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Dimensions, matroids, and dense pairs of first-order structures

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ABSTRACT

A structure M is pregeometric if the algebraic closure is a pregeometry in all structures elementarily equivalent to M. We define a generalisation: structures with an existential matroid. The main examples are superstable groups of Lascar U-rank a power of ω and d-minimal expansion of fields. Ultraproducts of pregeometric structures expanding an integral domain, while not pregeometric in general, do have a unique existential matroid.

Generalising previous results by van den Dries, we define dense elementary pairs of structures expanding an integral domain and with an existential matroid, and we show that the corresponding theories have natural completions, whose models also have a unique existential matroid. We also extend the above result to dense tuples of structures.

1. Introduction

A theory T is called pregeometric [14,13] if, in every model $\mathbb{K}$ of T, acl satisfies the Exchange Principle, denoted by EP (and, therefore, acl is a pregeometry on $\mathbb{K}$); if T is complete, it suffices to check that acl satisfies the EP in one ω-saturated model of T. The theory T is geometric if it is pregeometric and eliminates the quantifier $\exists^\infty$. We call a structure $\mathbb{K}$ (pre)geometric if its theory is (pre)geometric (thus, $\mathbb{K}$ is pregeometric iff there exists an ω-saturated elementary extension $\mathbb{K}'$ of $\mathbb{K}$ such that acl satisfies the EP in $\mathbb{K}'$). Note that a pregeometric expansion of a field is geometric ([10, 1.18]; see also Lemma 3.47).

In the remainder of this introduction, all theories and all structures expand a field; in the body of the article we will sometimes state definitions and results without this assumption.

Geometric structures are ubiquitous in model theory: if $\mathbb{K}$ is either o-minimal, or strongly minimal, or a p-adic field, or a pseudo-finite field (or more generally a perfect PAC field; see [9,14, 2.12]), then $\mathbb{K}$ is geometric.

However, ultraproducts of geometric structures (even strongly minimal ones) are not geometric in general. We will show that there is a more general notion, structures with existential matroids, which instead is preserved under taking ultraproducts. In more detail, we consider structures $\mathbb{K}$ with a matroid cl that satisfies some natural conditions (cl is an "existential matroid"). Our assumption that $\mathbb{K}$ expands a field implies that there is at most one existential matroid on $\mathbb{K}$. An (almost) equivalent notion has already been studied by van den Dries [25]: we will show that an existential matroid on $\mathbb{K}$ induces a (unique) dimension function on $\mathbb{K}$-definable sets, satisfying the axioms in [25], and conversely, any such dimension function, satisfying a slightly stronger version of the axioms, will be induced by a (unique) existential matroid. Moreover, a superstable group $\mathbb{K}$ of U-rank a power of ω is naturally endowed by an existential matroid (van den Dries [25, 2.25] noticed this already in the case when $\mathbb{K}$ is a differential field of characteristic 0).

Given a geometric structure $\mathbb{K}$, there is an abstract notion of dense subsets of $\mathbb{K}$, which specialises to the usual topological notion in the case of o-minimal structures or of p-adic fields. More precisely, a subset X of $\mathbb{K}$ is dense in $\mathbb{K}$ if every *infinite* $\mathbb{K}$-definable subset of $\mathbb{K}$ intersects X [16, Section 1.2]. If T is a complete geometric theory, then the theory of dense

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elementary pairs of models of T is complete and consistent (the proof of this fact was already in [26], but the result was stated there only for o-minimal theories).

We consider here the more general case when T is a complete theory with an existential matroid. We show that there is a corresponding abstract notion of density in models of T. Given T as above, consider the theory of pairs $\langle \mathbb{K}', \mathbb{K} \rangle$, where $\mathbb{K} \prec \mathbb{K}' \models T$ and $\mathbb{K}$ is a proper dense subset of $\mathbb{K}'$; the theory of such pairs will not be complete in general, but we will show that it will become complete (and consistent) if we add the additional condition that $\mathbb{K}$ is cl-closed in $\mathbb{K}'$ (that is, $\mathrm{cl}(\mathbb{K}) \cap \mathbb{K}' = \mathbb{K}$); we thus obtain the (complete) theory T^d. Moreover, T^d also has an existential matroid. This allows us to repeat the above construction, and consider dense cl-closed pairs of models of T^d, which turn out to coincide with nested dense cl-closed triples of models of T; iterating many times, we can thus study nested dense cl-closed n-tuples of models of T.

Of particular interest are two cases of structures with an existential matroid: cl-minimal structures and d-minimal topological structures.

A structure $\mathbb{K}$ (with an existential matroid) is cl-minimal if there is only one "generic" 1-type over every subset of $\mathbb{K}$ (see Section 10); the prototypes of such structures are given by strongly minimal structures and connected superstable groups of U-rank a power of ω. If T is the theory of $\mathbb{K}$, we show that the condition that $\mathbb{K}$ is dense in $\mathbb{K}'$ is superfluous in the definition of T^d, and that T^d is also cl-minimal.

A first-order topological structure $\mathbb{K}$ (expanding a topological field) is d-minimal if it is Hausdorff, it has an ω-saturated elementary extension $\mathbb{K}'$ such that every definable unary subset of $\mathbb{K}'$ is the union of an open set and finitely many discrete sets, and it satisfies a version of the Kuratowski–Ulam theorem for definable subsets of $\mathbb{K}^2$ (see Section 9; the "d" stands for "discrete"). Examples of d-minimal structures are p-adic fields, o-minimal structures, and d-minimal structures in the sense of Miller. We show that a d-minimal structure has a (unique) existential matroid, and that the notion of density given by the matroid coincides with the topological one. Moreover, if T is the theory of a d-minimal structure, then T^d is the theory of dense elementary pairs of models of T (the condition that $\mathbb{K}$ is a cl-closed subset of $\mathbb{K}'$ is superfluous); hence, in the case when T is o-minimal, we recover [26, Theorem 2.5]. However, if T is d-minimal, T^d will not be d-minimal. Moreover, while ultraproducts of o-minimal structures and of p-adic fields are d-minimal, ultraproducts of d-minimal structures are not d-minimal in general. Under some mild assumptions, if $\langle \mathbb{K}', \mathbb{K} \rangle$ is a dense pair of d-minimal structures, then $\mathbb{K}'$ is the open core of $\langle \mathbb{K}', \mathbb{K} \rangle$ (Theorem 13.11).

We show that, if $\mathbb{K}$ has an existential matroid, then $\mathbb{K}$ is a perfect field; therefore, the theory exposed in this article does not apply to differential fields of finite characteristic, or to separably closed nonperfect fields.

