The Characters of the Wreath Product Group Acting on the Homology Groups of the Dowling Lattices

PHIL HANLON

Department of Mathematics, California Institute of Technology, Pasadena, California 91125

Received October 19, 1982

Let G be a finite group, n a positive integer, $Q_n(G)$ the Dowling lattice of rank n based on G and $W_n$ the wreath product group $G \wr S_n$. It is easily seen that $W_n$ acts as a group of automorphisms of $Q_n(G)$. This action lifts to a representation of $W_n$ on each homology group of $Q_n(G)$. The character values of these representations are computed. Let $\sigma$ be an element of $W_n$. Consider $\sigma$ as an $n \times n$ permutation matrix $\delta$ whose nonzero entries have been replaced by elements of G. If $C$ is a cycle of $\delta$, the weight of $C$ is the product of the elements of G which lie in the cycle C. The type of $C$ is the conjugacy class of G containing the weight of C. Let $c_{l,u}$ denote the number of $l$-cycles of $\sigma$ of type $u$. The conjugacy class of $\sigma$ in $W_n$ depends only on the numbers $c_{l,u}$. For each $i = 0, 1, \ldots, n - 1$ let $(Q_n(G))_i$ be the geometric lattice obtained from $Q_n(G)$ by deleting ranks $i + 1$ through $n - 1$ (so $(Q_n(G))_{n-1} = Q_n(G)$). Let $\beta_i$ denote the character of the representation of $W_n$ on the unique non-vanishing reduced homology group of $(Q_n(G))_i$. For each $\sigma \in W_n$ let $B_{\sigma}(\lambda)$ be the polynomial $B_{\sigma}(\lambda) = \sum_{i=0}^{n-1} \beta_i(\sigma) \lambda^{n-1-i}$. It is shown that

$$B_{\sigma}(\lambda) = (-1)^n \prod_{l,u} c_{l,u}^{-1} \left( \frac{1}{\lambda + 1} \left( F(l, u, 1) - F(l, u, -\lambda) \right) \right) \quad (*)$$

where $F(l, u, \lambda) = (-1/|G|) \sum_{t \in G} b(t, u) \mu(t) \lambda^{l/t}$ and where $b(t, u)$ is the number of solutions $h \in G$ to $h^t = u$. The formula (*) allows for the explicit computation of the characters $\beta_i$. Using this information, several facts about the characters are deduced. For example, it is shown that the trivial character appears exactly once in each $\beta_i$ and it is shown that $\beta_{n-1}$ can be realized in a simple way as a sum of induced characters. © 1984 Academic Press, Inc.

INTRODUCTION

This paper requires little background knowledge other than a slight familiarity with characters of finite groups and homology groups of posets. For more information about characters of wreath products, the reader is encouraged to consult James and Kerber [8].

The paper is divided into three sections. In Section 1 we define the Dowling lattices $Q_n(G)$ and prove some technical results which will be of
later use. In Section 2 we carefully study the structure of the sublattices of \( Q_n(G) \) fixed by an arbitrary element of the wreath product \( G \wr S_n \). This section works towards an exact determination of the fixed point characteristic polynomial for any element of \( G \wr S_n \). In the last section, the Lefschetz fixed-point theorem is used to convert the information from Section 2 into information about character values. The character values are applied to the study of the corresponding representations.

1. The Dowling Lattices

Let \( n \) be a positive integer and \( G \) a finite group. We will consider a lattice \( Q_n(G) \), the Dowling lattice of rank \( n \) associated with \( G \), which is constructed from the partition lattice \( \Pi_n \) and the group \( G \). There are several descriptions of these lattices in the literature (see Doubilet, Rota and Stanley [3], Dowling [4], and Zaslavsky [10]). Of these, the description due to Zaslavsky is the most convenient for us to use and we begin here with that description.

**Definition 1.1.** A \( G \)-partite graph \( \gamma \) with vertex set \( T \) consists of a collection \( \{\gamma(g) : g \in G\} \) of disjoint subsets of \( T \times T \) satisfying:

- (a) For all \( i \in T \), \( (i, i) \in \gamma(e) \),
- (b) if \( (i, j) \in \gamma(g) \) and \( (j, k) \in \gamma(h) \) then \( (i, k) \in \gamma(gh) \),
- (c) if \( (i, j) \in \gamma(g) \) then \( (j, i) \in \gamma(g^{-1}) \).

We call the pairs in \( \gamma(g) \) the \( g \)-edges of \( \gamma \). A pair is an edge of \( \gamma \) if it is a \( g \)-edge for some \( g \). Two points \( x \) and \( y \) of are connected (or lie in the same connected component) if \( (x, y) \) is an edge of \( \gamma \). If the vertex set \( T \) is understood then \( \gamma_{G} \) denotes the \( G \)-partite graph with \( \gamma_{G}(e) = \{(i, i) : i \in T\} \) and \( \gamma_{G}(g) = \emptyset \) for \( g \) not equal to \( e \).

A \( G \)-partite graph \( \gamma_1 \) is a subgraph of \( \gamma_2 \) if \( \gamma_1(g) \subseteq \gamma_2(g) \) for all \( g \in G \). This implies that the vertex set of \( \gamma_1 \) is contained in the vertex set of \( \gamma_2 \) though equality need not hold.

The Dowling lattice \( Q_n(G) \) consists of all pairs \( \langle S, \gamma \rangle \), where \( S \) is a subset of \( n \) and where \( \gamma \) is a \( G \)-partite graph with vertex set \( n - S \). We say \( \langle S, \gamma \rangle \preceq \langle T, \rho \rangle \) if the following two conditions hold:

1. \( S \subseteq T \),
2. If \( C \) is a connected component of \( \gamma \) then either \( C \) is contained in \( T \) or \( C \) is a subgraph of \( \rho \).

As an example let \( n = 2 \) and let \( G \) be the two element group \( G = \{\pm 1\} \). Then a connected \( G \)-partite graph is exactly a complete bipartite graph with
(-1)-edges joining points in distinct parts and (+1)-edges joining points in the same part. A general G-partite graph is a disjoint union of complete bipartite graphs and we visualize them in this way drawing dotted lines to indicate the (-1)-edges. For convenience we do not indicate the (+1)-edges of the form \((i, i)\). With these conventions \(Q_2(G)\) appears below:

\[
\begin{array}{c}
\langle\{1,2\},\lambda\rangle \\
\langle\{1\},\lambda\rangle & \langle\{2\},\lambda\rangle & \langle\phi,\lambda\rangle \\
\langle\phi,-1\rangle & \langle\phi,-2\rangle \\
\langle\phi,\lambda\rangle
\end{array}
\]

The main structural theorems concerning these lattices are due to Dowling. We will need the following result.

**Theorem 1.1.** (Dowling [4]). Let \(G\) be a finite group and let \(n\) be a positive integer. Then \(Q_n(G)\) is a geometric, supersolvable lattice of rank \(n\) with Birkhoff polynomial given by

\[
\chi(\lambda) = \prod_{i=0}^{n-1} (\lambda - 1 - |G| i).
\]

Let \(\gamma = \{\gamma(g) : g \in G\}\) be an arbitrary collection of disjoint subsets of \(T \times T\). We form the closure \(\bar{\gamma} = \{\bar{\gamma}(g) : g \in G\}\) of \(\gamma\) with respect to conditions (a), (b) and (c) of Definition 1.1 in the obvious way. Then \(\bar{\gamma}\) is a \(G\)-partite graph exactly when the sets \(\bar{\gamma}(g)\) are disjoint. If this is the case we say that \(\gamma\) is consistent. We will need the following simple test for consistency which is given in terms of directed paths and cycles of \(\gamma\). A directed \(v_0, v_{s+1}\) path \(p\) in \(\gamma\) is a sequence of edges \((v_0, v_1) \in \gamma(g_0), (v_1, v_2) \in \gamma(g_1), \ldots, (v_s, v_{s+1}) \in \gamma(g_s)\) with \(v_0, \ldots, v_s\) distinct. The weight of \(p\) is the group element \(g_0 g_1 \cdots g_s\). A directed cycle \(\delta\) is a directed \(v_0, v_{s+1}\) path where \(v_0 = v_{s+1}\).

**Lemma 1.1.** Let \(\gamma = \{\gamma(g) : g \in G\}\) be a collection of disjoint subsets of \(T \times T\) having the property that \((i, j) \in \gamma(g)\) if and only if \((j, i) \in \gamma(g^{-1})\). If every cycle of \(\gamma\) has weight \(e\) then \(\gamma\) is consistent.

**Proof.** We first claim that if an edge \((i, j)\) is added to \(\gamma(g)\) when forming \(\bar{\gamma}(g)\) then there is an \(i, j\) path in \(\gamma\) of weight \(g\). To see this, proceed inductively on the number of edges added to \(\gamma\) prior to \((i, j)\). Suppose this new edge \((i, j)\) is added to \(\gamma(g)\) because for some \(k\) we have that \((i, k)\) is a \(gh^{-1}\)-
edge and \((k, j)\) is an \(h\)-edge. By induction there is an \(i, k\) path \(p\) of weight \(gh^{-1}\) in \(\gamma\) and a \(k, j\) path \(q\) of weight \(h\) in \(\gamma\). Putting these paths together (they may cross) gives us a figure of the form

![Diagram of a figure with paths](image)

As every cycle of \(\gamma\) has weight \(e\) we see that the weight of the \(p\)-path from \(v_3\) to \(k\) is the inverse of the weight of the \(q\)-path from \(v_3\) to \(k\). Similarly for the \(p\) and \(q\) paths from \(v_1\) to \(v_2\) and \(v_2\) to \(v_3\). Thus the \(i, j\) path in \(\gamma\) obtained by following \(p\) to \(v_1\) then \(q\) to \(j\) has weight \(g\) as was to be shown.

Next suppose that \((i, j)\) is added to \(\gamma(g)\) by virtue of the fact that \((j, i)\) is a \(g^{-1}\)-edge. Again by our induction hypothesis there is a \(j, i\) path in \(\gamma\) of weight \(g^{-1}\). Since \((s, t) \in \gamma(g)\) if and only if \((t, s) \in \gamma(g^{-1})\) we can invert this path to get an \(i, j\) path in \(\gamma\) of weight \(g\). This proves the claim.

Now to prove the lemma assume that \(\gamma\) is inconsistent. So we have \((i, j) \in \gamma(g) \cap \gamma(h)\) for \(g \neq h\). By the above claim there is an \(i, j\) path \(p\) in \(\gamma\) of weight \(g\) and a \(j, i\) path \(q\) of weight \(h^{-1}\). Following \(p\) from \(i\) to \(j\) and \(q\) from \(j\) to \(i\) gives a union of cycles in \(\gamma\) the product of whose weights is not the identity. This is a contradiction.

It should be noted the Lemma 1.1 is false if we drop the condition that \((i, j) \in \gamma(g)\) if and only if \((j, i) \in \gamma(g^{-1})\). As a simple counterexample, let \(G = \{\pm 1\}\), let \(T = \{a, b, c\}\) and let \(\gamma\) be the set of edges pictured below:

![Diagram of a counterexample](image)

This \(\gamma\) has no cycles so certainly the conditions of Lemma 1.1 are satisfied. But \(\gamma\) is inconsistent.

