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On the Koszul Property of the Homogeneous Coordinate Ring of a Curve

Alexander Polishchuk

Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138

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1. INTRODUCTION

This paper is devoted to the Koszul property of the homogeneous coordinate algebra of a smooth complex algebraic curve in projective space (the notion of a Koszul algebra is some homological refinement of the notion of a quadratic algebra; for a precise definition see next section). It grew out the attempt to understand the methods of Finkelberg and Vishik in their paper [10] proving this property for the canonical algebra of a curve in the case when it is quadratic. The basic ingredient of their proof is the following lemma on special divisors.

LEMMA 1 [11]. *Let C be a non-hyperelliptic non-trigonal curve which is not a plane quintic. Then there exists a divisor D of degree $g - 1$ on C such that $|D|$ and $|K(-D)|$ are base-point-free linear systems of dimension 1, where K is the canonical class.*

In the cited paper of Green and Lazarsfeld it is used for the vector bundle proof of Petri's theorem which asserts that if C satisfies the conditions of this lemma then the canonical algebra of C is quadratic. In this note we will show that the Koszul property can be derived from this lemma by a purely homological technique combined with a simple statement concerning the Koszul property of the homogeneous coordinate algebra of a finite set of points in projective space—in particular we obtain a new proof of Petri's theorem. It turns out that the same technique works for the proof of the Koszul property of the homogeneous coordinate ring of a curve of genus g embedded by a complete linear system of degree $\geq 2g + 2$ (this result is due to Butler [8]) and also for embeddings defined by the complement to some very special linear systems (the simplest case being that of tetragonal systems) in the canonical class. In particular, we prove the Koszul property for a general tetragonal curve of genus $g \geq 9$

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embedded by $K(-T)$, where T is the tetragonal series. It is worthy to mention here that if C is a non-hyperelliptic curve and L is a linear bundle of degree $2g+1$ such that $H^0(L \otimes K^{-1}) = 0$ then the coordinate algebra of C in the embedding defined by L is quadratic (see, e.g., [15]). It is still unknown (at least to me) whether it is Koszul or not. It seems that the following general question is also open: whether the homogeneous coordinate algebra of a projectively normal smooth connected complex curve is Koszul provided it is quadratic?

To illustrate our technique we present here the proof of Petri's theorem based on the lemma mentioned above.

THEOREM 2. *Assume that there exists a divisor D on C of degree $g-1$ such that $|D|$ and $|K(-D)|$ are base-point-free linear series of dimension* 1. *Then the canonical algebra R is quadratic.*

Proof. Choose a divisor $P_1 + \cdots + P_{g-1}$ in the linear system $|D|$ such that these $g-1$ points are distinct (then they are in general linear position in the $(g-3)$-dimensional space they span). Let $V = H^0(K(-D)) \subset H^0(K) = R_1$ be the corresponding two-dimensional subspace. Denote the symmetric algebra $\mathrm{Sym}(R_1)$ by S and let $J \subset S$ be the homogeneous ideal of C so that $R = S/J$ by Noether's theorem (obviously C is non-hyperelliptic). Consider the exact sequence

$$0 \to VS \cap J \to J \to J/(VS \cap J) \to 0,$$

where VS is the ideal in S generated by V. It is easy to see that it is enough to check the following two statements:

1. $J/(VS \cap J)$ is generated over S by elements of degree 2;
2. $VS \cap J$ is generated over S by elements of degree 2 modulo VJ.

We will use the following lemma.

LEMMA 3. *Let $|D|$ be a base-point-free linear system of dimension* 1. *Then the natural homomorphism $H^0(D) \otimes H^0(K^n) \to H^0(K^n(D))$ is surjective for $n \geq 0$.*

The proof is left to reader.

Now for the proof of (1) we claim that $VS + J$ is exactly the homogeneous ideal of the set of points $P_1, \ldots, P_{g-1}$ so the statement follows from the quadratic property of an algebra of $g-1$ points in general position in $\mathbf{P}^{g-3}$ (see Section 3 for a more general result in this direction). Indeed, let A be the coordinate algebra of these points. Then A is the algebra quotient of R by the ideal $I = \bigoplus_{n \geq 1} H^0(K^n(-D))$. Now it follows from the above lemma that I is generated by $I_1 = V$ over R so $A = R/(VR) = S/(VS + J)$ and our assertion follows.