2. Notations and conventions

Let T be a complete theory in some language $\mathcal{L}$, with only infinite models. Let $\kappa > |T|$ be a "big" cardinal. We work inside a κ-saturated and strongly κ-homogeneous model $\mathbb{M}$ of T; we call $\mathbb{M}$ a monster model of T.

We denote by A, B, and C, subsets of $\mathbb{M}$ of cardinality less than κ, by $\bar{a}$, $\bar{b}$, and $\bar{c}$, finite tuples of elements of $\mathbb{M}$, and by a, b, and c, elements of $\mathbb{M}$. As usual, we will write, for instance, $\bar{a} \subset A$ to say that $\bar{a}$ is a finite tuple of elements of A, and $A\bar{b}$ to denote the union of A with the set of elements in $\bar{b}$.

Given a set X and $m \leq n \in \mathbb{N}$, denote by $\Pi^n_m : X^n \to X^m$ the projection onto the first m coordinates. Given $Y \subseteq X^{n+m}$, $\bar{x} \in X^n$, and $\bar{z} \in X^m$, denote the sections $Y_{\bar{x}} := \left\{\bar{t} \in X^m : \langle \bar{x}, \bar{t} \rangle \in Y\right\}$ and $Y^{\bar{z}} := \left\{\bar{t} \in X^n : \langle \bar{t}, \bar{z} \rangle \in Y\right\}$.

Denote by $\mathrm{Aut}(\mathbb{M}/B)$ the set of automorphisms of $\mathbb{M}$ which fix B point-wise. Denote by $\Xi(a/B)$ the set of conjugates of a over B; that is,

$$\Xi(a/C) := \{a^\sigma : \sigma \in \mathrm{Aut}(\mathbb{M}/B)\}.$$

3. Matroids

Let cl be a (finitary) closure operator on $\mathbb{M}$; that is, $\mathrm{cl} : \mathcal{P}(\mathbb{M}) \to \mathcal{P}(\mathbb{M})$ satisfies, for every $X \subseteq \mathbb{M}$,

Extension: $X \subseteq \mathrm{cl}(X)$;
Monotonicity: $X \subseteq Y$ implies that $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$;
Idempotency: $\mathrm{cl}(\mathrm{cl}\, X) = \mathrm{cl}(X)$;
Finite Character: $\mathrm{cl}(X) = \bigcup \{\mathrm{cl}(A) : A \subseteq X \;\&\; A \text{ finite}\}$.

The closure operator cl is a (finitary) **matroid** (a.k.a. **pregeometry**) if, moreover, it satisfies the Exchange Principle.

EP: $a \in \mathrm{cl}(Xc) \setminus \mathrm{cl}(X)$ implies $c \in \mathrm{cl}(Xa)$.

When $\mathbb{M}$ is not clear from the context, we will write $\mathrm{cl}^{\mathbb{M}}$ instead of cl.

Notice that the closure of a set A such that $|A| < \kappa$ might be a "proper class", that is, it might have cardinality $\geq \kappa$, and that this will indeed happen in many important examples in this article (more precisely, it will happen for all the existential matroids different from the algebraic closure).

Proviso. *For the remainder of this section,* cl *is a finitary matroid on* $\mathbb{M}$.

As is well known from matroid theory, cl defines notions of rank (which we denote by $\mathrm{rk}^{\mathrm{cl}}$), generators, independence, and basis (see e.g. [23, Appendix C]).[1]

Definition 3.1. A subset A of C **generates** C over B if $\mathrm{cl}(AB) = \mathrm{cl}(CB)$. A subset A of $\mathbb{M}$ is **independent** over B if, for every $a \in A$, $a \notin \mathrm{cl}\big(B \cup (A \setminus \{a\})\big)$.

Remark 3.2 (*Additivity of Rank*).

$$\mathrm{rk}^{\mathrm{cl}}(\bar{a}\bar{b}/C) = \mathrm{rk}^{\mathrm{cl}}(\bar{a}/\bar{b}C) + \mathrm{rk}^{\mathrm{cl}}(\bar{b}/C).$$

For the axioms of independence relations, we will use the nomenclature in [1].

Definition 3.3. Given an infinite set X, a **preindependence relation**[2] on X is a the ternary relation $\mathop{\smile\!\!\!\!\!\mid}\,$ on $\mathscr{P}(X)$ satisfying the following axioms.

Monotonicity: If $A \mathop{\smile\!\!\!\!\!\mid}_C B$, $A' \subseteq A$, and $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\mid}_C B'$.
Base Monotonicity: If $D \subseteq C \subseteq B$ and $A \mathop{\smile\!\!\!\!\!\mid}_D B$, then $A \mathop{\smile\!\!\!\!\!\mid}_C B$.
Transitivity: If $D \subseteq C \subseteq B$, $B \mathop{\smile\!\!\!\!\!\mid}_C A$, and $C \mathop{\smile\!\!\!\!\!\mid}_D A$, then $B \mathop{\smile\!\!\!\!\!\mid}_D A$.
Normality: If $A \mathop{\smile\!\!\!\!\!\mid}_C B$, then $AC \mathop{\smile\!\!\!\!\!\mid}_C B$.
Finite Character: If $A_0 \mathop{\smile\!\!\!\!\!\mid}_C B$ for every finite $A_0 \subseteq A$, then $A \mathop{\smile\!\!\!\!\!\mid}_C B$.

$\mathop{\smile\!\!\!\!\!\mid}\,$ is **symmetric** if, moreover, it satisfies the following axiom.

Symmetry: $A \mathop{\smile\!\!\!\!\!\mid}_C B$ iff $B \mathop{\smile\!\!\!\!\!\mid}_C A$.

Definition 3.4. The preindependence relation on $\mathbb{M}$ induced by cl is the ternary relation $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}$ on $\mathscr{P}(\mathbb{M})$ defined by the following: $X \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_Y Z$ if, for every $Z' \subset Z$, if Z' is independent over Y, then Z' remains independent over YX. If $X \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_Y Z$, we say that X and Z are independent over Y (w.r.t. cl).

Remark 3.5. If $X \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_Y Z$, then $\mathrm{cl}(XY) \cap \mathrm{cl}(ZY) = \mathrm{cl}(Y)$.

Lemma 3.6. *The relation $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}$ is a symmetric preindependence relation.*

Proof. The same as that given in [1, Lemma 1.29]. $\square$

Remark 3.7. The relation $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}$ also satisfies the following version of **antireflexivity**.

- $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_C B$ iff $\mathrm{cl}(A) \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_{\mathrm{cl}(C)} \mathrm{cl}(B)$;
- $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_X a$ iff $a \in \mathrm{cl}(X)$.

Remark 3.8. For every X and Y, $X \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_Y Y$.