2. The Fixed Point Dowling Lattices

Throughout this section \(n\) is a positive integer and \(G\) is a finite group. We let \(W_n\) denote the wreath product of \(G\) over \(S_n\). The elements of \(W_n\) can be taken to be all \(n \times n\) permutation matrices whose nonzero elements have
been replaced by elements of $G$. For $\sigma \in W_n$, let $\delta$ denote the underlying permutation matrix and for $i \in \mathfrak{n}$ let $\text{sgn}(\sigma, i)$ denote the element of $G$ which appears in the $i, \delta(i)$ entry of $\sigma$. As an example of this notation, let $G = \{\pm 1\}, n = 3$ and

$$\sigma = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

Then we have

$$\delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we have $\text{sgn}(\sigma, 1) = \text{sgn}(\sigma, 3) = -1$, whereas $\text{sgn}(\sigma, 2) = 1$.

The conjugacy classes of $W_n$ are easily described in terms of what we call $G$-partitions of $n$.

**DEFINITION 2.1.** Let $U$ denote the set of conjugacy classes of $G$. A $G$-partition of $n$ is a set $\{\lambda(u): u \in U\}$ where each $\lambda(u)$ is a partition of a non-negative integer $a_u$ with $\sum a_u = n$.

For example, let $G = \{\pm 1\}$ and let $n = 3$. There are two conjugacy classes $u_1$ and $u_{-1}$ of $G$. We write a $G$-partition as an ordered pair of partitions with $\lambda(u_{-1})$ in the first coordinate and $\lambda(u_1)$ in the second coordinate. With this notation, the $G$-partitions of 3 are $(1^3, \emptyset)$, $(1^2, \emptyset)$, $(3, \emptyset)$, $(1^2, 1)$, $(2, 1)$, $(1, 1^2)$, $(1, 2)$, $(\emptyset, 1^3)$, $(\emptyset, 12)$, $(\emptyset, 3)$.

Let $\sigma$ be an element of $W_n$. Then $\delta$ can be expressed as a product of disjoint cycles. If $C = (c_1, c_2, \ldots, c_l)$ is such a cycle then we say $C$ is a $g$-type $l$-cycle of $\sigma$ ($g \in G$) if

$$g = \prod_{i=1}^{l} \text{sgn}(\sigma, c_i).$$

When the group $G$ is nonabelian it is important to take the product (*') in the same order as the cycle $C$. If $g$ is in the conjugacy class $u \in U$ we say that $C$ is a $u$-conjugate cycle of $\sigma$. For $u \in U$, $l \leq n$ and $\sigma \in W_n$ define $m(u, l, \sigma)$ to be the number of $u$-conjugate $l$-cycles of $\sigma$.

Given $\sigma \in W_n$ we define a $G$-partition of $n, A_\sigma$, by $A_\sigma = \{\lambda(u): u \in U\}$ where

$$\lambda(u) = 1^{m(u, 1, \sigma)} 2^{m(u, 2, \sigma)} \cdots n^{m(u, n, \sigma)}.$$  

It can be shown that two elements $\sigma, \eta \in W_n$ are conjugate if and only if $A_\sigma = A_\eta$. Thus the conjugacy classes of $W_n$ correspond exactly to the $G$-partitions of $n$. 


When choosing a representative from a conjugacy class it will often be convenient to assume that each cycle is of the form

\[
\begin{pmatrix}
0 & e & 0 & \ldots & 0 \\
0 & 0 & e & \ldots & 0 \\
\vdots \\
0 & 0 & 0 & \ldots & e \\
g & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

where \( g \) is the type of the cycle. If every cycle of \( \sigma \) has this form we say \( \sigma \) is standard.

The group \( W_n \) acts as a group of automorphisms of \( Q_n(G) \). To describe this action we first define an action of \( W_n \) on the set of \( G \)-partite graphs. Let \( \gamma \) be a \( G \)-partite graph with vertex set \( T \) contained in \( \mathbb{N} \). Define \( \sigma(\gamma) \) to be the \( G \)-partite graph \( \{ \gamma_\sigma(g) : g \in G \} \) where the sets \( \gamma_\sigma(g) \) are formed according to the rule

\[
(i, j) \in \gamma(g) \text{ if and only if } (\sigma(i), \sigma(j)) \in \gamma_\sigma(h),
\]

where \( h = (\text{sgn}(\sigma, i))^{-1} g(\text{sgn}(\sigma, j)) \).

Note that the vertex set of \( \sigma(\gamma) \) is \( \sigma(T) = \{ \sigma(i) : i \in T \} \). It must be checked that \( \sigma(\gamma) \) is a \( G \)-partite graph, i.e., that \( \sigma(\gamma) \) satisfies conditions (b) and (c) of Definition 4.1. To see that condition (b) holds suppose that \( (\sigma(i), \sigma(j)) \in \gamma_\sigma(g) \) and \( (\sigma(j), \sigma(k)) \in \gamma_\sigma(h) \). Then \( (i, j) \in \gamma(g_1) \) and \( (j, k) \in \gamma(h_1) \) where

\[
g_1 = (\text{sgn}(\sigma, i)) g(\text{sgn}(\sigma, j))^{-1}
\quad \text{and}
\]

\[
h_1 = (\text{sgn}(\sigma, j)) h(\text{sgn}(\sigma, k))^{-1}.
\]

(2.1)

By transitivity of \( \gamma \) we have \( (i, j) \in \gamma(g_1, h_1) \) and so \( (\sigma(i), \sigma(j)) \in \gamma_\sigma(d) \) where

\[
d = (\text{sgn}(\sigma, i))^{-1} g_1 h_1(\text{sgn}(\sigma, k)).
\]

Using Eqs. (2.1) it is easy to see that \( d = gh \) as was to be shown. The verification of condition (c) is equally straightforward and is left to the reader.

Now we describe an action of \( W_n \) on \( Q_n(G) \). Let \( \langle S, \gamma \rangle \) be in \( Q_n(G) \) and let \( \sigma \) be in \( W_n \). Define \( \sigma(\langle S, \gamma \rangle) \) to be \( \langle \delta S, \sigma(\gamma) \rangle \). It is easy to check that \( W_n \) acts in this way as a group of lattice automorphisms of \( Q_n(G) \). For each \( \sigma \in W \) we let \( (Q_n(G))_\sigma \) denote the sublattice of \( Q_n(G) \) consisting of all \( x \in Q_n(G) \) with \( \sigma x = x \).

**Definition 2.2.** Let \( \sigma \in W_n \) and let \( \rho \) be the rank function of \( Q_n(G) \). The **Birkhoff polynomial of** \( (Q_n(G))_\sigma \) **with respect to** \( \rho \) is the polynomial \( \chi_\sigma(\lambda) \) defined by

\[
\chi_\sigma(\lambda) = \sum_x \mu_\sigma(0, x) \lambda^{n-\rho(x)}
\]
where the sum is over all $x$ in $(Q_n(G))_\sigma$ and where $\mu_\sigma$ denotes the Möbius function of $(Q_n(G))_\sigma$.

Note that if $\sigma$ is the identity of $W_n$ then $\chi_\sigma(\lambda)$ is the usual Birkhoff polynomial of $Q_n(G)$ which is given in Theorem 1.1. The rest of this section is devoted to generalizing Theorem 1.1 by computing $\chi_\sigma(\lambda)$ for arbitrary $\sigma \in W_n$. Unfortunately the lattices $(Q_n(G))_\sigma$ fail to be either supersolvable or geometric in general. For example, with $G = \{ \pm 1 \}$, $n = 4$ and

$$\sigma = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.$$ 

We see that $(Q_n(G))_\sigma$ (pictured below) is not even ranked.

Despite the apparent difficulties we will find that the polynomials $\chi_\sigma(\lambda)$ have a simple description in terms of $\sigma$. To derive $\chi_\sigma(\lambda)$ we must closely examine the action of $W_n$ on $Q_n(G)$. This examination proceeds in stages. First we classify the atoms of $(Q_n(G))_\sigma$ and this classification allows us to reduce to the case where $\sigma$ consists of cycles all of the same length and type. We next consider the case where $\sigma$ has exactly one cycle and use that case to settle the case where $\sigma$ has several cycles all of the same length and type:

The Lattice $(Q_n(G))_\sigma$
Let $\sigma$ be in $W_n$ and let $F = \langle u, \gamma \rangle$ be in $(Q_n(G))_n$. If $C = (c_1, \ldots, c_t)$ is a cycle of $\sigma$ with the property that an edge of $\gamma$ joins two points in $C$ then we say $C$ is diagonalized by $F$. In this case there is a unique minimal $t > 0$ such that $(c_1, c_{t+1})$ is an edge of $\gamma$. Clearly $t$ divides $l$. We call $l/t$ the period of $C$ with respect to $F$.

Assume $C$ is a $g$-type $l$-cycle of $\sigma$, that $h$ is in $G$ and that $t > 0$ is a divisor of $l$. Define sets $\gamma(d), d \in G$ as follows:

1. $\gamma(h) = \{(c_i, c_{i+t}): 1 \leq i \leq l - t\}$,
2. $\gamma(g^{-1}h) = \{(c_{i-t}, c_i): 1 \leq i \leq l\}$,
3. $\gamma(h^{-1}) = \{(c_{i+t}, c_i): 1 \leq i \leq l - t\}$,
4. $\gamma(h^{-1}g) = \{(c_{i-t}, c_{i+t}): 1 \leq i \leq l\}$,
5. $\gamma(d) = \emptyset$ for $d \neq h, g^{-1}h, h^{-1}, h^{-1}g$.

Let $\tilde{\Delta}(C, l/t, h) = \{\gamma(d): d \in G\}$ and let $\Delta(C, l/t, h)$ be the closure of $\tilde{\Delta}(C, l/t, h)$.

**Lemma 2.1.** Let $C$ be a $g$-type $l$-cycle of $\sigma$, let $t$ be a divisor of $l$ and let $h$ be an element of $G$. A necessary and sufficient condition for $\Delta(C, l/t, h)$ to be a $G$-partite graph is that $h' = g$ in $G$.

**Proof.** Suppose $\Delta(C, l/t, h) = \{\delta(d): d \in G\}$ is a $G$-partite graph. It contains the following edges:

$$(c_1, c_{l/t+1}) \in \delta(h)$$
$$(c_{l/t+1}, c_{2l/t+1}) \in \delta(h)$$

$$\vdots$$
$$(c_{(t-2)l/t+1}, c_{(t-1)l/t+1}) \in \delta(h)$$
$$(c_{(t-1)l/t+1}, c_{1}) \in \delta(g^{-1}h).$$

By transitivity $(c_1, c_1) \in \delta(h^{-1}g^{-1}h)$. But also $(c_1, c_1) \in \delta(e)$ so $h^{-1}g^{-1}h = e$ or equivalently $h'^{-1} = h^{-1}g$. Thus $h' = g$.

Conversely suppose $h' = g$. Consider the directed cycles of $\tilde{\Delta}(C, l/t, h)$. Each has one of three forms:

(a) The cycle has length 2 and consists of edges $(i, j) \in \gamma(d)$ and $(j, i) \in \gamma(d^{-1})$ for $d = h, g^{-1}h, h^{-1}$ or $h^{-1}g$.