It remains to check (2). For this consider the multiplication map μ_S: $V \otimes S(-1) \to S$. It is enough to verify that $\mu_S^{-1}(J)/(V \otimes J(-1))$ is generated over S by elements of degree 2. We can rewrite this as follows: let μ_R: $V \otimes R(-1) \to R$ be another multiplication map then $\ker \mu_R$ is generated by elements of degree 2 over R. But this kernel is easily seen to be equal to $\bigoplus_{n \geq 2} H^0(K^{n-2}(D))$ (by the base-point-pencil trick applied to $K(-D)$), so we are done by the lemma above. ∎

2. SOME HOMOLOGICAL ALGEBRA

In this section we prove a general criterion for Koszul property of a graded algebra given its Koszul algebra quotient and some information about this quotient as a module over the original algebra.

Recall that graded (associative but not necessary commutative) algebra $A = A_0 \oplus A_1 \oplus \cdots$ over the field k is called Koszul if $A_0 = k$ and $\mathrm{Ext}^n(k, k(-m)) = 0$ for $n \neq m$. Here the functor Ext is taken in the category of graded (left) A-modules; k is a trivial A-module concentrated in degree 0. For graded A-module $M = \oplus M_i$ we define the shifted module as $M(l) = \oplus M_{i+l}$. Note that the above condition on Ext's for $n = 1$ means that A is generated by elements of degree 1, for $n = 1$ and $n = 2$—that A is quadratic (i.e., in addition, it has defining relations of degree 2). For equivalent definitions via the exactness of the Koszul complex and distributivity of some lattices, see [17, 3, 6].

The following theorem is the generalization of Lemma 7.5 in [7], where it was proved in the particular case when $A = k$.

THEOREM 4. *Let R be graded algebra over k with $R_0 = k$ and A be its Koszul (graded) algebra quotient. Assume that there exists a complex $K^\cdot$ of free right R-modules of the form*

$$\cdots \to V^2 \otimes R(-2) \to V^1 \otimes R(-1) \to R$$

(so $K^i = V^i \otimes R(-i)$, where V^i are finite-dimensional k-linear spaces, $V^0 = k$) such that $H_0(K^\cdot) = A$, $H_p(K^\cdot)_j = 0$ for $p \geq 1$, $j > p + 1$. Then R is Koszul and A has a linear-free resolution as an R-module that is a resolution of the form

$$\cdots \to U^2 \otimes R(-2) \to U^1 \otimes R(-1) \to R \to A \to 0.$$

Proof. Note that the conditions on the homology $H_p(K^\cdot)$ for $p \geq 1$ in formulation of the theorem mean that $H_p(K^\cdot)$ as R-module is an extension of some multiples of trivial modules $k(-p)$ and $k(-p-1)$. Indeed, this follows directly from the fact that $H_p(K^\cdot)_j = 0$ for $j < p$. Now we prove by induction on $n \geq 0$ that $\mathrm{Ext}^n_R(k, k(-m)) = 0$ if $m \neq n$. Case $n = 0$ is trivial. Assume that $n > 0$, $\mathrm{Ext}^i_R(k, k(-m)) = 0$ for $i < n$, $m \neq i$, and let us prove the assertion for n. Considering the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^q_A\big(\mathrm{Tor}^R_p(A,k), k(-m)\big) \Rightarrow \mathrm{Ext}^{p+1}_R(k, k(-m))$$

it is easy to see that it is enough to prove that $\mathrm{Tor}^R_j(A,k)_i = 0$ if $i \neq j$, $j \leq n$ (here we use the Koszul property of A). The latter is equivalent to $\mathrm{Ext}^j_R(A, k(-i)) = 0$ for $i \neq j$, $j \leq n$ (here Ext is taken in the category of right R-modules). Consider the spectral sequence associated with complex $K^\cdot$ and cohomological functor $\mathrm{Hom}_{D^-(R)}(\cdot, k(-m))$, where $D^-(R)$ is the derived category of complexes bounded from the right:

$$E_2^{p,q} = \mathrm{Ext}^q_R\big(H_p(K^\cdot), k(-m)\big) \Rightarrow \mathrm{Hom}^{p+q}_{D^-(R)}(K^\cdot, k(-m)).$$

The limit on the right can be computed easily with help of another spectral sequence with E_1 obtained by applying the same functor to the terms of the complex $K^\cdot$. Due to the form of this complex it degenerates at E_1 which gives an equality $\mathrm{Hom}^n(K^\cdot, k(-m)) = 0$ if $n \neq m$. On the other hand, for $p \geq 1$ the group $H_p(K^\cdot)$ is an extension of the direct sums of several copies of $k(-p)$ and $k(-p-1)$, and we obtain by assumption that $E_2^{p,q} = \mathrm{Ext}^q_R(H_p(K^\cdot), k(-m)) = 0$ if $p \geq 1$, $q < n$, $m \neq p+q$, $m \neq p+q+1$. This implies that the terms $E_2^{0,q}$ must survive and contribute to E_∞ if $q \leq n$, $q \leq m-1$. Indeed, all the differentials $d_r\colon E_r^{r-1,q-r} \to E_r^{0,q}$ are zero for $r \geq 2$, $q \leq \min(n, m-1)$ because $E_r^{r-1,q-r}$ is zero for these values of r and q. Therefore our computation of the limit implies an equality $E_2^{0,q} = \mathrm{Ext}^q_R(A, k(-m)) = 0$ for $q \leq \min(n, m-1)$. On the other hand, $\mathrm{Ext}^q_R(A, k(-m)) = 0$ for $q > m$ so we are done. ∎