Remark 3.9. T.f.a.e.:

1. $X \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_Y Z$;
2. for every Z' such that $Y \subseteq Z' \subseteq \mathrm{cl}(YZ)$, we have $\mathrm{cl}(XZ') \cap \mathrm{cl}(YZ) = \mathrm{cl}(Z')$;
3. there exists $Z' \subseteq Z$ which is a basis of ZY/Y, such that Z' remains independent over XY;
4. for every $Z' \subseteq Z$ which is a basis of ZY/Y, Z' remains independent over XY;
5. if $X' \subseteq X$ is a basis of YX/Y and $Z' \subseteq Z$ is a basis of YZ/Y, then X' and Z' are disjoint, and $X'Z'$ is a basis of XZ over Y;
6. for every X' finite subset of X, $\mathrm{rk}^{\mathrm{cl}}(X'/YZ) = \mathrm{rk}^{\mathrm{cl}}(X'/Y)$.

Lemma 3.10. *The preindependence relation $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}$ also satisfies the following stronger form of the **Local Character** axiom.*

For every A and B there exists a subset C of B such that $|C| \leq |A|$ and $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_C B$.

Proof. Let A and B be given. Let $B' \subseteq B$ be a basis of AB over A, $A' \subseteq A$ be a basis of AB over B, and $C \subseteq B$ be a basis of B over B'. Notice that CA' is a basis of AB/B' and A is a set of generators of AB/B'; hence, by the EP, $|C| \leq |A|$. Moreover, by Remark 3.9(3), $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_C B$. $\square$

Lemma 3.11. *Assume that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_C \bar{d}$ and that $\bar{a}\bar{d} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_C \bar{b}$. Then, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_C \bar{b}\bar{d}$ and $\bar{d} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_C \bar{b}\bar{a}$.*

Proof. Cf. [1, 1.9]. Since $\bar{a}\bar{d} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_C \bar{b}$, we have $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_{C\bar{d}} \bar{b}$, which implies that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_{C\bar{d}} \bar{b}\bar{d}$, which, together with $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_C \bar{d}$, implies that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}_C \bar{b}\bar{d}$. $\square$

1 Sometimes in geometric model theory the "rank" is called "dimension" and/or the "dimension" (defined later) is called "rank"; however, since in many interesting cases (e.g. algebraically closed fields and o-minimal structures, with the acl matroid) what we call the dimension of a definable set induced by the matroid coincides with the usual notion of dimension given geometrically, our choice of nomenclature is clearly better.

2 Preindependence relations as defined here are slightly different than the ones defined in [1]. However, as we will see later, if cl is definable, then $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}$ is a preindependence relation in Adler's sense.

Lemma 3.12. *Let $\langle I, \leq\rangle$ be a linearly ordered set, $\left(\bar{a}_i : i \in I\right)$ be a sequence of tuples in $\mathbb{M}^n$, and $C \subset \mathbb{M}$. Then, t.f.a.e.:*

1. *For every $i \in I$, we have $\bar{a}_i \mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} (\bar{a}_j : j < i)$;*
2. *For every $i \in I$, we have $\bar{a}_i \mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} (\bar{a}_j : j \neq i)$.*

Proof. Assume, for contradiction, that (1) holds, but $a_i \not\mathop{\smile\!\!\!\!\!\!\vert}\nolimits_{C} (\bar{a}_j : j \neq i)$, for some $i \in I$. Since $\mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}$ satisfies Finite Character, w.l.o.g., $I = \{1, \dots, m\}$ is finite. Let m' be such that $i < m' \leq m$ is minimal with $\bar{a}_i \not\mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} (\bar{a}_j : j \leq m' \,\&\, j \neq i)$; w.l.o.g., $m = m'$. Let $\bar{d} := (a_j : j \neq i \,\&\, j < m)$. By assumption, $\bar{a}_i \mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} \bar{d}$ and $\bar{d}\bar{a}_i \mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} \bar{a}_m$. Then, by Lemma 3.11, we have $\bar{a}_i \mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} \bar{d}\bar{a}_m$, which is absurd. $\square$

Definition 3.13. We say that a sequence $(\bar{a}_i : i \in I)$ satisfying one of the above equivalent conditions is an **independent sequence** over C.

Remark 3.14. Let $(a_i : i \in I)$ be a sequence of elements of $\mathbb{M}$. There is a clash of terminology with the previous definition of independence; more precisely, let $J := \{i \in I : a_i \notin \mathrm{cl}(C)\}$; then, $(a_i : i \in I)$ is an independent sequence over C according to $\mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}$ iff all the a_j are pairwise distinct for $j \in J$, and the set $\left\{a_j : j \in J\right\}$ is independent over C according to cl. Hopefully, this will not cause confusion.

3.1. Definable matroids

Definition 3.15. Let $\phi(x, \bar{y})$ be an $\mathcal{L}$-formula. We say that ϕ is x-**narrow** if, for every $\bar{b}$ and every a, if $\mathbb{M} \models \phi(a, \bar{b})$, then $a \in \mathrm{cl}(\bar{b})$ (cf. Remark 3.42). We say that cl is **definable** if, for every A,

$$\mathrm{cl}(A) = \bigcup \left\{\phi(\mathbb{M}, \bar{a}) : \phi(x, \bar{y}) \text{ is } x\text{-narrow}, \bar{a} \in A^n, n \in \mathbb{N}\right\}.$$

Proviso. *For the rest of the section,* cl *is a definable matroid.*

Remark 3.16. For every A and every $\sigma \in \mathrm{Aut}(\mathbb{M})$, $\sigma(\mathrm{cl}(A)) = \mathrm{cl}(\sigma(A))$.

Lemma 3.17. 1. *$\mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}$ satisfies the **Invariance** axiom: if $A \mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{B} C$ and $\langle A', B', C'\rangle \equiv \langle A, B, C\rangle$, then $A' \mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{B'} C'$.*

2. *$\mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}$ satisfies the following stronger form of the **Strong Finite Character** axiom: if $A \not\mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} B$, then there exist finite tuples $\bar{a} \subseteq A$, $\bar{b} \subseteq B$, and $\bar{c} \subseteq C$, and a formula $\phi(\bar{x}, \bar{y}, \bar{z})$ without parameters, such that*
   - *$\mathbb{M} \models \phi(\bar{a}, \bar{b}, \bar{c})$;*
   - *if $\bar{c}' \subseteq C$ and $\mathbb{M} \models \phi(\bar{a}', \bar{b}, \bar{c}')$, then $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} B$.*
3. *For every $\bar{a}$, B, and C, if $\mathrm{tp}(\bar{a}/BC)$ is finitely satisfied in C, then $\bar{a} \mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} B$.*

Proof. (1) By Remark 3.16.