(b) The cycle has length $t$ and consists of edges $(c_{i+sl/t}, c_{i+(s+1)l/t})$ $s = 0, 1, \ldots, t-1$. Here all edges but one have weight $h$ and the exceptional edge has weight $g^{-1}h$.

(c) The inverse of a type (b) cycle.
It is clear that every two-cycle from (a) has weight $e$ and the relation $h' = g$ insures that every cycle from (b) and (c) has weight $e$. By Lemma 1.1 we have that $\Delta(C, l/t, h)$ is a $G$-partite graph.

The next lemma is merely an observation but it is worth writing down for future reference.

**Lemma 2.2.** Let $\gamma$ be a $G$-partite graph and let $\sigma$ be an element of $W_n$ which satisfies $\sigma(\gamma) = \gamma$. Suppose $C$ is an $l$-cycle of $\sigma$ which is diagonalized by $\gamma$. Let $t$ be a divisor of $l$ and suppose $(c_1, c_{i+1}) \in \gamma(h)$. Then $\Delta(C, l/t, h)$ is a $G$-partite graph fixed by $\sigma$.

At this point we are ready to describe completely the atoms of $(Q_n(G))_\sigma$. For convenience we let $L_\sigma$ denote $(Q_n(G))_\sigma$. We proceed by constructing three obvious types of atoms and then we show that every atom is one of these three types.

**Construction 1.** Let $C$ be a $g$-type $l$-cycle of $\sigma$. Suppose that for every prime $p$ dividing $l$ there are no solutions to $d^p = g$ in $G$. Then $\langle C, \gamma, \sigma \rangle$ is an atom in $L_\sigma$.

To see this suppose $\langle S, \gamma \rangle$ is a nonzero element of $L_\sigma$ which is less than $\langle C, \gamma, \sigma \rangle$. Then $S \subseteq C$ but $C$ is a cycle of $\sigma$ so we must have $S = \emptyset$. Thus $\gamma$ is a proper diagonalization of $C$ and this contradicts Lemma 2.2. These are called class 1 atoms.

**Construction 2.** Let $C$ be a $g$-type $l$-cycle of $\sigma$ let $p$ be a prime divisor of $l$ and let $d$ be a solution to $d^p = g$. Then $\langle \emptyset, \Delta(C, p, d) \rangle$ is an atom in $L_\sigma$.

To see this note that every connected component of $\Delta(C, p, d)$ consists of $p$ points permuted cyclically by $\sigma^{l/p}$. Also the connected components of $\Delta(C, p, d)$ are permuted cyclically by $\sigma$. Clearly a nontrivial subgraph of $\Delta(C, p, d)$ is impossible. These are called class 2 atoms.

**Construction 3.** Let $C = (c_1, \ldots, c_l)$ be a $g$-type $l$-cycle of $\sigma$ and let $D = (d_1, \ldots, d_l)$ be an $f$-type $l$-cycle of $\sigma$ where $g$ and $f$ are conjugate in $G$. Let $h$ be a solution to $h^{-1} gh = f$. Let $m \in l$. Define $\gamma$ according to

$$\gamma(h) = \{(c_i, d_{m+i}): 1 \leq i \leq l-m\}$$

$$\gamma(hf) = \{(c_i, d_{m+i-1}): l-m+1 \leq i \leq l\}.$$

Let $\overline{\gamma}$ be the closure of $\gamma$. Then $\langle \emptyset, \overline{\gamma} \rangle$ is an atom of $L_\sigma$.
Note that the image of the $hf$-edge $(c_1, d_m)$ under $\sigma$ is the $g^{-1}hf$-edge $(c_1, d_{m+1})$. But $(c_1, d_{m+1}) \in \gamma(h)$ so we must have $g^{-1}hf = h$ or $f = h^{-1}gh$.

It is easy to see that this is an atom of $L_\sigma$. Transitivity contributes no edges to $\gamma$ so in fact we have

$$\gamma(h) = \gamma(h), \gamma(hf) = \gamma(hf), \gamma(e) = \{(i, i): i \in \mathbb{N}\} \text{ and}$$

$$\gamma(h^{-1}) = \{(d_{m+1}, c_i): 1 \leq i \leq l - m\}$$

$$\gamma(f^{-1}h^{-1}) = \{(d_{m+1}, c_i): l - m + 1 \leq i \leq l\}.$$

So if $\eta$ is a nonzero subgraph of $\gamma$ then $\eta$ must contain some edge of $\gamma(h) \cup \gamma(g^{-1}h)$. But $\sigma$ permutes the edges in that union cyclically so $\eta$ must contain all of $\gamma(h) \cup \gamma(g^{-1}h)$. Thus $\eta = \gamma$ which shows $\gamma$ is an atom. These are called class 3 atoms.

**Theorem 2.1.** Let $F$ be an atom of $L_\sigma$. Then $F$ is one of the three types constructed above.

**Proof.** Let $F = \langle S, \gamma \rangle$. Assume first that $S$ is nonempty so $S$ contains a cycle $C$ of $\sigma$. Clearly $\langle C, \gamma_C \rangle \leq \langle S, \gamma \rangle$ so we must have equality. Suppose $C$ is a $g$-type $l$-cycle. If $h^p = g$ has a solution for some $p$ dividing $l$ then $(\emptyset, A(C, p, h))$ is a atom of $L_\sigma$ strictly less than $F$. Thus $d^p = g$ has no solutions for all primes $p$ dividing $l$ and $F$ is a class 1 atom.

Assume that $S$ is empty and that $F$ diagonalizes a cycle $C$. Then it follows easily from Lemma 2.2 that $F$ is a class 2 atom.

So assume $F$ diagonalizes no cycles of $\sigma$. Let $C = (c_1, \ldots, c_l)$ and $D = (d_1, \ldots, d_n)$ be cycles of $\sigma$ with the property that some edge of $\gamma$ joins a point in $C$ to a point in $D$. It is easy to see that if $(c_i, d_j)$ is an edge of $\gamma$ then so is $(c_{i+t}, d_{j+t})$ and $(c_{i+t}, d_{j+s})$ for every multiple $t$ of $\text{gcd}(l, p)$. So if $(l, p) \neq l$ then by transitivity, two distinct points of $C$ are joined by an edge. But this contradicts the assumption that $F$ diagonalizes no cycles of $\sigma$. So $(l, p) = l$ and similarly $(l, p) = p$. Hence $l = p$.

Let $C$ be a $g$-type cycle and $D$ an $f$-type cycle and suppose $(c_1, d_{1+m}) \in \gamma(h)$. Applying $\sigma$ $l$-times we obtain $(c_1, d_{1+m}) \in \gamma(g^{-1}hf)$ so $h = g^{-1}hf$. Thus
g and f are conjugate and h is a solution to \( h^{-1}gh = f \). Thus \( F \) is a class 3 atom and that completes the proof.

Theorem 2.1 has a very surprising corollary. Loosely put, this corollary says that to compute \( \chi_\sigma(\lambda) \) we may as well assume that all cycles of \( \sigma \) have the same length and same type. We state this formally in the next corollary.

Let \( \sigma \in W_n, l \in \mathfrak{n} \) and \( u \in U \). Let \( n_{l,u} \) denote \( l \cdot m(u, l, \sigma) \) so \( n_{l,u} \) is the total number of elements of \( \mathfrak{n} \) which lie in \( u \)-conjugate \( l \)-cycles. Let \( \sigma_{l,u} \) denote the element of \( G \wr S_{n_{l,u}} \) consisting of the \( u \)-type \( l \)-cycles of \( G \). Let \( L_{l,u} \) denote the fixed point sublattice \( (Q_{n_{l,u}}(G))_{\sigma_{l,u}} \). Let \( J_\sigma \) and \( J_{l,u} \) denote the join sub-semilattices of \( L_\sigma \) and \( L_{l,u} \) (respectively) consisting of all joins of atoms. Lastly let \( \chi_{l,u}(\lambda) \) and \( \mu_{l,u} \) denote the Birkhoff polynomial and M"obius function of \( L_{l,u} \). Let \( r \) denote the rank of \( J_\sigma \).

**Corollary 2.1.** With notation as above:

(a) \( J_\sigma = \bigsqcup_{l,u} J_{l,u} \).
(b) \( \chi_\sigma(\lambda) = \lambda^{n-r} \prod_{l,u} \chi_{l,u}(\lambda) \).
(c) \( \mu_\sigma(0, 1) = \prod_{l,u} \mu_{l,u}(0, 1) \).

**Proof:** Note that each atom of \( L_\sigma \) involves only cycles of the same length and type. Thus each atom of \( J_\sigma \) is in fact an atom of some \( J_{l,u} \). This establishes a bijection from the atoms of \( J_\sigma \) to the atoms of \( \bigsqcup_{l,u} J_{l,u} \). It is easy to check that this bijection lifts to a lattice isomorphism.

Statements (b) and (c) follow from (a) together with the following general fact about lattices. Let \( M \) be a finite lattice and let \( M_1 \) be the join sub-semilattice of \( M \) consisting of 0 together with all joins of atoms of \( M \). Let \( \mu \) be the M"obius function of \( M \) and \( \mu_1 \) the M"obius function of \( M_1 \). Then for all \( x \in M, \mu(0, x) = 0 \) unless \( x \in M_1 \) and then \( \mu(0, x) = \mu_1(0, x) \).

As an example of Theorem 2.1 consider \( G = \{ \pm 1 \}, n = 4 \) and

\[
\sigma = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]

Each cycle \( C \) of \( \sigma \) has type \(-1\) and so it is easily seen that there are no class 2 atoms in \( (Q_n(G))_\sigma \). There are two class 1 atoms and four class 3 atoms.

**Class 1 atoms.**

1. \( \langle \{1, 2\}, \gamma_{\sigma}\rangle \)
2. \( \langle \{3, 4\}, \gamma_{\sigma}\rangle \).
Class 3 atoms. (As these are of the form \( \langle \varnothing, \gamma \rangle \) we only specify \( \gamma \).)

1.

\[
\begin{array}{c}
1. \\
3 & 4 \\
1 & 2 \\
4 & 3 \\
3 & 4 \\
1 & 2 \\
4 & 3 \\
\end{array}
\]

With \( n \) and \( G \) as above, let

\[
\sigma = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

This time we have eight atoms (see figure 1). Of these four are class 2 atoms, and four are class 3 atoms. The class 3 atoms are identical to the class 3 atoms above and the class 2 atoms are
Using Corollary 2.1 we can reduce the general problem of determining $\chi_\sigma(\lambda)$ to the special case where $\sigma$ consists of $\alpha$ $u$-conjugate $l$-cycles. We begin with $\alpha = 1$.

**Lemma 2.3.** Suppose $\sigma$ consists of a single $g$-type $l$-cycle $C$. Then

$$\mu_\sigma(0, 1) = -\frac{1}{|G|} \sum_{h \in G} \sum_{t \mid l} \sum_{x} \mu(t) \chi(h^t) \chi(g)$$

where $\sum_x$ is over the irreducible characters $\chi$ of the group $G$.