3. APPLICATION TO THE COORDINATE RING OF A CURVE

Now we are going to apply the theorem of the previous section to the homogeneous coordinate algebra of a curve. So let C be a curve and let L be a very ample linear bundle on C. We are interested in the algebra $R = R_L = \bigoplus_{n \geq 0} H^0(L^n)$. Our approach is as follows: we consider the algebra quotient A of R associated with some effective divisor D on C, namely, $A = A_D = R/J_D$, where $J_D = \bigoplus_{n \geq 1} H^0(L^n(-D))$ is an ideal in R. To apply the above theorem to this situation we have to construct a complex of free R-modules, which is an "almost resolution" of A and has the required form, and to check the Koszul property of the algebra A. The latter is simple, provided that points of the divisor D are "sufficiently linear independent" in the embedding defined by L. To construct the desired complex we assume that $\mathcal{O}(D)$ and $L(-D)$ are base-point-free so that we have the following exact triples:

$$0 \to \mathcal{O}(-D) \to V \otimes \mathcal{O} \to \mathcal{O}(D) \to 0$$
$$0 \to L^{-1}(D) \to U \otimes \mathcal{O} \to L(-D) \to 0,$$

where V and U are some vector space of dimension 2. Now the complex K has the form

$$\cdots \to U \otimes R(-3) \xrightarrow{d_3} V \otimes R(-2) \xrightarrow{d_2} U \otimes R(-1) \xrightarrow{d_1} R,$$

where d_1 is induced by the composition of maps

$$U \to H^0(L(-D)) \to H^0(L) = R_1,$$

the nth component of d_{2k} is the composition

$$V \otimes H^0(L^{n-2k}) \to H^0(L^{n-2k}(D)) \to U \otimes H^0(L^{n-2k+1})$$

and that of d_{2k+1} is the composition

$$U \otimes H^0(L^{n-2k-1}) \to H^0(L^{n-2k}(-D)) \to V \otimes H^0(L^{n-2k}).$$

Using the exact triples above one can compute easily the homology of K. Indeed,

$$\ker(d_{2k+1})_n = \ker\big(U \otimes H^0(L^{n-2k-1}) \to H^0(L^{n-2k}(-D))\big)$$
$$\simeq H^0(L^{n-2k-2}(D)).$$

Therefore,

$$H_{2k+1}(K)_n \simeq \operatorname{coker}\big(V \otimes H^0(L^{n-2k-2}) \to H^0(L^{n-2k-2}(D))\big)$$
$$\simeq \ker\big(H^1(L^{n-2k-2}(-D)) \to V \otimes H^1(L^{n-2k-2})\big).$$

It follows that if $H^1(L^2(-D)) = 0$ then $H_{2k+1}(K)_n = 0$ for $n \geq 2k+4$. Furthermore, if the map α: $H^1(L(-D)) \to V \otimes H^1(L)$ is injective then $H_{2k+1}(K)_{2k+3} = 0$ as well. In an analogous way we obtain that if $H^1(L(D)) = 0$ and β: $H^1(D) \to U \otimes H^1(L)$ is injective then $H_{2k}(K)_{\geq 2k+2} = 0$. Note that the condition $H^1(L(D)) = 0$ implies surjectivity of α so in this case injectivity is equivalent to the equality of dimensions: $h^1(L(-D)) = 2h^1(L)$. Analogously if $H^1(L^2(-D)) = 0$ then injectivity of β is equivalent to equality $h^1(D) = 2h^1(L)$. It easy to see that if all these conditions hold and, in addition, $h^0(L(-D)) = 2$, that is, $U \simeq H^0(L(-D))$, then the image of d_1 is equal to the ideal J_D. Thus all the homological conditions of Theorem 1 are satisfied, except for the Koszul property of an algebra A. At this step we use the following result of Kempf [14].