(2) Assume that $A \not\mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} B$. Hence, there exists $\bar{b} \in B^n$ independent over C, such that $\bar{b}$ is not independent over AC. Hence, there exist $\bar{a} \subset A$ and $\bar{c} \subset C$ finite tuples, such that, w.l.o.g., $b_1 \in \mathrm{cl}(\bar{c}\bar{a}\tilde{b})$, where $\tilde{b} := \langle b_2, \dots, b_n\rangle$. Let $\alpha(x, \tilde{x}, \bar{y}, \bar{z})$ be an x-narrow formula, such that $\mathbb{M} \models \alpha(b_1, \tilde{b}, \bar{c}, \bar{a})$. If $\bar{a}' \subset \mathbb{M}$ and $\bar{c}' \subseteq C$ satisfy $\alpha(\bar{b}, \bar{c}', \bar{a}')$, then $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}_{C} B$.

(3) Follows as in [1, Remark 2.3]. $\square$

Definition 3.18 (*[1, Definition 1.1]*)**.** Let $\mathop{\smile\!\!\!\!\!\!\vert}\nolimits$ be a preindependence relation on $\mathbb{M}$. We say that $\mathop{\smile\!\!\!\!\!\!\vert}\nolimits$ is an **independence relation** on $\mathbb{M}$ if, moreover, it satisfies Invariance, Local Character, and the following.

Extension: If $A \mathop{\smile\!\!\!\!\!\!\vert}\nolimits_{C} B$ and $D \supseteq B$, then there exists $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!\vert}\nolimits_{C} D$.

Adler also defines the following axiom.

Existence: For any A, B, and C, there exists $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!\vert}\nolimits_{C} B$.

Corollary 3.19. *If $\mathop{\smile\!\!\!\!\!\!\vert}\nolimits^{\mathrm{cl}}$ satisfies either the Extension or the Existence axiom, then it is an independence relation (and it satisfies the Existence axiom).*

Proof. See [1, Thm. 2.5]. $\square$

Definition 3.20. The matroid cl satisfies **Existence** if the following holds.

For every a, B, and C, if $a \notin \mathrm{cl}(B)$, then there exists $a' \equiv_B a$ such that $a' \notin \mathrm{cl}(BC)$.

The following lemma will be quite useful in the following.

Lemma 3.21. *T.f.a.e.:*

1. cl *satisfies Existence.*
2. *For every a, B, and C, if $\Xi(a/B) \subseteq \mathrm{cl}(BC)$, then $a \in \mathrm{cl}(B)$.*
3. *For every a, $\bar{b}$, and $\bar{c}$, if $a \notin \mathrm{cl}(\bar{b})$, then there exists $a' \equiv_{\bar{b}} a$ such that $a' \notin \mathrm{cl}(\bar{b}\bar{c})$.*

4. *For every* a, $\bar{b}$, *and* $\bar{c}$, *and every* x*-narrow formula* $\psi(x, \bar{y}, \bar{z})$, *if* $\mathbb{M} \models \psi(a', \bar{b}, \bar{c})$ *for every* $a' \equiv_{\bar{b}} a$, *then* $a \in \mathrm{cl}(\bar{b})$.
5. *For every formula (without parameters)* $\phi(x, \bar{y})$ *and every* x*-narrow formula* $\psi(x, \bar{y}, \bar{z})$, *if* $\mathbb{M} \models \forall \bar{y}\, \exists \bar{z}\, \forall x \left(\phi(x, \bar{y}) \rightarrow \psi(x, \bar{y}, \bar{z})\right)$, *then* ϕ *is* x*-narrow.*
6. *For every* a *and* B, *if* $\mathrm{rk}^{\mathrm{cl}}\left(\Xi(a/B)\right)$ *is finite, then* $a \in \mathrm{cl}(B)$.
7. *For every* a *and* B, *if* $\mathrm{rk}^{\mathrm{cl}}\left(\Xi(a/B)\right) < \kappa$, *then* $a \in \mathrm{cl}(B)$.
8. $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ *is an independence relation.*

Proof. The only nontrivial fact is (5 $\Rightarrow$ 4), which is proved by a compactness argument. $\square$

Remark 3.22. If cl satisfies Existence, then $\mathrm{acl}\, A \subseteq \mathrm{cl}\, A$.

Lemma 3.23. *Assume that* $\mathrm{cl}(A)$ *is an elementary substructure of* $\mathbb{M}$, *for every* $A \subset \mathbb{M}$. *Then,* cl *satisfies Existence, and therefore* $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ *is an independence relation. Hence, if* T *has definable Skolem functions and* cl *extends* acl, *then* cl *satisfies Existence.*

Proof. Let $\Xi(a/B) \subseteq \mathrm{cl}(BC)$. We want to prove that $a \in \mathrm{cl}(B)$. Let B' and C' be elementary substructures of $\mathbb{M}$, such that $B \subseteq B' \subseteq \mathrm{cl}(B)$, $B'C \subseteq C' \subseteq \mathrm{cl}(BC)$, $|B'| < \kappa$, and $|C'| < \kappa$ (B' and C' exist by hypothesis on cl). By substituting B with B' and C with C', w.l.o.g., we can assume that $B \preceq C \prec \mathbb{M}$. By saturation, there exist an x-narrow formula $\phi(x, \bar{y}, \bar{z})$, $\bar{b} \subset B$, and $\bar{c} \subset C$, such that $\Xi(a/B) \subseteq \phi(\mathbb{M}, \bar{b}, \bar{c})$. Let $p := \mathrm{tp}(a/B)$, let $q \in S_1(C)$ be an heir of p, and let a' be a realisation of q. Since $\phi(x, \bar{b}, \bar{c}) \in q$, there exists $\bar{b}' \in B$ such that $\phi(x, \bar{b}, \bar{b}') \in p$. Hence, $a' \in \mathrm{cl}(B)$; since $a' \equiv_B a$, $a \in \mathrm{cl}(B)$. $\square$

Definition 3.24. The **trivial matroid** cl^0 is given by $\mathrm{cl}^0(X) = \mathbb{M}$ for every $X \subseteq \mathbb{M}$. The trivial matroid cl^0 is a definable matroid and satisfies Existence. It induces the trivial preindependence relation $\mathop{\smile\!\!\!\!\!\!|}^{0}$, such that $A \mathop{\smile\!\!\!\!\!\!|}^{0}_B C$ for every A, B, and C. Notice that $\mathop{\smile\!\!\!\!\!\!|}^{0}$ is an independence relation.

Definition 3.25. We say that cl is an **existential matroid** if cl is a definable matroid, satisfies Existence, and is nontrivial (i.e., different from cl^0).

Notice that every existential matroid cl defines an independence relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$, and is uniquely determined by $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ (Remark 3.7); however, not every independence relation is induced by some matroid.

