotic results are required. Once these are known, the asymptotic search procedure can be specified and used instead of the procedure requiring the value of N initially. Thus golden section search is often used instead of Fibonacci search, and golden block search instead of block search [8]. The asymptotic results for the present case will be derived under the assumption that $k_n = k$ for all n and $\tau = 0$, 1 and 2.

The method of solving linear recursion relationships with constant coefficients is sufficiently well known (for example, see Jeske [4]) that it will be omitted here.

The general solution for L_n is:

$$(7.1) \quad L_n = \sum_{i=1}^{\tau+1} d_i a_i^n,$$

where the a_i are the $\tau + 1$ roots of the equation,

$$(7.2) \quad a^{\tau+1} - a^\tau - k = 0,$$

and the d_i are determined by the initial conditions,

$$L_n = nk + 1, \qquad n = 0, \ldots, \tau.$$

The asymptotic solution is

$$(7.3) \quad L_n^a = a_1^n,$$

where the roots of (7.3) are ordered so that a_1 is the largest real root.

This result may be used for the asymptotic search strategy in the usual manner [6]. If the initial interval is [0, 1], then the asymptotic search strategy is:

$$(7.4) \quad x_{in}^a = l'_{n-1} + i a_1^{-(n+\tau)}, \qquad i = 1, \ldots, k.$$

The interval of uncertainty after n blocks is a_1^{-n}.

If N had been known initially, the length of the final interval of uncertainty, using (3.3), would be $(L_N)^{-1}$. The ratio of the final interval of uncertainty if N is known initially, to that after N experiments when N is unknown initially, is $a_1{}^N/L_N$. Since

$$\lim_{N\to\infty} L_N = d_i a_1{}^N,$$

then

$$\lim_{N\to\infty} a_1{}^N/L_N = d_1^{-1}.$$

This provides a measure of comparison between the effectiveness of the two search techniques.

For $\tau = 0$, $\quad a_1 = k + 1$, $\quad d_1 = 1$.

For $\tau = 1$,

$$a_1 = (1 + \sqrt{1 + 4k})/2,$$

$$a_2 = (1 - \sqrt{1 + 4k})/2.$$

Using the initial conditions $L_0 = 1$, $L_1 = k + 1$,

$$d_1 = \frac{1}{2} + \frac{2k + 1}{2\sqrt{1 + 4k}},$$

$$d_2 = \frac{1}{2} - \frac{2k + 1}{2\sqrt{1 + 4k}}.$$

For $\tau > 1$, (7.2) is difficult to solve in general. Specific results for each value of k and τ can be calculated by solving (7.2). For example, for $\tau = 2$, these results are $a_1 = 1.466, 1.696, 1.864$, and 2.0 for $k = 1, 2, 3$, and 4, respectively. To obtain d_1, the other roots of (7.2) are required.

8. Summary

The sequential minimax search strategy to optimize one-dimensional unimodal functions has been derived using blocks containing any number of experiments evaluating derivatives.

The search is applicable for any time delay that may be present in the system. The optimal arrangement of experiments in the blocks is specified. The results have been extended to asymptotic searches in which N need not be known initially. The search strategy specified is also minimax for locating the zero of a monotone function. The appendix extends the work to an application for use in some non-linear programming algorithms.

APPENDIX

The following problem is very similar to the one described in the main part of the paper, although the physical situation is quite distinct. The problem to be described is found in many non-linear programming algorithms (for example, see [7]).

Let $\mathbf{x} \in R^N$. Let there be T inequality constraints,

$$\text{(A1)} \qquad \begin{array}{l} c_1(\mathbf{x}) \leqslant 0 \\ \quad \vdots \\ c_T(\mathbf{x}) \leqslant 0. \end{array}$$

$$F = \{\mathbf{x} : c_i(x) \leqslant 0, \qquad i = 1,..., T\}.$$
$$I = \{\mathbf{x} : c_i(x) > 0, \qquad \text{for any } i = 1,..., T\}.$$

Let $\mathbf{x}^1 \in F$ and $\mathbf{x}^2 \in I$. Let $S = \{\mathbf{x} : \mathbf{x} = \lambda \mathbf{x}^1 + (1 - \lambda)\, \mathbf{x}^2, 0 \leqslant \lambda \leqslant 1\}$. It is desired to determine the point in $S \cap F$ that is farthest from $\mathbf{x}^1$. This shall be called the *boundary point*, $\mathbf{x}_B$:

$$\mathbf{x}_B = \{\mathbf{x}' : \mathbf{x}' = \max_{\mathbf{x} \in F \cap S} | \mathbf{x} - \mathbf{x}^1 |\}.$$

The search problem is to specify a minimax strategy to determine an interval in which $\mathbf{x}_B$ must be contained. It will be shown that the time delay search, with slight modification, is applicable.