THEOREM 5 [14]. *Homogeneous coordinate algebra of the finite set of d distinct points in a general linear position in $\mathbf{P}^{d-p}$ is Koszul, provided that $p \leq d/2$.*

Remark. In the case $p \leq 3$ this is particularly easy; the points lie on a rational normal curve (see [12]) so the statement can be deduced easily from the main theorem of [4] (see also [9]). Note also that if $p = 1$ (resp. $p = 2$) then the set of points in question is a hyperplane section of a rational (resp. an elliptic) normal curve, so the Koszul property in these two cases follows from the same property of the coordinate algebra of a rational (resp. an elliptic) normal curve. So we can avoid the reference to the results above if we are interested in the case $h^1(L) \leq 1$ only.

Now we are ready to prove our main theorem.

THEOREM 6. *Let L be a very ample linear bundle on C of degree $\deg L \geq g + 3$ such that the corresponding embedding of C into $\mathbf{P}(H^0(L)^*)$ is projectively normal. Assume that there exists a divisor $D = P_1 + \cdots + P_d$ of the degree $d = \deg L - g - 1 + 2h^1(L)$ such that the linear series $|D|$ and $|L(-D)|$ are base-point-free of the dimensions $\deg L - 2g + 4h^1(L) - 1$ and 1, correspondingly. Assume also that $h^1(L(D)) = h^1(L^2(-D)) = 0$ and that any $h^1(L)$ points of D impose independent conditions on $K \otimes L^{-1}(D)$. Then algebra R_L is Koszul.*

Proof. By the Riemann–Roch theorem [12] we obtain that $h^1(D) = h^1(L(-D)) = 2h^1(L)$, so it follows from the discussion above that we only have to check the Koszul property of the set of points $P_1, \ldots, P_d$ embedded by L. The dimension of the linear subspace spanned by these points is equal to $h^0(L) - 3 = d - h^1(L) - 1$. The condition $p = h^1(L) + 1 \leq d/2$ is satisfied by the assumption, so by Theorem 5 it is sufficient to verify that any $d - h^1(L)$ of $P_1, \ldots, P_d$ impose independent conditions on $|L|$. But this is equivalent to the property we have assumed that any $h^1(L)$ of them impose independent conditions on $K \otimes L^{-1}(D)$. ∎

Remark. The condition of the theorem concerning independence of any $h^1(L)$ points is satisfied automatically if $h^1(L) \leq 1$. If $h^1(L) = 2$ and $\deg L = 2g - 6$ so that the dimension of $|D|$ is 1 then the sufficient condition is that $K \otimes L^{-1}(2D)$ is very ample.

COROLLARY 1. *If C is a non-hyperelliptic, non-trigonal curve which is not a plane quintic then the canonical algebra R_K is Koszul.*

Proof. It follows from Green and Lazarsfeld's lemma [11] and the theorem above.

COROLLARY 2. *For any curve of genus g and any linear bundle L of degree $\geq 2g+2$ algebra R_L is Koszul.*

Proof. In this case $h^1(L) = 0$ and we can choose D as above in the linear system $L(-P_1 - \cdots - P_{g+1})$ for general $g+1$ points $P_1, \ldots, P_{g+1}$. Note also that the Koszul property in this case follows trivially from Kempf's theorem [14] applied to a general hyperplane section of C. ∎

COROLLARY 3. *Under the assumptions of the theorem the following natural map is surjective:*

$$H^0(L) \otimes H^0(D) \to H^0(L(D)).$$

Also if we define a vector bundle M_D from the exact triple

$$0 \to M_D \to H^0(D) \otimes \mathcal{O} \to \mathcal{O}(D) \to 0$$

then $M_D \otimes L$ is generated by global sections.

Proof. Both statements follow from the second part of Theorem 4. ∎

It remains to analyze the case $h^1(L) \geq 2$ of our theorem. Put $L = K(-A)$, $\dim |A| = r$. Then the condition $h^0(D) \geq 2$ implies the following inequality: $\deg A \leq 4r$. On the other hand, considering the natural map

$$|A| \times |D| \to |K \otimes L^{-1}(D)|,$$

we obtain another inequality,

$$r + h^0(D) - 1 \leq h^1(L(-D)) - 1,$$

which is equivalent to $\deg A \geq 3r$. Also considering the map

$$|D| \times |L(-D)| \to |L|,$$

we obtain the restriction $r \leq g/3 - 1$. In the case $r = 1$ we obtain from the above inequalities that $|A|$ is either trigonal or tetragonal system. Furthermore, it is easy to see that the former case is impossible so $A = T$ is a tetragonal system. One can check that T should be base-point-free, otherwise C fails to be cut out (even set-theoretical) by quadrics in the embedding defined by $|L|$. Also the fact that L is very ample implies that C is not hyperelliptic. To satisfy the conditions of the theorem we should have a decomposition of L into the sum of two divisors of degree $g-3$, each defining the base-point-free linear system of dimension 1. In the next section we will give some examples when this situation occurs.