Examples 3.26. 1. Given $n \in \mathbb{N}$, the **uniform matroid** of rank n is defined as follows: $\mathrm{cl}^n(X) := X$, if $|X| < n$, or $\mathbb{M}$ if $|X| \geq n$. The uniform matroid cl^n is a definable matroid, but does not satisfy Existence in general (unless $n = 0$).
2. Define $\mathrm{id}(X) := X$. Then, id is a definable matroid, but it does not satisfy Existence in general. The preindependence relation induced by id is given by $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{id}}_B C$ iff $A \cap C \subseteq B$.

Remark 3.27. Let $\mathbb{M}'$ be another monster model of T. We can define an operator $\mathrm{cl}^{\mathbb{M}'}$ on $\mathbb{M}'$ in the following way:

$$\mathrm{cl}^{\mathbb{M}'}(X') := \bigcup \left\{\phi(\mathbb{M}', \bar{a}') : \phi(x, \bar{y})\ x\text{-narrow} \ \&\ \bar{a}' \subset X'\right\}.$$

Then, $\mathrm{cl}^{\mathbb{M}'}$ is a definable matroid. If cl satisfies Existence, then $\mathrm{cl}^{\mathbb{M}'}$ also satisfies Existence. We will call $\mathrm{cl}^{\mathbb{M}'}$ the **extension** of cl to $\mathbb{M}'$.

Remark 3.28. Notice that the definitions of "definable" (3.15) and "existential" (3.25 and 3.20) make sense also for finitary closure operators (and not only for matroids).

However, we will not need such more general definitions.

Proviso. *For the remainder of this section,* cl *is an existential matroid.*

Summarising, we have the following. $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ *is an independence relation, satisfying the Strong Finite Character axiom.* In particular, if $\mathbb{M}$ is a pregeometric structure, then $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ is an independence relation.

3.2. Dimension

Definition 3.29. Given a set $V \subseteq \mathbb{M}^n$, definable with parameters from A, the **dimension** of V (w.r.t. to the matroid cl) is given by

$$\dim^{\mathrm{cl}}(V) := \max\left\{\mathrm{rk}^{\mathrm{cl}}(\bar{b}/A) : \bar{b} \in V\right\},$$

with $\dim^{\mathrm{cl}}(V) := -\infty$ iff $V = \emptyset$. More generally, the dimension of a partial type p with parameters from A is given by

$$\dim^{\mathrm{cl}}(p) := \max\left\{\mathrm{rk}^{\mathrm{cl}}(\bar{b}/A) : \bar{b} \models p\right\}.$$

The following remark shows that the above notion is well posed; in its proof, it is important that cl satisfies Existence.

Remark 3.30. Let V be a type-definable subset of $\mathbb{M}^n$. Then, $\dim^{\mathrm{cl}}(V) \leq n$, and $\dim^{\mathrm{cl}}(V)$ does not depend on the choice of the parameters.

Remark 3.31. For every $d \leq n \in \mathbb{N}$, the set of complete types in $S_n(A)$ of $\dim^{cl}$ greater or equal to d is closed (in the Stone topology). That is, $\dim^{cl}$ is continuous in the sense of [20, Section 17.b].

Remark 3.32. $\dim^{cl}(\mathbb{M}^n) = n$. Moreover, $\dim^{cl}$ is monotone; if $U \subseteq V \subseteq \mathbb{M}^n$, then $\dim^{cl}(U) \leq \dim^{cl}(V)$.

Lemma 3.33. *Let p be a partial type over A. Then,*

$$\dim^{cl}(p) := \min\left\{\dim^{cl}(V) : V \text{ is } A\text{-definable} \,\&\, V \in p\right\}.$$

Moreover, if p is a complete type, then, for every $\bar{b} \models p$, $\mathrm{rk}^{cl}(\bar{b}/A) = \dim^{cl}(p)$.

Proof. Let $d := \dim^{cl}(p)$, $e := \min\left\{\dim^{cl}(V) : V \text{ is } A\text{-definable} \,\&\, V \in p\right\}$, and $\bar{b} \models p$ be such that $d = \mathrm{rk}^{cl}(\bar{b}/A)$. Let $V \in p$ be such that $\dim^{cl}(V) = e$; then, $\bar{b} \in V$, and therefore $e \geq \mathrm{rk}^{cl}(\bar{b}/A) = d$.

For the opposite inequality, first assume that p is a complete type. W.l.o.g., $\tilde{b} := \langle b_1, \dots, b_d\rangle$ are cl-independent over A, and therefore $b_i \in \mathrm{cl}(A\tilde{b})$ for every $i = d+1, \dots, n$. For every $i \leq n$, let $\phi_i(x, \bar{y}, \bar{z})$ be an x-narrow formula such that $\mathbb{M} \models \phi(b_i, \tilde{b}, \bar{a})$ (where $\bar{a} \subset A$); define $\psi(\bar{x}, \bar{z}) := \bigwedge_{i=1}^{n} \phi_i(x_i, x_1, \dots, x_d, \bar{z})$, and $V := \psi(\mathbb{M}^n, \bar{a})$. Then, for every $\bar{b}' \in V$, $\mathrm{rk}^{cl}(\bar{b}'/A) \leq d$, and therefore $\dim^{cl}(V) \leq d$. Moreover, $\bar{b} \in V$; hence $V \in p$, and therefore $e \leq d$.

For the general case when p is a partial type, let P be the set of complete types over A extending p. Then, by the previous result on complete types, for every $q \in P$, there exists an A-definable set W_q such that $W_q \in q$ and $\dim^{cl}(W_q) = \dim^{cl}(q) \leq d$. By compactness, there exists $V \in p$ such that $V \subseteq W$, where $W := W_{q_1} \cup \dots \cup W_{q_l}$. Hence,

$$e \leq \dim^{cl}(V) \leq \dim^{cl}(W) \leq \max_{i \leq l}\left(\dim^{cl}(W_i)\right) \leq d. \quad \square$$

Definition 3.34. Given $p \in S_n(B)$ and $q \in S_n(C)$, with $B \subseteq C$, we say that q is a **nonforking extension** of p (w.r.t. cl) if q extends p and $\dim^{cl}(q) = \dim^{cl}(p)$. We write $q \mathop{\smile\!\!\!\!\!\!|}^{cl}_B C$ if q is a nonforking extension of $q \restriction_B$.

Remark 3.35. Let $B \subseteq C$ and $q \in S_n(C)$. Then, $q \mathop{\smile\!\!\!\!\!\!|}^{cl}_B C$ iff, for some (for all) $\bar{a}$ realising q, $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{cl}_B C$.

Remark 3.36. Let $p \in S_n(B)$ and $B \subseteq C$. Then, for every $q \in S_n(C)$ extending p, $\dim^{cl}(q) \leq \dim^{cl}(p)$. Moreover, there exists $q \in S_n(C)$ which is a nonforking extension of p.