**Proof.** Note that $L_\sigma$ consists of $(C, \gamma_{\emptyset})$ together with all diagonalizations of $C$. If $\gamma$ is a diagonalization of $C$ with period $t$ then $[0, \langle \emptyset, \gamma \rangle]$ in $L_\sigma$ is isomorphic to the lattice of divisors of $t$ by Lemma 2.1. Also the number of diagonalizations of $C$ with period $t$ is exactly the number of solutions in $G$ to $h^t = g$. Call this number $b(t, g)$. Putting this together we have

$$\mu_\sigma(0, 1) = -\sum_{t \mid l} \mu(t) b(t, g). \tag{***}$$

By the orthogonality relations, for all $h \in G$ we have

$$\sum_x \chi(h^{-1}) \chi(g) = \begin{cases} |C_\sigma(g)| & \text{if } h^{-1} \text{ and } g \text{ are conjugate} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\sum_{h \in G} \sum_x \chi(h^{-1}) \chi(g) = |G| b(t, g)$.

Combining this with (***$)$ proves the result.

We move on now to general $\alpha \geq 1$. We will use the following result due to Crapo [2].

**Theorem 2.2 (Crapo's Complementation Theorem).** Let $M$ be a finite lattice and let $x \in M$. Let $\mu$ be the Möbius function of $M$. Then

$$\mu(0, 1) = \sum_{c, c'} \mu(0, c) z(c, c') \mu(c', 1)$$

where $\sum_{c, c'}$ is over all complements of $x$ and where $z(c, c')$ is 1 or 0 depending on whether $c$ is or is not less than or equal to $c'$.

We are now ready to completely determine $\mu_\sigma(0, 1)$. This is most elegantly done in terms of the polynomials $F(l, u, \lambda)$ defined below.
DEFINITION 2.3. Let \( l \in \mathbb{N} \) and let \( g \in G \). Define the polynomial \( F(l, g, \lambda) \) by

\[
F(l, g, \lambda) = \frac{-1}{l|C_{\sigma}(g)|} \left( \frac{1}{|G|} \sum_{h \in G} \sum_{\chi} \mu(t) \chi(h') \chi(g) \lambda^{l/t} \right).
\]

where \( \mu(t) \) is the value of the ordinary number theoretic Möbius function and where \( \sum_{\chi} \) is over the irreducible characters \( \chi \) of \( G \). If \( u \in U \), let \( \chi(l, u, \lambda) = \chi(l, g, \lambda) \) for \( g \in u \).

THEOREM 2.3. Let \( l \in \mathbb{N} \) and let \( g \in G \). For each \( \alpha \geq 1 \) let \( m(\alpha) \) be \( \mu(0, 1) \) for \( \sigma \) consisting of \( \alpha \) g-type l-cycles. Let \( M(z) = \sum_{\alpha \geq 1} (m(\alpha) z^{\alpha}/\alpha!) \). Then

a) \( M(z) = (1 + l|C_{\sigma}(g)|) z^{f(l, g, 1)} \),

b) \( (m(\alpha)/\alpha!(l|C_{\sigma}(g)|)^{\alpha}) = \left( f(l, g, 1) \right) \).

Proof. Assume \( \sigma \) consists of \( \alpha \) g-type l-cycles. Fix one such cycle \( C_1 \). We apply Theorem 2.2 to the element \( F = \langle C_1, \gamma_1 \rangle \). Our first step is to analyze the complements of \( F \).

Let \( X = \langle S, \gamma \rangle \) be a complement of \( F \). As \( F \wedge X = 0 \) we have \( S \cap C_1 = \emptyset \). Thus \( S \) is a union of some number \( r \) of the \( \alpha - 1 \) cycles of \( \sigma \) distinct from \( C_1 \). Let \( C_2, \ldots, C_{\alpha - r} \) be the \( \alpha - r - 1 \) cycles different from \( C_1 \) which are not in \( S \).

Note that \( \langle C_1, \gamma_1 \rangle \cup \langle S, \gamma \rangle = \langle S \cup D, \gamma_1 \rangle \), where \( \gamma_1 \) is \( \gamma \) minus the connected components of \( \gamma \) which contain points of \( C \) and where \( D \) is the set of points which lie in a connected component containing a point of \( C \). As \( F \cup X = 1 \), we see that every connected component of \( \gamma \) contains a point of \( C \).

On the other hand, no connected component contains more than one point of \( C \). For if this were the case then \( \gamma \) would diagonalize \( C \) and so \( \langle \emptyset, \delta(C, t, h) \rangle \) would be less than or equal to \( X \wedge F \).

Thus every point of each cycle \( C_2, \ldots, C_{\alpha - r} \), is adjacent to exactly one point of \( C_1 \) in \( \gamma \). The graph \( \gamma \) appears below:
The interval from $X$ to 1 in $L_a$ is isomorphic to the interval from 0 to \langle C_1, \gamma_0 \rangle in L_{C_1}$. This is clear because whatever diagonalizations happen to a cycle $C_i$, $1 \leq i \leq a - r$, in the interval $[X, 1]$ will happen to all other $C_j$, $1 \leq j \leq r$, as well by transitivity. Thus

$$\mu_a(X, 1) = m(1).$$

Next note that the interval from 0 to $X$ is isomorphic to the direct product of the intervals $[0, \langle S, \gamma_0 \rangle]$ and $[0, \langle \emptyset, \gamma \rangle]$. Also the interval $\langle \emptyset, \gamma \rangle$ is isomorphic to the partition lattice $\prod_{a-r}$. This is because any subgraph of $\gamma$ fixed by $\sigma$ must simply separate the cycles, i.e., partition $C_1, ..., C_{a-r}$ into disjoint subsets. Thus

$$\mu_a(0, X) = m(r)((-1)^{a-r-1} ((a - r - 1)!)�.$$

Lastly note that distinct complements $\langle S, \gamma \rangle$ are incomparable so by Theorem 2.2,

$$m(a) = \sum_{r=0}^{a-1} \binom{a-1}{r} m(1) m(r) ((-1)^{a-r-1} ((a - r - 1)!) f(a-r)$$

where $f(a-r)$ is the number of graphs $\gamma$ of the form pictured above.

To construct such a graph pick an element $c_1 \in C_1$. We can choose from each cycle $C_2, ..., C_{a-r}$ any one of $l$ elements to be adjacent to $c_1$. After this choice is made, we must weight the edge from $c_1$ to each of these points by some group element. If $C_i$ is an $f$-type cycle and we want $(c_1, c_j) \in \gamma(h)$ for $x_i \in C_i$ then we need $h^{-1}gh = f$. We know $g$ and $f$ are conjugate so there are exactly $|C_0(g)|$ choices for $h$. In total this gives $l |C_0(g)|^{a-r-1}$ possible $\gamma$. It is easy to check that the closure of every $\gamma$ constructed in this way is a $G$-partite graph (using Lemma 1.1). To summarize

$$m(a) = \sum_{r=0}^{a-1} \binom{a-1}{r} m(1) m(r) ((-1)^{a-r-1} (a - r - 1)!) f(a-r)$$

Multiplying by $z^{a-1}/(a-1)!$ gives

$$\frac{m(a) z^{a-1}}{(a-1)!} = m(1) \sum_{r=0}^{a-1} \frac{m(r) z^r}{r!} (-l |C_0(g)| z)^{a-r-1}.$$

In generating function form this reads

$$M'(z) = \frac{m(1) M(z)}{1 + l |C_0(g)| z}$$
or

\[ \frac{M'(z)}{M(z)} = m(1) \frac{1}{(1 + l \mid C_G(g) \mid z)}. \]

Integrating this from 0 to z and noting that \( M(0) = 1 \) gives

\[ M(z) = (1 + l \mid C_G(g) \mid z)^{m(1)/l \mid C_G(g) \mid}. \]

Applying Lemma 2.3 completes the result.

Part (b) follows from the binomial theorem.

It should be noted that for our purpose, the significance of the factor \( a! \mid IC_G(g) \mid^a \) which appears on the left hand side of part (b) in Theorem 2.3 is that this is the size of the centralizer in \( W_n \) of an element consisting of exactly \( a \) \( g \)-type \( l \)-cycles.

It is clear that \( F(Z, g, \lambda) \) is a class function. For \( u \in U \), define \( F(l, u, \lambda) \) to be \( F(l, g, \lambda) \) for \( g \in u \). For each \( l, u \) let \( x_i(u) \) be an indeterminate.

**Definition 2.4.** For \( \sigma \in W_n \) define the cycle indicator \( Z(\sigma) \) of \( \sigma \) by

\[ Z(\sigma) = \prod_{i, u} (x_i(u))^{m(u, l, \sigma)}. \]

Define the Möbius function cycle index of \( W_n, Z_u(W_n) \) by

\[ Z_u(W_n) = \frac{1}{\mid W_n \mid} \sum_{\sigma \in W_n} \mu_\sigma(0, 1) Z(\sigma) \]

and let \( Z(u) = \sum_{n=0}^{\infty} Z_u(W_n) \).

**Corollary 2.2.** \( Z(u) = \prod_{l, u} (1 + x_i(u))^{F(l, u, 1)}. \)

**Proof.** This follows immediately from Theorem 2.3 (b).

At this point we proceed onto the more general problem of determining \( \chi_\sigma(\lambda) \). As noted earlier we may assume \( \sigma \) consists of cycles of the same length and type thanks to Corollary 2.1.

**Lemma 2.4.** Let \( \sigma \) consist of \( r \) cycles \( C_1, \ldots, C_r \) all of the length \( l \) and of types \( g_1, \ldots, g_r \) all of which are conjugate. Write \( C_1 = (c_1, \ldots, c_l) \). Then for every choice of

(a) a diagonalization \( \Delta(C_1, t, h) \) of \( C_1 \),
(b) \( (x_1, \ldots, x_r) \in C_1 x_1 \cdots x_r C_r \),
(c) \( f_1, \ldots, f_r \in G \) satisfying \( f_1^{-1} g_i f_1 = g_i \),
there exists a unique G-partite graph γ fixed by σ with \((c_1, x_i) \in γ(f_i), i = 2, ..., r\), and with \(\langle \emptyset, γ \rangle \wedge \langle C_1, γ_γ \rangle = \langle \emptyset, A(C_1, t, h) \rangle\) in \(L_γ\). Moreover for such γ the interval \(\langle \emptyset, γ \rangle \) in \(L_γ\) is isomorphic to the fixed-point partition lattice \((\prod_r )_η\) where η is a permutation of \(\{1, 2, ..., rt\}\) consisting of r cycles all of length t.

**Proof.** Suppose you have chosen (a), (b) and (c) and suppose γ satisfies \((c_1, x_i) \in γ(f_i)\) and \(\langle \emptyset, γ \rangle \wedge \langle C_1, γ_γ \rangle = \langle \emptyset, A(C_1, t, h) \rangle\). Fix \(i, 2 \leq i \leq r\), and write \(C_i = (b_1, ..., b_y, x_i, b_{y+2}, ..., b_1)\). As γ is invariant under σ we have that \((c_2, b_{y+2}) \in γ(d_i)\), where \(d_i\) is uniquely determined in terms of \(g_1, g_i, x_i\) and \(f_i\). Similarly \((c_{i+s}, b_{y+1+s}) \in γ(d_s)\), where \(d_s\) is uniquely determined. So we are forced to have in γ the edges from \(C_1\) to \(C_t\) pictured below:

![Diagram](image)

Next we use the fact that γ diagonalizes \(C_1\) with period t. This gives us edges in \(C_1\) between \(c_j\) and \(c_{j+s/t}\) for \(s = 0, 1, ..., t - 1\) and \(j = 1, 2, ..., l\). Moreover these edges are uniquely determined as the edges which appear in \(A(C_1, t, h)\). By transitivity these give us edges from \(b_j\) to \(b_{j+s/t}\) for \(s = 0, 1, ..., t - 1\) and \(j = 1, 2, ..., l\).