Remark. So far I do not know much about the case $h^0(L) > 2$. I hope that there should be examples when the above technique applies to this case too.

4. TETRAGONAL CURVES

In this section we study the case of the embedding of a tetragonal curve C by the complete linear system $|K(-T)|$, where T is a (base-point-free) tetragonal series. As a tetragonal series is not unique in general it is natural to consider the moduli space $\mathscr{M}_g^t$ of pairs (C, T), where T is such a series on a non-hyperelliptic curve C of genus g. By the well-known construction (see [18]) T gives an embedding of C into three-dimensional rational normal scroll X. Furthermore, it is easy to see that C is a complete intersection of two divisors on X. Thus there is a stratification of $\mathscr{M}_g^t$ by the type of scroll X and the type of complete intersection, and all the strata are irreducible. Note that the Hilbert series of $R_{K(-T)}$ is constant over $\mathscr{M}_g^t$, so by the well-known result the set of pairs (C, T) for which $R_{K(-T)}$ is Koszul is an intersection of countably many open subsets in $\mathscr{M}_g^t$. The problem to be solved is to find all the strata $\mathscr{N}$ such that for a general pair $(C, T) \in \mathscr{N}$ the algebra $R_{K(-T)}$ is Koszul (we say that such a stratum is Koszul). Here "general" means "in the complement of countably many proper subvarieties" so a stratum satisfies this property if it contains at least one such pair. Note that if a stratum $\mathscr{N}_1$ is Koszul and it is contained in the closure of a stratum $\mathscr{N}_2$ then $\mathscr{N}_2$ is also Koszul. The analogous problem for quadratic algebras is easy and we will see that the most degenerate (with respect to complete intersection type) quadratic strata are Koszul, so it is natural to expect that this is true in general. Also we construct examples of pairs (C, T) for which the algebra $R_{K(-T)}$ is Koszul on some other strata, considering ramified double coverings of hyperelliptic curves and applying the method of the previous section. As a consequence we obtain that for a general tetragonal curve of genus $g \geq 9$ the algebra $R_{K(-T)}$ is Koszul (note that for a general tetragonal curve T is unique).

We begin with recalling the construction of a three-dimensional rational normal scroll containing tetragonal curve C. Let T be a base-point-free tetragonal system on C. It defines a 4-sheeted covering $\pi: C \to \mathbf{P}^1$. Applying the relative duality to the canonical non-vanishing section $\mathscr{O} \to \pi_* \mathscr{O}$ we obtain the surjective homomorphism $\pi_* K_C \to \mathscr{O}(-2)$. Let V be its kernel so that we have the following exact triple on $\mathbf{P}^1$:

$$0 \to V \to \pi_* K_C \to \mathcal{O}(-2) \to 0.$$

It follows that the homomorphism $V \to \pi_* K_C$ induces an isomorphism of global sections and therefore the corresponding homomorphism $\pi^* V \to K_C$ is surjective. Thus we obtain a morphism $\phi: C \to X$, where $X = \mathbf{P}(V^\vee)$ such that $\phi^* \mathcal{O}_X(1) \simeq K_C$. Note that it follows from the exact triple above that $h^0(V(-i)) = h^0(K_C(-iT))$. In particular, $h^0(V) = g$, $h^0(V(-1)) = g - 3$ so that $V \geq 0$ in the sense that all linear direct summands in V have the form $\mathcal{O}(l)$ with $l \geq 0$. Therefore $\mathcal{O}_X(1)$ is base-point-free and defines a morphism from X to $\mathbf{P}^{g-1}$, inducing the canonical morphism by composition with ϕ. It follows that ϕ is an embedding (we have assumed that C is non-hyperelliptic). Furthermore, $h^0(V(-2)) = g - 6$ if and only if $h^0(2T) = 3$ and assuming that we obtain that $V(-1) \geq 0$ and, hence, $\mathcal{O}_X(1)$ is very ample.