Lemma 3.37. *Let $\mathop{\smile\!\!\!\!\!\!|}^{f}$ be Shelah's forking relation on $\mathbb{M}$. Then, for every A, B, and C subsets of $\mathbb{M}$, if $A \mathop{\smile\!\!\!\!\!\!|}^{f}_B C$, then $A \mathop{\smile\!\!\!\!\!\!|}^{cl}_B C$. In particular, if $\mathbb{K} \prec \mathbb{M}$, $\mathbb{K} \subseteq C$, and $q \in S_n(C)$, and q is either a heir or a coheir of $q \restriction_{\mathbb{K}}$, then $q \mathop{\smile\!\!\!\!\!\!|}^{cl}_{\mathbb{K}} C$.*

Proof. The fact that $\mathop{\smile\!\!\!\!\!\!|}^{f}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{cl}$ is a particular case of [1, Remark 1.20]. For the case when q is a coheir of $q \restriction_{\mathbb{K}}$, see also Lemma 3.17(3). $\square$

Corollary 3.38. *Assume that T is supersimple and that $p \in S_n(A)$ for some $A \subseteq \mathbb{M}$. Then, $\mathrm{SU}(p) \geq \dim^{cl}(p)$, where SU is the SU-rank (see [27, Section 5.1]).*

Remark 3.39. Given $B \supseteq A$, let $N_n(B/A)$ be the set of all n-types over B that do not fork over A. Since $\mathop{\smile\!\!\!\!\!\!|}^{cl}$ satisfies Strong Finite Character (cf. Lemma 3.17(2)), $N_n(B/A)$ is closed in $S_n(B)$.

Lemma 3.40. *For every complete type p, $\dim^{cl}(p)$ is the maximum of the cardinalities n of chains of complete types $p = q_0 \subset q_1 \subset \dots \subset q_n$, such that each q_{i+1} is a forking extension of q_i.*

Proof. Let A be the set of parameters of p, and $\bar{b} \models p$. Let $d := \dim^{cl}(p)$; w.l.o.g., $\tilde{b} := \langle b_1, \dots, b_d\rangle$ are independent over A. For every $i \leq d$, let $A_i := Ab_1 \dots b_i$, and $q_i := \mathrm{tp}(\bar{b}/A_i)$. Then, $p = q_0 \subset \dots \subset q_d$, and each q_{i+1} is a forking extension of q_i.

Conversely, assume that $p = q_0 \subset \dots \subset q_n$, and that each q_{i+1} is a forking extension of q_i.

Claim 1. *For every $i \leq n$, $\dim^{cl}(q_{n-i}) \geq i$; in particular, $\dim^{cl}(p) \geq n$.*

By induction on i. The case $i = 0$ is clear. Assume that we have proved the claim for i; we want to show that it holds for $i+1$. Since q_{n-i} is a forking extension of $q_{n-(i+1)}$, $\dim^{cl}(q_{n-i}) < \dim^{cl}(q_{n-(i+1)})$, and we are done. $\square$

Remark 3.41. Let $V \subseteq \mathbb{M}^n$ be nonempty and definable with parameters $\bar{a}$. Then, either $\dim^{cl}(V) = 0 = \mathrm{rk}^{cl}(V/\bar{a})$, or $\dim^{cl}(V) > 0$ and $\mathrm{rk}^{cl}(V) \geq \kappa$.

Remark 3.42. A formula $\phi(x, \bar{y})$ is x-narrow iff, for every $\bar{b} \in \mathbb{M}^n$, $\dim^{cl}\left(\phi(\mathbb{M}, \bar{b})\right) \leq 0$.

Remark 3.43. Let $\phi(x, \bar{y})$ be a formula without parameters, and $\bar{a} \in \mathbb{M}^n$. Then, $\dim^{cl}(\phi(\mathbb{M}, \bar{a})) = 0$ iff there exists an x-narrow formula $\psi(x, \bar{y})$ such that $\forall x \left(\phi(x, \bar{a}) \to \psi(x, \bar{a})\right)$. Therefore, define

$$\Gamma_\phi(\bar{y}) := \left\{\neg\theta(\bar{y}) : \theta(\bar{y}) \text{ formula without parameters s.t. } \forall \bar{a}\left(\theta(\bar{a}) \to \dim^{cl}(\phi(\mathbb{M}, \bar{a})) = 0\right)\right\},$$

$$U^1_\phi := \left\{\bar{a} \in \mathbb{M}^n : \dim^{cl}(\phi(\mathbb{M}, \bar{a})) = 1\right\}.$$

Then, $U^1_\phi = \left\{\bar{a} \in \mathbb{M}^n : \mathbb{M} \models \Gamma_\phi(\bar{a})\right\}$, and in particular U^1_ϕ is type-definable (over the empty set).

More generally, let $k \leq m$, $\bar{x} := \langle x_1, \dots, x_m\rangle$, and let $\phi(\bar{x}, \bar{y})$ be a formula without parameters. Define

$$U^{\geq k}_\phi := \left\{\bar{a} \in \mathbb{M}^n : \dim^{cl}(\phi(\mathbb{M}^m, \bar{a})) \geq k\right\}.$$

Then, $U^{\geq k}_\phi$ is type-definable.

Lemma 3.44 (*Fibre-Wise Dimension Inequalities*). *Let $U \subseteq \mathbb{M}^{m_1}$, $V \subseteq \mathbb{M}^{m_2}$, and $F : U \to V$ be definable, with parameters from C. Let $X \subseteq U$ and $Y \subseteq V$ be type-definable, such that $F(X) \subseteq Y$. Define $f := F \restriction X : X \to Y$. For every $\bar{b} \in Y$, let $X_{\bar{b}} := f^{-1}(\bar{b}) \subseteq X$, and $m := \dim^{\mathrm{cl}}(Y)$.*

1. *If, for every $\bar{b} \in Y$, $\dim^{\mathrm{cl}}(X_{\bar{b}}) \leq n$, then $\dim^{\mathrm{cl}}(X) \leq m + n$.*
2. *If f is surjective and, for every $\bar{b} \in Y$, $\dim^{\mathrm{cl}}(X_{\bar{b}}) \geq n$, then $\dim^{\mathrm{cl}}(X) \geq m + n$.*
3. *If f is surjective, then $\dim^{\mathrm{cl}}(X) \geq m$.*
4. *If f is injective, then $\dim^{\mathrm{cl}}(X) \leq m$.*
5. *If f is bijective, then $\dim^{\mathrm{cl}}(X) = m$.*

Proof. (1) Assume, for contradiction, that $\dim^{\mathrm{cl}}(X) > m + n$. Let $\bar{a} \in X$ be such that $\mathrm{rk}^{\mathrm{cl}}(\bar{a}/C) > m + n$, and $\bar{b} := F(\bar{a})$. Since $\bar{a} \in X_{\bar{b}}$, and $X_{\bar{b}}$ is type-definable with parameters $C\bar{b}$, $\mathrm{rk}^{\mathrm{cl}}(\bar{a}/\bar{b}C) \leq n$. Hence, by Remark 3.2, $\mathrm{rk}^{\mathrm{cl}}(\bar{a}/C) \leq \mathrm{rk}^{\mathrm{cl}}(\bar{a}\bar{b}/C) \leq m+n$, which is absurd.