This is true for any of the cycles \(C_i, i = 2, ..., r\). Thus we have a uniquely determined subgraph γ' contained in γ consisting of t connected components each containing \(l/t\) elements from every cycle:

![Diagram](image)

If γ contains any edge not in γ' then such an edge must join points in distinct connected components of γ'. By transitivity we have edges between the points in \(C_1\) contained in those distinct components. But these edges are not in \(A(C_1, t, h)\) and this contradicts \(\langle \emptyset, γ \rangle \wedge \langle C_1, γ_γ \rangle = \langle \emptyset, A(C_1, t, h) \rangle\).

So \(γ = γ'\) and this shows γ is unique. It remains to show that γ is a G-partite graph. To do so, let γ₁ be the set of all edges in γ which involve a
point of $C_1$. Clearly $(a, b) \in \gamma_1(d)$ iff $(b, a) \in \gamma_1(d^{-1})$ and clearly $\gamma_1 = \gamma'$. Also the cycles of $\gamma_1$ are of two types:

1. The two-cycles $(a, b) \in \gamma_1(d), (b, a) \in \gamma_1(d^{-1})$.
2. Cycles of $\Delta(C_1, t, h)$.

Clearly all the cycles (1) have weight $e$ and since $\Delta(C_1, t, h)$ is a $G$-partite graph, all cycles (2) also have weight $e$. By Lemma 2.1, $\gamma_1$ is consistent so $\gamma$ is a $G$-partite graph.

To see that $[0, \langle \emptyset, \gamma \rangle]$ is isomorphic to $(\prod_{t_i})_\eta$ we define a map $\phi$ from the former to the latter in the following way. Fix attention on the connected component $D$ of $\gamma$ containing $c_1$. Label the points of $D \cap C_i$ with the numbers $\tau(i - 1) + 1, \ldots, \tau t$ starting with $x_i$ and labeling in the order of the cycle (say $x_1 = c_j$). Let $\eta$ be the permutation $(1, 2, \ldots, t)(t + 1, \ldots, 2t), \ldots, ((r - 1)t, \ldots, rt)$. Define $\phi: [0, \langle \emptyset, \gamma \rangle] \to (\prod_{t_i})_\eta$ by letting $\phi(\langle \emptyset, \rho \rangle)$ be the partition whose blocks are the connected components of $D \cap \rho$. It is easy to see that this is a lattice isomorphism.

We will use the following earlier result of this author [6].

**Theorem 2.4.** Let $\mu$ be the Möbius function of $(\prod_{t_i})_\eta$, where $\eta$ is a permutation consisting of $r$ cycles of length $t$. Then

$$\mu(0, 1) = \mu(1) = (-1)^{r-1} (r - 1)! \frac{t^r}{r^t}$$

(here $\mu(t)$ refers to the classic number theoretic Möbius function).

Let $u \in U$ and let $\sigma$ consist of $a$ $u$-conjugate $l$-cycles $C_1, \ldots, C_a$. Let $P$ be the set of the $al$ points contained in the cycles of $\sigma$ and let $\gamma$ be a $\sigma$-invariant $G$-partite graph with vertex set $P$. Then $\gamma$ defines a partition $\prod(\gamma)$ of $\alpha$ by the rule that $b_1$ and $b_2$ are in the same block of $\prod(\gamma)$ if and only if there exist points in $C_{b_1}$ and $C_{b_2}$ which are in the same connected component of $\gamma$.

We use the notation $S_\alpha$ to denote the symmetric group on $\alpha$ letters. $Z(S_\alpha)$ will denote the ordinary cycle index of $S_\alpha$ as a permutation group of $\alpha$.

**Theorem 2.6.** Suppose $\sigma$ consists of $a$ $u$-conjugate $l$-cycles. Then

$$\chi_\sigma(\lambda) = a! (l \mid C_G(u))! \alpha \left( F(l, u, 1) - F(l, u, \lambda) \right)$$

**Proof.** We begin by considering the contribution made to $\chi_\sigma(\lambda)$ by all elements of the form $\langle \emptyset, \gamma \rangle$ which have $\prod(\gamma)$ equal to a fixed partition $\beta$ of $\alpha$. Let $\beta = R_1/ \cdots / R_k$ with $r_i = |R_i|$. Let $S(\beta)$ be the sum

$$S(\beta) = \sum_{\gamma} \mu_\sigma(0, \langle \emptyset, \gamma \rangle) \lambda^{a \beta - \mu(\emptyset, \gamma)}$$

(2.2)

where the sum is over those $\gamma$ fixed by $\sigma$ with $\prod(\gamma) = \beta$. 
Fix attention on one such $\gamma$. Let $\gamma_i$ be the subgraph of $\gamma$ consisting of all points in a cycle $C_b$ for $b \in R_i$ and let $\sigma_i$ be the permutation with cycles $C_b$ for $b \in R_i$. Clearly $\gamma_i$ is $\sigma_i$-invariant and

$$[0, \langle \emptyset, \gamma \rangle] \cong \bigsqcup_i [0, \langle \emptyset_i, \gamma_i \rangle]$$

where the later intervals are from the lattices $L_{\sigma_i}$. Also $\gamma$ is just the disjoint union of the graphs $\gamma_i$ so $\gamma$ is uniquely determined by $\gamma_1, \ldots, \gamma_k$. We further refine the sum (2.2) by summing over all possible $\gamma_1, \ldots, \gamma_k$.

Let $R_i = \{b_1, \ldots, b_r\}$. According to Lemma 2.4 we can construct all possible $\gamma_i$ by choosing

(a) A diagonalization $\Delta(C_{b_1}, t, h)$ of $C_{b_1}$,
(b) $(x_{2}, \ldots, x_{r}) \in C_{b_2} \times \cdots \times C_{b_r}$,
(c) $f_2, \ldots, f_r \in G$ which satisfy $f_j g_j f_j^{-1} = g_j$.

We have $l/t$ choices for each $x_j$ from part (b) and we have $|C_G(g)|$ choices for each $f_j$ in part (c). If $\gamma_i$ is so constructed then

1. $\rho(\langle \emptyset, \gamma_i \rangle) = (r_i^{-1}) l/t = lr_i - l/t$,
2. $\mu_i(0, \langle \emptyset, \gamma_i \rangle) = \mu(t)(-1)^{r_i-1} (r_i - 1)! t^{r_i-1}$.

Thus

$$\sum_{\gamma_i} \mu_i(0, \langle \emptyset, \gamma_i \rangle) \lambda^{r_i-\rho(\emptyset, \gamma_i)}$$

$$= \sum_{t | l} \left( b(t, g_1) \mu(t)(-1)^{r_i-1} (r_i - 1)! t^{r_i-1} \right) \times \left( \frac{l}{t} \right)^{r_i-1} |C_G(g_1)|^{r_i-1} \lambda^{l/t}$$

where $b(t, g_1)$ is the number of solutions $h \in G$ to $h^t = g_1$. Rewriting the above we obtain

$$(r_i - 1)! (-l | C_G(g_1) |)^{r_i-1} \sum_{t | l} \mu(t) b(t, g_1) \lambda^{l/t}$$

which is equal to

$$(r_i - 1)! (-l | C_G(g_1) |)^{r_i-1} F(l, g_1 \lambda).$$

So

$$S(\beta) = \prod_{i=1}^k (-l | C_G(u_i) |)^{r_i} (r_i - 1)! F(l, g_1 \lambda). \quad (2.3)$$
Note that in \( L_\alpha \) we have \([0, \langle S, \gamma \rangle] \cong [0, \langle S, \gamma_\emptyset \rangle] \times [0, \langle \emptyset, \gamma \rangle]\). Thus

\[
\chi_\alpha(\lambda) = \sum_{S \subseteq \alpha} \mu_\alpha(0, \langle S, \gamma_\emptyset \rangle) \cdot \left( \sum_\beta S(\beta) \right)
\]

where \( \sum_\beta \) is over all partitions of \( \alpha - S \).

For a partition \( R_1/\cdots/R_k \), the number of permutations with cycle sets \( R_1, \ldots, R_k \) is exactly \( \prod_{i=1}^k (r_i - 1)! \). Thus referring to Eq. (2.3)

\[
\sum_\beta S(\beta) = (\alpha - |S|)! \cdot Z(S_{\alpha - |S|})[x_i \rightarrow (-I\vert C_\emptyset(u))]^i \times F(l, u, \lambda).
\]

(Here \( Z(S_h)[x_i \rightarrow X_i] \) means to replace every occurrence of \( x_i \) in \( Z(S_h) \) by \( X_i \), or

\[
\sum_\beta S(\beta) = (\alpha - |S|)! \cdot (-I\vert C_\emptyset(u))^{a-|S|} \cdot Z(S_{\alpha - |S|})[x_i \rightarrow F(l, u, \lambda)].
\]

Combining this with Theorem 2.3 (b) we have

\[
\chi_\alpha(\lambda) = \sum_{S \subseteq \alpha} \left( |S|! \cdot (I\vert C_\emptyset(u))^{a-|S|} \left( \begin{array}{c} F(l, u, 1) \\ |S| \end{array} \right) \right) \times ((\alpha - |S|)! \cdot (-1)^{a-|S|} \cdot (l\vert C_\emptyset(u))^{a-|S|} \cdot Z(S_{\alpha - |S|})[x_i \rightarrow F(l, u, \lambda)])
\]

It is well-known that for any \( X, Z(S_h)[x_i \rightarrow X] = (x^r - 1) \). Applying this we have

\[
\chi_\alpha(\lambda) = \alpha! \cdot (l\vert C_\emptyset(u))^{a} \sum_{s=0}^{a} \left( \begin{array}{c} F(l, u, 1) \\ s \end{array} \right) \times \left( \begin{array}{c} F(l, u, \lambda) + \alpha - s - 1 \\ \alpha - s \end{array} \right) (-1)^{a-s}.
\]

We use the following two well-known identities:

\[
\binom{X}{r} = (-1)^r \binom{-X + r - 1}{r}
\]

\[
\sum_{r=0}^{k} \binom{X}{r} \binom{Y}{k-r} = \binom{X+Y}{k}
\]
Applying the first identity gives

\[ \chi_\sigma(\lambda) = \alpha! \left( \prod_{i \geq 0} \frac{F(l, u, 1)}{\alpha - s} \right). \]

Applying the second identity completes the proof.

It is worth noting that by the definition of \( \chi_\sigma(\lambda) \) we must have \( \chi_\sigma(0, 1) = \mu_\sigma(0, 1) \) and \( \chi_\sigma(1) = 0 \). The reader can easily check that the function \( \chi_\sigma(\lambda) \) given by Theorem 2.5 satisfies both these properties.