Let $p: X \to \mathbf{P}^1$ be the projection. The push forward by p of the natural homomorphism $\mathcal{O}(2) \to \mathcal{O}(2)|_C$ gives rise to a homomorphism $f: S^2 V \to \pi_* K_C^2$, where $S^2 V \simeq p_* \mathcal{O}(2)$ is the second symmetric power of V. We claim that f is surjective. Indeed, it suffices to check this pointwise so we have to verify that for each $q \in \mathbf{P}^1$ the four points of $\pi^{-1}(q)$ impose independent conditions on quadrics in the corresponding projective plane $p^{-1}(q)$. But this is true because these points span that plane by the geometric form of the Riemann–Roch theorem [12]. Let us denote the kernel of f by E. This is a rank-2 vector bundle on $\mathbf{P}^1$ of degree $g - 5$ and it fits in the following exact sequence

$$0 \to E \to S^2 V \to \pi_* K^2 \to 0.$$

Now we claim that the corresponding homomorphism $p^* E \to \mathcal{O}_X(2)$ vanishes exactly along $C \in X$. Indeed, as any divisor in $|T|$ contains four points no three of which lie on a line, they are cut out by two quadrics in the corresponding plane. More than that, comparing arithmetical genera we conclude that C as a subscheme of X coincides with the zero-locus of the corresponding regular section of $p^* E^\vee(2)$. Now $E \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$, where $a + b = g - 5$, so C is in fact a complete intersection of two divisors $S_1 \in |\mathcal{O}_X(2)(-aH)|$ and $S_2 \in |\mathcal{O}_X(2)(-bH)|$, where $H = p^* \mathcal{O}(1)$.

Assume now that $K_C(-T)$ is projectively normal. Then the homomorphism $S^2 H^0(K_C(-T)) \to H^0(K_C^2)$ is surjective. As $V(-1) \geq 0$ it follows that the natural homomorphism $S^2 H^0(V(-1)) \to H^0(S^2 V(-2))$ is surjective too. Hence the homomorphism $S^2 V(-2) \to \pi_*(K_C^2)(-2)$ induces a surjection on global sections and consequently $H^1(E(-2)) = 0$ that is $a, b \geq 1$. Conversely, it easy to see that if C is a complete intersection as above with $a, b \geq 1$ then $K_C(-T) = (\mathcal{O}_X(1)(-H))|_C$ is projectively normal. We gather these results in the following proposition.

PROPOSITION 7. *Let C be a non-hyperelliptic curve of genus g with a base-point-free tetragonal system T on it. Then we can present C as a complete intersection of two divisors from the linear systems $|\mathcal{O}_X(2)(-aH))|$ and $|\mathcal{O}_X(2)(-bH)|$ on a three-dimensional rational normal scroll $X = \mathbf{P}(V^{\vee})$, where $V \geq 0$ is a rk-3 vector bundle of degree $g - 3$ on $\mathbf{P}^1$, in such a way that $T = H|_C$ (here H is the pull-back of $\mathcal{O}(1)$ from $\mathbf{P}^1$, $a + b = g - 5$). Furthermore, $h^0(2T) = 3$ if and only if $V(-1) \geq 0$ and if this condition is satisfied then $K_C(-T)$ is projectively normal if and only if $a \geq 1$, $b \geq 1$.*

Remark. Note that if $K_C(-T)$ is very ample then $a, b \geq 0$. In any case $a, b \geq -1$.

Our next remark is that under notations of the previous proposition $R_{K_C(-T)}$ is quadratic if and only if $a, b \geq 2$ (this is very easy to verify using exact sequences of the restriction to a divisor). So it makes the following conjecture reasonable.

Conjecture 1. Under assumptions and notations above, if $a, b \geq 2$ then the algebra $R_{K_C(-T)}$ is Koszul.

Remark. It is easy to see that this is true at least when $a = 2$ or $b = 2$. Indeed, first we can prove that the coordinate algebra of any divisor $S \in |\mathcal{O}(2)(-aH)|$ under the embedding defined by $\mathcal{O}(1)(-H)$ is Koszul, provided that $a \geq 2$. The reason is that its hyperplane section is a curve of genus $g - 5 - a$ embedded by the complete linear system of degree $2g - 10 - a$ so we can apply Corollary 2 above. Then our curve C is an intersection of S with a quadric so its coordinate algebra is Koszul by the result of [5]. This proves the conjecture for $g = 9, 10$. To prove the Koszul property for a general pair (C, T) of some stratum with $a, b \geq 2$ it would be sufficient to prove that this stratum can be degenerated into one with $a = 2$ or $b = 2$.

Our method of proving the Koszul property suggests the following conjecture which implies the previous one.

Conjecture 2. Under the same assumptions there exists a decomposition of $K_C(-T)$ into the sum of two base-point-free pencils of degree $g - 3$.

Now we consider the specific case when our tetragonal curve C is a double covering of a hyperelliptic curve. We are going to construct some examples when the required decomposition of $K_C(-T)$ into the sum of two pencils exists. For this we will use the well-known connection between linear bundles over the double covering and rk-2 bundles with a Higgs field (which is a twisted endomorphism of a bundle) over the base of the covering (see, for example, [13]).