It will be assumed that the constraints are well behaved, i.e., that if $\mathbf{x}^0 \in F \cap S$ then $\mathbf{x} \in F$ for $\mathbf{x}$ such that $\mathbf{x} \in S$ and such that $| \mathbf{x}^1 - \mathbf{x} | \leqslant | \mathbf{x}^1 - \mathbf{x}^0 |$.

Without loss of generality, assume $\mathbf{x}^1 = \mathbf{0}$ and $\mathbf{x}^2 = (l_n, 0,..., 0)$. Each point in S may be specified by its first component and this abbreviation will be used (i.e., denote $\mathbf{x}^1$ by 0, $\mathbf{x}^2$ by l_n).

The search schemes considered are those that satisfy the following rules :

(A-2)

(1) Evaluate $c_1(x_1)$ initially.

(2) If $c_t(x_n) > 0$, $t = 1,..., T$; $n = 1,..., N$, evaluate $c_t(x_{n+1})$.

(3) If $c_t(x_n) \leqslant 0$, $t = 1,..., T - 1$; $n = 1,..., N$; evaluate $c_{t+1}(x_n)$. If $t = T$, evaluate $c_1(x_{n+T})$ unless there exists a previous measurement such that $c_g(x_m) \leqslant 0$ for $x_{n+T} \leqslant x_m$, where $g = 1,..., T - 1$. In this case, evaluate $c_{f+1}(x_{n+T})$ where $f = \max\{g : c_g(x_m) \leqslant 0\}$.

The optimization problem is to determine the x_n, $n = 1,\ldots, N$ so that a minimax strategy is obtained. The minimax strategy is:

(A-3) $x_n = l_n' + L_{N-\tau-n}$,

where $T = \tau + 1$ and $L_{N-\tau-n}$ is determined by (3.1) and (3.2) with $k_n = 1$, $n = 1,\ldots, N$. Also,

$$l_n' = \max_{x_m \in F \cap S, m=1,\ldots,n} (x_m, 0).$$

An outline of the proof strategy will be given. The proof is very similar to the technique used in [2]. Suppose N tests are to be made. Assume that the maximum interval that can be searched with n experiments to give a unit final interval of uncertainty is of length λ_n. If $c_1(x_1) > 0$, then $n - 1$ experiments remain to search $[0, x_1]$. Thus, it is required that:

(A-4) $x_1 \leqslant \lambda_{n-1}$.

Similarly, if $c_t(x_1) \leqslant 0$, $t = 1,\ldots, T$, then it is required that

(A-5) $\lambda_n - x_1 \leqslant \lambda_{n-T}$.

Combining (A-4) and (A-5), we obtain

(A-6) $\lambda_n \leqslant \lambda_{n-T} + \lambda_{n-1}$.

It can be shown that this constraint is required for all n such that $N \geqslant n \geqslant T$. This gives an upper bound on the length of interval that may be feasibly searched with N experiments, $\lambda_N \leqslant \lambda_{N-1} + \lambda_{N-T}$. In addition, the initial conditions are easily deduced to be

(A-7) $\lambda_n = 1, \quad n = 0,\ldots, T - 1.$

Since (A-6) gives an upper bound on the interval that may be feasibly searched, if it can be shown that a feasible strategy exists where (A-6) is satisfied as an equality for $n \geqslant T$ and (A-7) is satisfied, then that strategy will be the minimax strategy.

Note that if $T = \tau + 1$ and $k_n = 1$, $n = 1,\ldots, N$, then (A-6), if satisfied as an equality, is the same as (3.2). Thus the minimax strategy can be related to the minimax strategy for the time delay case. Using the initial conditions (A-7), $T = \tau + 1$, $k_n = 1$ for $n = 1,\ldots, N$, and (A-6) as an equality, then $\lambda_n = L_{n-\tau}$.

The minimax search strategy is given by evaluating the constraints as indicated by (A-2), with

$$\begin{aligned} x_n &= l'_{n-1} + \lambda_{N-n}, \qquad n = 1,\ldots, N \\ &= l'_{n-1} + L_{N-\tau-n}. \end{aligned}$$

This strategy can be shown to be feasible by the usual inductive proof method, which for brevity will be omitted here.

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