**Definition 2.3.** Define \( Z(W_n; \lambda) \), the characteristic polynomial cycle index of \( W_n \), by

\[ Z(W_n; \lambda) = \frac{1}{|W_n|} \sum_{\sigma \in W_n} \chi_\sigma(\lambda) Z(\sigma). \]

**Corollary 2.3.** \( \sum_{n=0}^{\infty} Z(W_n; \lambda) = \prod_{l, u} \left( 1 + x_l(u) \right)^{F(l, u, 1)} - F(l, u, \lambda) \)

As an example let \( G \) be the non-abelian group \( S_3 \). The conjugacy classes of \( G \) are indexed by the partitions 1, 2, 12 and 3. To avoid confusion we call these conjugacy classes \( A, B \) and \( C \) and we denote the cycle index variables \( x_l(A), x_l(B) \) and \( x_l(C) \) by \( a_l, b_l \) and \( c_l \). The three irreducible characters of \( G \) are given in the character table below:

<table>
<thead>
<tr>
<th></th>
<th>( A = 1^3 )</th>
<th>( B = 12 )</th>
<th>( C = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_0 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

To apply the results of this section it is necessary to compute the polynomials \( F(l, u, \lambda) \). One can easily check that

\[ F(l, A, \lambda) = -\frac{1}{6l} \sum_{i \geq 1} f_A(t) \mu(t) \lambda^{i/l} \]

where \( f_A(t) \) is the function given by

\[ f_A(t) = \begin{cases} 
6 & \text{if } (6, t) = 6 \\
3 & \text{if } (6, t) = 3 \\
4 & \text{if } (6, t) = 2 \\
1 & \text{if } (6, t) = 1.
\]
It is equally straightforward to show that

\[ F(l, B, \lambda) = -\frac{1}{2l} \sum_{t | l \text{ odd}} \mu(t) \lambda^{l/t} \]

and

\[ F(l, C, \lambda) = -\frac{1}{3l} \sum_{t | l \text{ } t \neq 0 \mod 3} \mu(t) \lambda^{l/t}. \]

Using these functions, together with Theorem 2.6 it is easy to compute the fixed point characteristic polynomials \( \chi_\sigma(\lambda) \) for any \( \sigma \). The case of \( n = 3 \) appears in the table below:

<table>
<thead>
<tr>
<th>( Z(\sigma) )</th>
<th>( \chi_\sigma(\lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1^3 )</td>
<td>((\lambda - 1)(\lambda - 7)(\lambda - 13))</td>
</tr>
<tr>
<td>( a_1 a_2 )</td>
<td>((\lambda - 1)^2 (\lambda - 3))</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>((\lambda - 1)^2 (\lambda + 2))</td>
</tr>
<tr>
<td>( a_1^2 b_1 )</td>
<td>((\lambda - 1)^2 (\lambda - 7))</td>
</tr>
<tr>
<td>( a_2 b_1 )</td>
<td>((\lambda - 1)^2 (\lambda - 3))</td>
</tr>
<tr>
<td>( a_1^2 c_1 )</td>
<td>((\lambda - 1)^2 (\lambda - 7))</td>
</tr>
<tr>
<td>( a_2 c_1 )</td>
<td>((\lambda - 1)^2 (\lambda - 3))</td>
</tr>
<tr>
<td>( a_1 b_1^2 )</td>
<td>((\lambda - 1)^2 (\lambda - 3))</td>
</tr>
<tr>
<td>( a_1 b_2 )</td>
<td>((\lambda - 1)^2 (\lambda + 1))</td>
</tr>
<tr>
<td>( a_1 b_1 c_1 )</td>
<td>((\lambda - 1)^3)</td>
</tr>
<tr>
<td>( a_1 c_1^2 )</td>
<td>((\lambda - 1)^2 (\lambda - 4))</td>
</tr>
<tr>
<td>( a_1 c_2 )</td>
<td>(\lambda (\lambda - 1)^2)</td>
</tr>
<tr>
<td>( b_1^3 )</td>
<td>((\lambda - 1)(\lambda - 3)(\lambda - 5))</td>
</tr>
<tr>
<td>( b_1 b_2 )</td>
<td>((\lambda - 1)^2 (\lambda + 1))</td>
</tr>
<tr>
<td>( b_3 )</td>
<td>(\lambda (\lambda - 1)(\lambda + 1))</td>
</tr>
<tr>
<td>( b_1^2 c_1 )</td>
<td>((\lambda - 1)^2 (\lambda - 3))</td>
</tr>
<tr>
<td>( b_2 c_1 )</td>
<td>((\lambda - 1)^2 (\lambda + 1))</td>
</tr>
<tr>
<td>( b_1 c_1^2 )</td>
<td>((\lambda - 1)^2 (\lambda - 4))</td>
</tr>
<tr>
<td>( b_1 c_2 )</td>
<td>(\lambda (\lambda - 1)^2)</td>
</tr>
<tr>
<td>( c_1^3 )</td>
<td>((\lambda - 1)(\lambda - 4)(\lambda - 7))</td>
</tr>
<tr>
<td>( c_1 c_2 )</td>
<td>(\lambda (\lambda - 1)^2)</td>
</tr>
<tr>
<td>( c_3 )</td>
<td>(\lambda^3 - 1)</td>
</tr>
</tbody>
</table>
To compute just the values of the fixed point Möbius functions we must compute \( F(l, A, 1) \), \( F(l, B, 1) \) and \( F(l, C, 1) \).

Let \( l \) be a positive integer written as \( l = rs \), where \( r = 2^{e} \cdot 3^{w} \) and \( (s, 6) = 1 \). Then

\[
F(l, A, 1) = -\frac{1}{6l} \left( \sum_{t|r} f_A(t) \mu(t) \right) \left( \sum_{u|l} \mu(u) \right)
\]

Thus

\[
F(l, A, 1) = \begin{cases} 
-\frac{1}{6} & \text{if } s = 1 \\
\frac{1}{2^{v+1}} & \text{if } l = 2^v, v > 1 \\
\frac{1}{3^{w+1}} & \text{if } l = 3^w, w > 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Similarly

\[
F(l, B, 1) = \begin{cases} 
-\frac{1}{2^{v+1}} & \text{if } l = 2^v, v \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
F(l, C, 1) = \begin{cases} 
\frac{1}{3^{w+1}} & \text{if } l = 3^w, w \geq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

So

\[
\sum_{n} Z_a(W_n) = (1 + a_1)^{-1} \prod_{i=0}^{\infty} \left( \frac{1 + a_{2i}}{1 + b_{2i}} \right)^{1/2^{i+1}} \\
\times \prod_{w=0}^{\infty} \left( \frac{1 + a_{3w}}{1 + c_{3w}} \right)^{1/3^{w+1}}.
\]

Note that if we substitute \( a_i = b_i = c_i = x_i \), we obtain \((1 + x_1)^{-1}\). This is true for any group \( G \) (see Example 3.3).

Later we will use the following two facts.

**Corollary 2.4.** Let \( n \in \mathbb{N} \) and let \( W_n \) denote \( G \wr S_n \). Then

(a) \( \frac{(-1)^n |W_n|}{|W_n|} \sum_{\sigma \in W_n} \mu_\sigma(0, 1) = 1 \)

(b) \( \frac{(-1)^n |W_n|}{|W_n|} \sum_{\sigma \in W_n} ((1/2) \chi_\sigma(-1)) = n \).
Proof: We first prove (a). Consider the power series $f(z)$ defined by

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{|W_n|} \left( \sum_{\sigma \in W_n} \mu_\sigma(0,1) \right) (z)^n.$$  

By Corollary 2.2 we have

$$f(z) = \prod_{l, u} (1 + (-z)^l)^{F(l, u, 1)} = \prod_{l} (1 + (-z)^l)^{\sum_{u} F(l, u, 1)}.$$  

Now

$$\sum_u F(l, u, 1) = -\frac{1}{l} \frac{1}{|G|^2} \sum_{h, g} \sum_{t|l} \mu(t) \chi(h^t) \chi(g)$$

$$= -\frac{1}{l} \frac{1}{|G|} \sum_g \sum_{t|l} \mu(t) b(t, g)$$

(recall that $b(t, g)$ is the number of $t$th roots of $g$ in $G$). But every $h \in G$ is the $t$th root of exactly one $g \in G$ (namely $h^t$) hence

$$\sum_u F(l, u, 1) = -\frac{1}{l} \sum_{t|l} \mu(t) = \begin{cases} -1, & l = 1 \\ 0, & l \neq 1. \end{cases}$$

So $f(z) = (1 - z)^{-1}$ which proves part (a).

To prove (b) the computation is very much the same. Define $g(z)$ by

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{|W_n|} \left( \sum_{\sigma \in W_n} \left( \frac{1}{2} \chi_\sigma(-1) \right) \right) z^n.$$  

Then by Corollary 2.3 we have

$$g(z) = \frac{1}{2} \prod_{l} (1 + (-z)^l)^{F(l, u, 1) - F(l, u, -1)}$$

$$\sum_u (F(l, u, 1) - F(l, u, -1)) = -\frac{1}{l} \sum_{t|l} \mu(t)(1 - (-1)^{l/t})$$

$$= \begin{cases} -2 & \text{if } l = 1 \\ 1 & \text{if } l = 2 \\ 0 & \text{if } l \neq 1, 2. \end{cases}$$

It follows that

$$g(z) = \frac{1}{2} \frac{(1 + z^2)}{(1 - z)^2} = \frac{1}{2} + \sum_{n=1}^{\infty} n z^n.$$
3. Characters

Let $G$ be a fixed finite group and let $W_n$ denote the wreath product of $G$ over $S_n$. Let $R(G)$ be the set of all sequences $\psi = ((\psi)_0, (\psi)_1, (\psi)_2, \ldots)$, where $(\psi)_n$ is a class function of $W_n$. Make $R(G)$ into a ring by defining addition of sequences component-wise and by defining multiplication $*$ as

$$(\psi \ast \theta)_n = \sum_{m=0}^{n} \text{ind}_{W_m \times W_{n-m}}^W((\psi)_m \times (\theta)_{n-m}).$$

It is easy to check that $R(G)$ is a commutative ring with identity, called the induction ring.

For $\psi, \varphi \in R(G)$ define $Z(\psi)$ by

$$Z(\psi) = \sum_{n=0}^{\infty} \frac{1}{|W_n|} \sum_{\sigma \in W_n} (\psi)_n (\sigma) Z(\sigma)$$

where $Z(\sigma)$ is the cycle indicator of $\sigma$ as defined in Section 2.

**Example 3.1.** Define $\mu, \chi \in R(G)$ as follows; for $\sigma \in W_n$ let

$$(\mu)_n (\sigma) = \mu_\sigma(0, 1)$$

$$(\chi)_n (\sigma) = \chi_\sigma(\lambda)$$

where $\mu_\sigma$ and $\chi_\sigma$ are the fixed-point Möbius function and the fixed-point Birkhoff polynomials as defined in Section 2. The notation $Z(\mu)$ and $Z(\chi)$ defined above is consistent with the notation used in Section 2.

**Lemma 3.1.** Let $\psi, \theta \in R(G)$. Then

$$Z(\psi \ast \theta) = Z(\psi) Z(\theta).$$

This lemma is straightforward to prove and will be of use later.

Our interest in this section is the study of certain characters of $W_n$ which arise from the lattices $Q_n(G)$. We next define these characters and state a theorem which relates the values of these characters to the class functions $\mu_\sigma$ and $\chi_\sigma$.