So let C_h be the hyperelliptic curve of genus g_h, $\pi_h\colon C_h \to \mathbf{P}^1$ be the corresponding double covering, $\Gamma = \pi_h^*\mathscr{O}(1)$ be the hyperelliptic system. Then $K_h = \mathscr{O}((g_h - 1)\Gamma)$ is the canonical class of C_h. Now we consider a Higgs bundle (F, ϕ), where $F = \mathscr{O}(D_1) \oplus \mathscr{O}(D_2)$, D_i being the divisors of degree $g_h - 2$ and $\phi\colon F \to F(M)$ is a homomorphism defined by the sections $s_1 \in H^0(M(D_2 - D_1))$ and $s_2 \in H^0(M(D_1 - D_2))$ (other entries of ϕ being zero)—here M is some linear bundle which has a sufficiently large degree to be estimated later. These data define the line bundle $\mathscr{O}(D)$ over the double covering $\pi\colon C \to C_h$ such that $\pi_*\mathscr{O}(D) \simeq F$. Simple computation shows that $\deg D = g - 3$, where $g = 2g_h - 1 + \deg M$ is the genus of C. Now we look under what conditions $|D|$ is a base-point-free linear system of dimension 1. First we should have that F is globally generated outside the ramification divisor $s = s_1 s_2 \in H^0(M^2)$. Hence $h^0(D_1) = h^0(D_2) = 1$ and the unique divisor of $|D_i|$ is contained in the zero divisor of s. Furthermore, at the point x of ramification, global sections of F should generate the unique ϕ-invariant one-dimensional quotient of the stalk F_x. It is easy to see that these conditions are satisfied if we put $s_i = u_i t_i$, where $u_i \in H^0(D_i)$ are non-zero sections ($i = 1, 2$), $t_1 \in H^0(M(D_2 - 2D_1))$, $t_2 \in H^0(M(D_1 - 2D_2))$, provided that the zero divisors of u_1, u_2, t_1, t_2 are all disjoint. Note that the change of D by $K_C(-T - D)$, where $T = \pi^*\Gamma$ leads to the change of D_i by $(g_h - 2)\Gamma - D_i$ with essentially the same Higgs field. In particular if we choose $D_2 = (g_h - 2)\Gamma - D_1$ then we will have $\mathscr{O}(2D) \simeq K_C(-T)$. Now it is clear that if M is a general linear bundle of degree at least $2g_h - 1$ then we can find (F, ϕ) as above such that corresponding divisor D on C satisfies the conditions of Theorem 2. Indeed, then $\deg M(D_2 - 2D_1) \geq g_h + 1$ and we can use the fact that general bundle of degree $\geq g_h + 1$ is base-point-free. Also, as we have mentioned above under suitable choices we'll have $\mathscr{O}(2D + T) \simeq K_C$ which implies the last condition of Theorem 2 (see the remark after it).

Now in order to verify projective normality of $K_C(-T)$ we have to compute the discrete invariants described above (namely the bundles V and E on $\mathbf{P}^1$) of the pair (C, T) obtained in this way. First, we have the canonically splitting exact triple

$$0 \to K_h \otimes M \to \pi_* K_C \to K_h \to 0.$$

Now V is the kernel of composition

$$(\pi_h)_*(\pi_* K_C) \to (\pi_h)_* K_h \to \mathscr{O}(-2).$$

It follows that V fits into the splitting exact sequence

$$0 \to (\pi_h)_* M(g_h - 1) \to V \to \mathscr{O}(g_h - 1) \to 0.$$

Now it is easy to check that $(\pi_h)_*\pi_*(K_C^2) \simeq ((\pi_h)_*(M^2) \oplus (\pi_h)_*M)(2g_h - 2)$ and the natural map

$$S^2((\pi_h)_*M \oplus \mathscr{O}) \simeq S^2V(-2g_h + 2) \to (\pi_h)_*\pi_*(K_C^2)(-2g_h + 2)$$
$$\simeq (\pi_h)_*(M^2) \oplus (\pi_h)_*M$$

is induced by the natural maps $\psi\colon S^2((\pi_h)_*M) \to (\pi_h)_*(M^2)$, $\mathscr{O} \to (\pi_h)_*(M^2)$ and the identity map of $(\pi_h)_*M$. It follows that there is an exact sequence

$$0 \to \ker\psi \to E(-2g_h + 2) \to \mathscr{O} \to 0,$$

provided that ψ is surjective. Assume that $\deg M \geq 2g_h + 1$ then ψ is surjective and $\ker\psi \simeq \mathscr{O}(\deg M - 2g_h - 2)$; hence we obtain that $E(-2g_h + 2) \simeq \mathscr{O} \oplus \mathscr{O}(\deg M - 2g_h - 2)$. At last, note that as M is general the splitting type of $(\pi_h)_*M$ is either (i, i) or $(i, i + 1)$, depending on the parity of deg M.