(2) Let $\bar{b} \in Y$ be such that $\dim^{\mathrm{cl}}(\bar{b}/C) = m$. Let $\bar{a} \in X_{\bar{b}}$ be such that $\dim^{\mathrm{cl}}(\bar{a}/\bar{b}C) \geq n$. Then, by Remark 3.2, $\mathrm{rk}^{\mathrm{cl}}(\bar{a}\bar{b}/C) \geq m + n$. However, since $\bar{a} = F(\bar{b})$, $\bar{a} \subset \mathrm{cl}(\bar{b}C)$, and therefore $\mathrm{rk}^{\mathrm{cl}}(\bar{b}/C) = \mathrm{rk}^{\mathrm{cl}}(\bar{a}\bar{b}/C) \geq m + n$.

(3) Follows from (2) applied to $n = 0$. The other assertions are clear. □

Remark 3.45. Let cl' be another existential matroid on $\mathbb{M}$. T.f.a.e.:

1. $\mathrm{cl} \subseteq \mathrm{cl}'$;
2. $\mathrm{rk}^{\mathrm{cl}} \geq \mathrm{rk}^{\mathrm{cl}'}$;
3. $\dim^{\mathrm{cl}} \geq \dim^{\mathrm{cl}'}$ on definable sets;
4. $\dim^{\mathrm{cl}} \geq \dim^{\mathrm{cl}'}$ on complete types;
5. for every definable set $X \subseteq \mathbb{M}$, if $\dim^{\mathrm{cl}}(X) = 0$, then $\dim^{\mathrm{cl}'}(X) = 0$.

T.f.a.e.:

1. $\mathrm{cl} = \mathrm{cl}'$;
2. $\mathrm{rk}^{\mathrm{cl}} = \mathrm{rk}^{\mathrm{cl}'}$;
3. $\dim^{\mathrm{cl}} = \dim^{\mathrm{cl}'}$ on definable sets;
4. $\dim^{\mathrm{cl}} = \dim^{\mathrm{cl}'}$ on complete types;
5. for every definable set $X \subseteq \mathbb{M}$, $\dim^{\mathrm{cl}}(X) = 0$ iff $\dim^{\mathrm{cl}'}(X) = 0$.

We will show that, for many interesting theories, there is at most one existential matroid. Define $T_{R\nmid 0}$ to be the theory of rings without zero divisors, in the language of rings $\mathcal{L}_R := (0, 1, +, \cdot)$.

Definition 3.46 (*[10, 1.18]*). If $\mathbb{K}$ expands a ring without zero divisors, let $F : \mathbb{K}^4 \to \mathbb{K}$ be the following function, definable without parameters in the language $\mathcal{L}_R$:

$$\langle x_1, x_2, y_1, y_2 \rangle \mapsto \begin{cases} t & \text{if } y_1 \neq y_2 \;\&\; t \cdot (y_1 - y_2) = x_1 - x_2; \\ 0 & \text{if there is no such } t. \end{cases}$$

Notice that F is well defined because, in a ring without zero divisors, if $y_1 \neq y_2$, then, for every x, there exists at most one t such that $t \cdot (y_1 - y_2) = x$.

Lemma 3.47 (*[10, 1.18]*). *Assume that T expands $T_{R\nmid 0}$. Let $A \subseteq \mathbb{M}$ be definable. Then, $\dim^{\mathrm{cl}}(A) = 1$ iff $\mathbb{M} = F(A^4)$.*

Proof. Assume for contradiction that $\dim^{\mathrm{cl}}(A) = 1$, but there exists $c \in \mathbb{M} \setminus F(A^4)$. Since $c \notin F(A^4)$, the function $\langle x_1, x_2 \rangle \mapsto c \cdot x_1 + x_2 : A^2 \to \mathbb{M}$ is injective. Hence, by Lemma 3.44, $\dim^{\mathrm{cl}}(\mathbb{M}) \geq \dim^{\mathrm{cl}}(A^2) = 2$, which is absurd.

Conversely, by Lemma 3.44 again, if $F(A^4) = \mathbb{M}$, then $\dim(A) = 1$. □

Theorem 3.48. *If T expands $T_{R\nmid 0}$, then cl is the only existential matroid on $\mathbb{M}$. If S is a definable subfield of $\mathbb{M}$ of dimension 1, then $S = \mathbb{M}$.*

Proof. Let $A \subseteq \mathbb{M}$ be definable. By the previous lemma, $\dim(A) = 1$ iff $F(A^4) = \mathbb{M}$. Since the same holds for any existential matroid cl' on $\mathbb{M}$, we conclude that, for every definable set $A \subseteq \mathbb{M}$, $\dim^{\mathrm{cl}}(A) = 0$ iff $\dim^{\mathrm{cl}'}(A) = 0$, and hence $\dim^{\mathrm{cl}} = \dim^{\mathrm{cl}'}$.

Given S a subfield of $\mathbb{M}$, $F(S^4) = S$. Hence, if $\dim^{\mathrm{cl}}(S) = 1$, then $S = \mathbb{M}$. □

Examples 3.49. 1. In the above theorem, we cannot drop the hypothesis that T expands $T_{R\nmid 0}$. Let M be a set with an equivalence relation E, such that E has infinitely many equivalence classes, all infinite, and let $\mathbb{M}$ be a monster elementary extension of $\langle M, E \rangle$. For every $a \in \mathbb{M}$, let Ea be the equivalence class of a, and define $\mathrm{cl}(A) := \bigcup_{a \in A} Ea$. Then, acl and cl are two different existential matroids on $\mathbb{M}$. The example can be improved, taking for instance a chain $E_1 \supset E_2 \ldots$ of equivalence relations, such that each E_i-equivalence class is the union of infinitely many E_{i+1}-equivalence classes; each equivalence relation will then induce a different existential matroid on $\mathbb{M}$.

2. In Theorem 3.48, we cannot even relax the hypothesis to "T expands the theory of a vector space". In fact, let $\mathbb{F}$ be an ordered field, considered as a vector space over itself, in the language $\langle 0, 1, +, <, \lambda_c \rangle_{c \in \mathbb{F}}$, and let T be its theory. Let T^d be the theory of dense pairs of models of T. [10, 5.8] show that T^d has elimination of quantifiers, and acl is a matroid on T^d. However, as the reader can verify, the small closure Scl is another existential matroid on T^d (cf. Section 8.4), and it is different from acl.