Let $Q$ be a ranked poset with unique minimal and maximal elements $0$ and $\hat{1}$. Define the order complex $\Delta(Q)$ to be the simplicial complex whose vertices are the elements of $Q - \{0, \hat{1}\}$ and whose faces are the chains $x_0 < x_1 < \cdots < x_k$ in $Q - \{0, \hat{1}\}$. Denote by $\tilde{H}_i(Q)$ the reduced simplicial homology group of $\Delta(Q)$ over $\mathbb{C}$.

We will deal exclusively with the case where $Q$ is a geometric lattice. In this case the homology groups are easily described. If the rank of $Q$ is 1 (so
Q just consists of 0 and 1) then $\mathcal{A}(Q)$ is empty so $\tilde{H}_{i-1}(Q) = \mathbb{C}$ and $\tilde{H}_i(Q) = 0$ for $i \neq -1$. The following result of Folkman deals with the general case.

**Theorem 3.1.** (Folkman [5]). Let $Q$ be a geometric lattice of rank $n \geq 2$. Then

$$
\tilde{H}_i(Q) = \begin{cases} 
0 & \text{if } i \neq n - 2 \\
\mathbb{C} & \text{if } i = n - 2.
\end{cases}
$$

Let $Q$ be a geometric lattice of rank $n$ and let $i$ be an integer with $0 \leq i \leq n - 1$. Let $Q_i$ denote the poset obtained from $Q$ by taking $0, 1$ and ranks $1, 2, \ldots, i$. It is well-known that $Q_i$ is a geometric lattice of rank $i + 1$. Note also that $Q = Q_{n-1}$ and that $Q_0 = \{0, 1\}$. Let $\tilde{H}(Q_i)$ denote the unique non-vanishing reduced homology group of $Q_i$.

Fix $W$ a group of automorphisms of $Q$. Then $W$ also acts as a group of automorphisms of each $Q_i$ hence $W$ acts on each of the reduced homology groups $\tilde{H}(Q_i)$. Let $\beta_i$ denote the character of $W$ acting on $\tilde{H}(Q_i)$ and for each $\sigma \in W$ let $B_\sigma(\lambda)$ be the polynomial.

$$
B_\sigma(\lambda) = \sum_{i=0}^{n-1} \beta_i(\sigma) \lambda^{(n-1)-i}.
$$

For each $\sigma \in W$ let $\chi_\sigma(\lambda)$ be as defined in Section 2 (there for the case of $Q$ a Dowling lattice). The next theorem relates the polynomials $B_\sigma(\lambda)$ to the $\chi_\sigma(\lambda)$.

**Theorem 3.2.** (Baclawski and Bjorner [1]). For each $\sigma \in W$ we have

$$
B_\sigma(\lambda) = (-1)^n \frac{\chi_\sigma(-\lambda)}{(\lambda + 1)}.
$$

This theorem gives our primary motivation for computing the polynomials $\chi_\sigma(\lambda)$. From the results of Section 2 we have a complete determination of the values of the characters $\beta_i$ in the case that $Q$ is a Dowling lattice and $W$ the wreath product group.

**Definition 3.1.** Define $\beta, A, \tau \in R(G)$ as follows:

1. Let $\beta_n$ be the character of $W_n$ acting on the unique non-vanishing reduced homology of $Q_n(G)$.
2. Let $A_n$ be the class function defined by $A_n(\sigma) = B_\sigma(\lambda)$ (see Theorem 3.2).
3. Let $\tau_n$ be the character of $W_n$ given by $\tau_n(\sigma) = B_\sigma(1) = \beta_0(\sigma) + \beta_1(\sigma) + \cdots + \beta_{n-1}(\sigma)$. (The reader is warned not to confuse the notation $\beta_n$ with $\beta_n$.)
From Theorems 2.6 and 3.2 we have that the sequences \( \beta, A, \tau \) depend only on the character table of \( G \). A bit more can be said.

**Theorem 3.3.** Let \( n \) and \( i \) be positive integers with \( i \leq n - 1 \).

Then the trivial representation occurs exactly once in \( \mathcal{H}(Q_i) \).

**Proof:** Let \( \varepsilon \) denote the trivial character of \( W_n \). Then

\[
\langle \varepsilon, (\beta)_n \rangle = \frac{1}{|W_n|} \sum_{\sigma \in W_n} (\beta)_n (\sigma)
\]

\[
= \frac{1}{|W_n|} \sum_{\sigma \in W_n} B_{\sigma}(0)
\]

\[
= \frac{(-1)^n}{|W_n|} \sum_{\sigma \in W_n} \chi_{\sigma}(0) \quad \text{by Theorem 3.2}
\]

\[-1 \quad \text{by Corollary 2.4.}
\]

Similarly

\[
\langle \varepsilon, (\tau)_n \rangle = \frac{(-1)^n}{2|W_n|} \sum_{\sigma \in W_n} \chi_{\sigma}(-1) = n \quad \text{by Corollary 2.4.}
\]

So the trivial character occurs once in \( (\beta)_n = \beta_{n-1} \) and \( n \) times in \( (\tau)_n = \beta_0 \mid \beta_1 + \cdots + \beta_{n-1} \). It is straightforward to check that \( \beta_i \) is contained in \( \beta_{i+1} \) for \( i = 0, 1, \ldots, n - 2 \) which completes the proof.

**Example 3.1.** Let \( G \) be trivial. Then \( W_n \) is the symmetric group \( S_n \) and \( Q_n(G) = \prod_{n+1} \). In this case we abbreviate \( F(l, e, \lambda) \) by \( F(l, \lambda) \). By definition we have

\[
F(l, \lambda) = -\frac{1}{l} \sum_{t \in T} \mu(t) \lambda^{1/t}
\]

so

\[
F(l, 1) = \begin{cases} 
-1 & \text{if } l = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

As above, let \( \beta_{n-1} \) be the character of \( S_n \) acting on \( \mathcal{H}_{k-1}(\prod_{n+1}) \). By Theorem 3.2 we have

\[
\beta_{n-1}(\sigma) = \begin{cases} 
\begin{align*}
n! \\
0
\end{align*} & \text{if } \sigma = e \\
0 & \text{otherwise.}
\end{cases}
\]

So \( \beta_{n-1} \) is the regular representation of \( S_n \), a result obtained earlier by Stanley (see [9]).
Next consider the character $\beta_0 + \beta_1 + \cdots + \beta_{n-1} = \tau_n$. We have

$$
\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} \tau_n(\sigma) Z(\sigma) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} B_\sigma(1) Z(\sigma)
$$

$$
= \sum_{n=0}^{\infty} \frac{(-1)^n}{2(n!)} \chi_\sigma(-1) Z(\sigma)
$$

by Theorem 3.2.

By Corollary 2.3 we have that the above sum is

$$
\frac{1}{2} (1 - x_1)^{-1} \prod_{l=1}^{\infty} (1 + (-1)^l x_l)^{1/l} \mu(l)(-1)^{1/l}.
$$

It is easy to check that

$$
\frac{1}{l} \sum_{t \mid l} \mu(t)(-1)^{1/l} = \begin{cases} 
-1 & \text{if } l = 1 \\
1 & \text{if } l = 2 \\
0 & \text{otherwise.}
\end{cases}
$$

Hence

$$
\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} \tau_n(\sigma) Z(\sigma) = \frac{1}{2} (1 - x_1)^{-1} (1 + x_2).
$$

So

$$
\tau_n(\gamma) = \begin{cases} 
\frac{(n+1)!}{2} & \text{if } \gamma = e \\
(n-1)! & \text{if } \gamma \text{ is of type } 1^{n-2}2 \\
0 & \text{otherwise.}
\end{cases}
$$

Let $\chi$ be the character of $S_{n+1}$ obtained by inducing the trivial character from a 2-cycle to the entire group. Then $\tau_n = \chi|_{S_n}$ again a result obtained earlier by Stanley.

**Example 3.2.** Let $G = \{+1\}$. For simplicity we will use the notation $a_i$ for $x_i(+1)$ and $b_i$ for $x_i(-1)$. In this case $W_n$ is the hyperoctahedral group and $Q_n(G)$ is the lattice of flats of the independence matroid of the root system for the Lie algebra $B_n$.

A simple computation shows that

$$
F(l, +1, \lambda) = -\frac{1}{2l} \left( \sum_{t \mid l} \mu(t) f_1(t) \lambda^{t/l} \right)
$$

where

$$
f_1(t) = \begin{cases} 
1 & \text{if } t \text{ is odd} \\
2 & \text{if } t \text{ is even} \end{cases}
$$
and

\[ F(l, -1, \lambda) = -\frac{1}{2l} \sum_{\text{odd } t} \mu(t) \lambda^{lt}. \]

Thus

\[ F(l, +1, 1) = \begin{cases} -1/2 & \text{if } l = 1 \\ 1/2^{a+1} & \text{if } l = 2^a \\ 0 & \text{otherwise} \end{cases} \]

and

\[ F(l, -1, 1) = \begin{cases} -1/2^{a+1}, & l = 2^a \\ 0 & \text{otherwise.} \end{cases} \]

So,

\[ Z(\beta) = (1 - a_1)^{-1/2} (1 - b_1)^{-1/2} \prod_{a=1}^{\infty} (1 + a z_a)^{1/2a - 1} (1 + b z_a)^{-1/2a + 1}. \]

It is easy to check that

\[ F(l, 1, 1) - F(l, 1, -1) = \begin{cases} -1 & \text{if } l = 1 \\ 1 & \text{if } l = 2 \\ 0 & \text{otherwise} \end{cases} \]

and

\[ F(l, -1, 1) - F(l, -1, -1) = \begin{cases} -1 & \text{if } l = 1 \\ 0 & \text{otherwise.} \end{cases} \]

So

\[ Z(\tau) = \frac{1}{2} (1 - a_1)^{-1} (1 - b_1)^{-1} (1 + a_2). \]

Thus

\[ (\tau)_n(\sigma) = \begin{cases} i! j! 2^{n-1} & \text{if } \sigma \text{ is of type } 1^i 1^{-j} \text{ or of type } 1^i 1^{-i/2} \\ 0 & \text{otherwise.} \end{cases} \]

Note that \((\tau)_n(\epsilon) = |W_n|/2\) so one might hope that \((\tau)_n\) could be realized as an induced character from a subgroup of \(W_n\) of order 2. This clearly is not the case.

**Example 3.3.** For each \(n\) and each \(i \leq n - 1\) define a character \(\rho_{i,n}\) of \(S_n\) by

\[ \rho_{i,n}(\gamma) = \sum_{\sigma \gamma} \beta_i(\sigma) \]
where the sum on the right is over all $\sigma \in W_n$ with $\delta$ equal to $\gamma$. Interestingly enough, the characters $\rho_{i,n}$ depend only on $n$ and not on the group $G$.

For each $\gamma \in S_n$ let $Y(\gamma)$ denote $\sum_{i=0}^{n-1} \rho_{i,n}(\gamma) x_i^{n-i}$. One can show that $
abla_n (1/n!) \sum_{\gamma \in S_n} Y(\gamma) Z(\gamma) = (1 - x_1)^{-1} \prod_{i \geq 1} (1 + (-1)^i x_i)^{P_i(A)}$, where $P_i(A) = \sum_{\mu(t)} \mu(t) \lambda^i t^i$. The proof of this result is analogous to the proof of Corollary 2.4 and is left to the reader.