Summarizing the discussion above we obtain the following statement.

THEOREM 8. *Let $\mathscr{N}$ be the stratum of the moduli space $\mathscr{M}_g^t$ $(g \geq 9)$ with one of the following splitting types of V and E:*

1. $(g_h - 1, g_h - 1 + i, g_h - 1 + i)$ *and* $(2g_h - 2, g_h - 3 + 2i)$, *where* $g = 3g_h + 2i$, $i \geq g_h/2$, $g_h \geq 2$;
2. $(g_h - 1, g_h - 2 + i, g_h - 1 + i)$ *and* $(2g_h - 2, g_h - 4 + 2i)$, *where* $g = 3g_h + 2i - 1$, $i \geq (g_h + 1)/2$, $g_h \geq 2$.

Then for general $(C, T) \in \mathscr{N}$ algebra $R_{K(-T)}$ is Koszul.

Remark. The condition $g \geq 9$ excludes the case $g_h = 2$, $i = 1$ in 1).

COROLLARY 4. *For a general tetragonal curve of genus $g \geq 9$ algebra $R_{K(-T)}$ is Koszul.*

Proof. It is sufficient to note that the pairs (C, T) constructed above (for which this algebra is Koszul) lie in the open subset U of $\mathscr{M}_g^t$ for which $a, b \geq 1$ (notation as above), and it is known (see [18]) that for such curves C the series T is unique. Hence U embeds as an open subset in the locus of tetragonal curves inside $\mathscr{M}_g$ which is irreducible, so U itself is irreducible. ∎

5. ON REGULARITY OF MODULES OVER A COMMUTATIVE KOSZUL ALGEBRA

In this section we give a geometric bound for the regularity of a module over a commutative Koszul algebra. The result is not new but our proof seems to be very simple so we present it here.

Let X be a projective scheme, $L = \mathcal{O}_X(1)$ be a very ample line bundle on X such that the corresponding algebra $R = R_L$ is Koszul. For a sheaf F on X we denote tensor product of F with the nth power of L by $F(n)$. With these assumptions we have the following.

THEOREM 9. *For any coherent sheaf F on X if $H^i(F(-i)) = 0$ for any $i > 0$ then the corresponding R-module $M = \bigoplus_{i \geq 0} H^0(F(i))$ has a linear free resolution that is a resolution of the form*

$$\cdots \to V_2 \otimes R(-2) \to V_1 \otimes R(-1) \to V_0 \otimes R \to M \to 0,$$

where V_i are some vector spaces.

Proof. Koszul property of R means that there is a resolution of the trivial module (of degree zero) which has form

$$\cdots \to Q_2 \otimes R(-2) \to Q_1 \otimes R(-1) \to Q_0 \otimes R \to k \to 0,$$

where $Q_0 = k$. It induces the following exact sequence of sheaves on X:

$$\cdots \to Q_2 \otimes \mathcal{O}(-2) \to Q_1 \otimes \mathcal{O}(-1) \to \mathcal{O} \to 0.$$

Now we can tensor it by $F(n)$ and consider the corresponding spectral sequence computing hypercohomology

$$E_1^{p,q} = Q_{-p} \otimes H^q(F(i+p)) \Rightarrow 0$$

with differentials d_r of bidegree $(r, -r+1)$. Now F is 0-regular in the sense of Castelnuovo and Mumford (see [16]) so we have $E_1^{p,q} = 0$ if $p + q \geq -i$, $q \geq 1$. It follows that the complex $E_1^{\cdot,0}$ is exact in terms $p > -i$. But this complex computes syzigies of M, namely its cohomology in pth term is $\mathrm{Tor}_p^R(k, M)(i + 2p)$. So we have proved that $\mathrm{Tor}_p(k, M)_j = 0$ for $j > p$ which is equivalent to the existence of linear free resolution in question. ∎

Remark. The statement of this theorem is essentially equivalent to the statement of the main theorem of [2] for the case of Koszul algebras R which have form R_L. Indeed, that theorem asserts that the regularity of a module M over R is bounded by its regularity as a module over a symmetric algebra S which surjects onto R. We have proved that for a modules which come from coherent sheaves the regularity is bounded by its geometric counterpart—the regularity in the sense of Castelnuovo and Mumford. Note that the regularity of an arbitrary module (not necessary coming from coherent sheaf) can be bounded easily using the bound for that special type of modules. Now the point is that for the case of symmetric algebra these two regularities are equal so we arrive to the formulation of [2].

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