There are interesting connections between the characters $(\beta)_n$ and certain characters which can be realized as induced characters. We derive these results next and compare them to similar results obtained by Stanley for the case of $S_{n+1}$ acting on the partition lattice $\mathcal{P}_{n+1}$.

We will compare $(\beta)_n$ to a character $(\psi)_n$ which we obtain by inducing a certain generic character to $W_n$ from the centralizers of the permutations in $W_n$.

To define $(\psi)_n$, consider a permutation $\sigma$ in $W_n$, i.e., $\sigma \in W_n \cap S_n$. We look first at the centralizer of $\sigma$. If $\sigma$ consists of a single cycle (necessarily an $n$-cycle) then $C = C_{W_n}(\sigma)$ has order $n |G|$ and consists of all group elements of the form $\sigma^a(gI)$, where $0 \leq a \leq n - 1$ and where $g \in G$. Here $gI$ is the scalar matrix with entries $g$ down the main diagonal. Define a character $\eta$ on $C$ by

$$\eta(\sigma^a(gI)) = z_n^a \text{ where } z_n = e^{2\pi i/n}.$$ 

It is worth noting that $z_n^a$ is a primitive $n$th root of 1 where $i = n/gcd(n, a)$.

Let $Z(\eta)$ be the cycle index of this character $\eta$, i.e.,

$$Z(\eta) = \frac{1}{|C|} \sum_{\rho \in C} \eta(\rho) Z(\rho).$$

The next lemma is an immediate consequence of the above discussion together with the fact that the sum of the primitive $n$th roots of 1 is $\mu(t)$.

**Lemma 3.2.** Suppose $\sigma$ consists of an $n$-cycle. Then

$$Z(\eta) = \frac{1}{n} \sum_{g \in G} \sum_{t \in \mathbb{Z}_n} \mu(t) x_t^{n/t}(g^t).$$

Suppose next that $\sigma$ consists of $c_l$ $l$-cycles. Then $C_{W_n}(\sigma)$ is isomorphic to the wreath product of $C$ over $S_l$, where $C$ is the centralizer of a single $l$-cycle. Lift $\eta$ from $C$ to $C_{W_n}(\sigma)$ in the following way. If $\rho \in C_{W_n}(\sigma)$ and $\gamma = (\gamma_1, \ldots, \gamma_s)$ is an $s$-cycle of $\rho$ define $w(\gamma)$ to be

$$w(\gamma) = \prod_{i=1}^{s} \eta(\text{sgn}(\rho, \gamma_i)).$$
Note that the entries \( \text{sgn}(\rho, \gamma_i) \) of \( \rho \) are from \( C \) so \( \eta(\text{sgn}(\rho, \gamma_i)) \) makes sense. Define \( \eta(\rho) \) to be the product of \( w(\gamma) \) taken over all cycles \( \gamma \) of \( \rho \). By a well-known result on the cycle index of a wreath product we have

\[
Z(\rho) = Z(S_{c_1}) \left[ \frac{1}{|G|} \sum_{g \in G} \sum_{\ell \in l} \mu(t) x_{i[]}^{l/\ell}(g^\ell) \right]
\]

where \( Z(S_{c_1}) \) is the ordinary cycle index of \( S_{c_1} \) (written with variables \( a_1, a_2, \ldots \)). Here \( Z(S_{c_1})[\_] \) is the composition product which means that each occurrence of \( a_n \) in \( Z(S_{c_1}) \) is replaced by

\[
\frac{1}{|G|} \sum_{g \in G} \sum_{\ell \in l} \mu(t) x_{i[]}^{l/\ell}(g^\ell).
\]

Lastly assume that \( \sigma \) is a general element of \( W_n \cap S_n \). Let \( c_1 \) be the number of \( l \)-cycles of \( \sigma \). Then \( C_{W_n}(\sigma) \) is isomorphic to the direct product of the groups \( C_{W_n_{c_1}}(\sigma_i) \), where \( \sigma_i \) is the permutation consisting of the \( l \)-cycles of \( \sigma \). The character \( \eta \) defined on each \( C_{W_n_{c_1}}(\sigma_i) \) lifts naturally to the direct product \( C_{W_n}(\sigma) \). It is clear that

\[
Z(\eta) = \prod_{i=1}^{n} Z(S_{c_1}) \left[ \frac{1}{|G|} \sum_{g \in G} \sum_{\ell \in l} \mu(t) x_{i[]}^{l/\ell}(g^\ell) \right].
\]

**Theorem 3.4.** For each \( \sigma \in S_n \) let \( \psi_\sigma \) denote the induction of the character \( \eta \) from \( C_{W_n}(\sigma) \) to \( W_n \). Let \( (\psi)_n = \sum_{\sigma \in S_n} \psi_\sigma \) and let \( \text{sgn} \) denote the sign character of \( W_n \) (i.e., \( \text{sgn}(\sigma) \) is the sign of \( \sigma \)). Then

\[
(\beta)_n \text{sgn}(\psi)_n.
\]

**Proof.** Let \( H \) be a subgroup of a group \( J \), let \( \theta \) be a character of \( H \) and let \( \theta \) be the induction of \( \theta \) to \( J \). Then by the definition of induced character we have

\[
\frac{1}{|J|} \sum_{j \in J} \theta(j) = \frac{1}{|H|} \sum_{h \in H} \theta(h).
\]

Thus

\[
\sum_{n=0}^{\infty} \frac{1}{|W_n|} \sum_{\rho \in W_n} \left( \sum_{\sigma \in S_n} \psi_\sigma(\rho) \right) Z(\rho)
\]

\[
= \prod_{l=1}^{\infty} \sum_{c_1=0}^{\infty} Z(S_{c_1}) \left[ \frac{1}{|G|} \sum_{g \in G} \sum_{\ell \in l} \mu(t) x_{i[]}^{l/\ell}(g^\ell) \right].
\]
Using the well-known identity (see Harary and Palmer [6])
\[ \sum_{c_r = 0}^{\infty} Z(S_c) = \exp \left( \sum_{m=1}^{\infty} \frac{a_m}{m} \right) \]
we have that (3.1) equals
\[ \prod_{l=1}^{\infty} \exp \left( \sum_{m=1}^{\infty} \frac{1}{ml|G|} \sum_{g \in G} \sum_{i|l} \mu(t) x_{ml}^i(g^i) \right) \]
\[ = \exp \left( \sum_{m,l,g} \frac{1}{ml|G|} \left( \sum_{i|l} \mu(t)(x_{ml}(g^i))^{i/l} \right) \right). \]

Let \( s = ml \) and let \( j = l/t \). Rewriting the above in terms of \( s \) and \( j \) we obtain
\[ \exp \left( \sum_{s,x} \frac{1}{s|G|} \sum_{t|s} \mu(t) \sum_{j=1}^{\infty} \frac{x_s^j(g^i)}{j} \right). \] (3.2)

Note that
\[ \sum_{j=1}^{\infty} \frac{x_s^j(g^i)}{j} = - \log(1 - x_s(g^i)). \]

Hence
\[ \sum_{g} \sum_{j=1}^{\infty} \frac{x_s^j(g^i)}{j} = \sum_{h \in G} \log(1 - x_s(h)) b(t, h) \]
where \( b(t, h) \) is the number of \( g \in G \) which satisfy \( g^i = h \). Thus we can rewrite (3.2) as
\[ \exp \left( \sum_{s,h} \log(1 - x_s(h)) \left( \frac{1}{s|G|} \sum_{t|s} \mu(t) b(t, h) \right) \right). \]

This in turn can be rewritten as
\[ \exp \left( \sum_{s,h} \log(1 - x_s(h)) F(s, h, 1) \left( \frac{|C_G(h)|}{|G|} \right) \right) \]
\[ = \exp \left( \sum_{s,u} \log(1 - x_s(u)) F(s, u, 1) \right) \]
where now the sum is over conjugacy classes \( u \). So
\[ Z(\psi) - \prod_{s,\psi} (1 - x_s(u))^{F(s, u, 1)} \]
hence
\[ Z(A\psi) = \prod_{s,u} (1 - (-1)^{s-1} x_s(u))^{F(s,u,1)} = Z(\beta). \]

Consider now the case where \( Q \) is the partition lattice \( \prod_n \) and \( W \) is the symmetric group \( S_n \). For \( \sigma \) an \( n \)-cycle in \( S_n \) we have that \( C_w(\sigma) = \langle \sigma \rangle \). Define the linear character \( \alpha \) on \( C_w(\sigma) \) by \( \alpha(\sigma) = e^{2\pi i/n} \). Let \( \hat{\alpha}_\sigma \) denote the induction of \( \alpha \) to \( S_n \). Note the similarity between \( \hat{\alpha}_\sigma \) and the character \( \hat{\psi}_\sigma \) defined for the previous result. Note also the similarity between the previous theorem and the next result due to Richard Stanley.

**Theorem 3.5.** (Stanley [9]). Let \( (\Gamma)_n = \sum_\sigma \hat{\alpha}_\sigma \), where the sum is over all \( n \)-cycles \( \sigma \) in \( S_n \). Let \( \text{sign} \) be the sign character of \( S_n \) and let \( (\beta')_n \) be the character of \( S_n \) acting on the non-vanishing reduced homology group of \( \prod_n \). Then
\[ (\beta')_n = \text{sign}(\Gamma)_n. \]

We can consider \( Q \) as the lattice of flats of the independence matroid of a root system for the Lie algebra \( A_n \). Then \( W \) is just the Weyl group of \( A_n \). When viewed in this way the \( n \)-cycles \( \sigma \in W \) are exactly the elements of the Weyl group with no eigenvalues equal to 1.

If we consider the Lie algebra \( B_n \) then the corresponding lattice is the Dowling lattice \( Q_\infty(G) \) with \( G = \{ \pm 1 \} \) (see Example 3.2). The character \( \hat{\psi}_\sigma \) is the natural analogue to the character \( \hat{\alpha}_\sigma \) used by Stanley. Define \( (\psi_1)_n \) to be \( \sum_\sigma \hat{\psi}_\sigma \) where the sum is over all \( \sigma \in W_n \) with no eigenvalues equal to 1. Stanley had conjectured that \( (\beta)_n \) should be \( (\text{sign} \psi_1)_n \). This is not the case, however \( \beta \) and \( \text{sign} \psi_1 \) are closely related in the induction ring.

**Theorem 3.6.** With notation as above, let \( (\rho)_n \) be the regular representation of \( W_n \). Then in \( R(G) \) we have
\[ \beta^*(\text{sign} \psi_1) = \rho. \]

**Proof.** Using exactly the kinds of computations which appear in the proof of Theorem 3.4 one can show that
\[ Z(\text{sign} \psi_1) = (1 - a_1)^{-1/2} (1 - b_1)^{1/2} \prod_{a=1}^\infty (1 + a_2a)^{-(a+1)/2} (1 + b_2a)^{(a+1)/2} \]
Thus \[ Z(\beta) Z(\text{sign} \psi_1) = (1 - a_1)^{-1} = Z(\rho). \]
I would like to thank Professor R. P. Stanley for many helpful discussions during the preparation of this paper.

REFERENCES