Corollary 3.50. *If $\mathbb{M}$ expands a field, then $\mathbb{M}$ must be a perfect field. In particular, the theory of separably closed (but nonalgebraically closed) fields, and the theory of differentially closed fields of finite characteristic do not admit an existential matroid.*

Proof. Cf. [25, 1.6]. If $\mathbb{M}$ is not perfect, then $\mathbb{M}^p$ is a proper definable subfield of $\mathbb{M}$, where $p := \mathrm{char}(\mathbb{M})$, and therefore $\dim^{\mathrm{cl}}(\mathbb{M}^p) = 0$. However, the map $x \mapsto x^p$ is a bijection from $\mathbb{M}$ to $\mathbb{M}^p$; therefore, $\dim^{\mathrm{cl}}(\mathbb{M}) = 0$, which is absurd. □

Corollary 3.51. *Let cl' be a nontrivial definable matroid on some monster model $\mathbb{M}'$. Assume that $\mathbb{M}'$ expands a model of $T_{R\restriction 0}$. Then, t.f.a.e.:*

1. cl' *is an existential matroid;*
2. *for every formula (without quantifiers) $\phi(x, \bar{y})$, ϕ is x-narrow (w.r.t. cl') iff, for every $\bar{b}$, $F\big((\phi(\mathbb{M}', \bar{b})^4\big) \neq \mathbb{M}'$.*

Proof. $(1 \Rightarrow 2)$ is clear.
$(2 \Rightarrow 1)$ follows from Lemma 3.21(5). □

Lemma 3.52. *Let $\mathbb{K}$ be a ring without zero divisors definable in $\mathbb{M}$, of dimension $n \geq 1$. Let $\mathbb{F} \subseteq \mathbb{K}$ be a definable subring such that $\mathbb{F}$ is a skew field. If $\dim^{\mathrm{cl}}(\mathbb{F}) = n$, then $\mathbb{K} = \mathbb{F}$.*

Proof. Assume, for contradiction, that there exists $c \in \mathbb{K} \setminus \mathbb{F}$. Define $h : \mathbb{F} \times \mathbb{F} \to \mathbb{K}$, $h(x, y) := x + cy$. Since $c \notin \mathbb{F}$ and $\mathbb{F}$ is a skew field, h is injective. Thus, $2n = \dim(\mathbb{F}^2) \leq \dim(\mathbb{K}) = n$, a contradiction. □

Corollary 3.53. *Let $\mathbb{K} \subseteq \mathbb{M}^n$ be a definable field, such that $\dim^{\mathrm{cl}}(\mathbb{K}) \geq 1$. Then, $\mathbb{K}$ is perfect.*

Proof. Let $p := \mathrm{char}\,\mathbb{K}$, and let $\phi : \mathbb{K} \to \mathbb{K}$ be the Frobenius automorphism $\phi(x) = x^p$. Since ϕ is injective, $\dim^{\mathrm{cl}}(\mathbb{K}^p) = \dim^{\mathrm{cl}}(\mathbb{K})$, and therefore $\mathbb{K}^p = \mathbb{K}$. □

The assumption that $\dim^{\mathrm{cl}}(\mathbb{K}) \geq 1$ is necessary; nonperfect definable fields of dimension 0 can exist. For instance, let $\mathbb{F}$ be a nonperfect field, P be an infinite set, and let $\mathbb{K}$ be the disjoint union of $\mathbb{F}$ and P, with the following dimension function (cf. Section 4).

$\dim(X) = 1$ iff $X \cap P$ is infinite, where X varies among the definable subsets of $\mathbb{K}$.

Then, $\mathbb{F}$ is a nonperfect field definable in $\mathbb{K}$ and of dimension 0.

Definition 3.54. Let $X \subseteq \mathbb{K}^n$ and $Y \subseteq \mathbb{K}^m$ be definable. Let $g : X \rightsquigarrow Y$ be a definable application (i.e., a multi-valued partial function), with graph G. For every $x \in X$, let $g(x) := \{y \in Y : \langle x, y \rangle \in G\} \subseteq Y$. Such an application g is a **Z-application** if, for every $x \in X$, $\dim^{\mathrm{cl}}\big(g(x)\big) \leq 0$.

Remark 3.55. Let $A \subseteq \mathbb{K}$, and let $b \in \mathbb{K}$. Then, $b \in \mathrm{cl}(A)$ iff there exists a $\emptyset$-definable Z-application $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ and $\bar{a} \in A$, such that $b \in f(\bar{a})$. Moreover, if $\bar{c} \in \mathbb{K}^n$, then $b \in \mathrm{cl}(A\bar{c})$ iff there exists an A-definable Z-application $f : \mathbb{K}^n \to \mathbb{K}$, such that $b \in f(\bar{c})$.

Definition 3.56. We say that $\dim^{\mathrm{cl}}$ is **definable** if, for every $d \in \mathbb{N}$ and for every X definable subset of $\mathbb{M}^m \times \mathbb{M}^n$, the set $\big\{\bar{a} \in \mathbb{M}^m : \dim^{\mathrm{cl}}(X_{\bar{a}}) = d\big\}$ is definable.

Lemma 3.57. *T.f.a.e.:*

1. $\dim^{\mathrm{cl}}$ *is definable;*
2. *for every X definable subset of $\mathbb{M}^m \times \mathbb{M}$, the set $X^{1,1} := \big\{\bar{a} \in \mathbb{M}^m : \dim^{\mathrm{cl}}(X_{\bar{a}}) = 1\big\}$ is also definable;*
3. *for every $k \leq n$, every m, and every X definable subset of $\mathbb{M}^m \times \mathbb{M}^n$, the set $X^{n,k} := \big\{\bar{a} \in \mathbb{M}^m : \dim^{\mathrm{cl}}(X_{\bar{a}}) = k\big\}$ is also definable, with the same parameters as X.*

Proof. $(3 \Rightarrow 1 \Rightarrow 2)$ is obvious.

$(2 \Rightarrow 1)$ We will prove by induction on n that, for every Y definable subset of $\mathbb{K}^n \times \mathbb{K}^m$, the set $Y^{n,\geq k} := \big\{\bar{a} \in \mathbb{M}^m : \dim^{\mathrm{cl}}(X_{\bar{a}}) \geq k\big\}$ is definable. The case $k = 0$ is clear. The case $k = 1$ follows from the assumption and the observation that, for every Z definable subset of $\mathbb{K}^n$, $\dim^{\mathrm{cl}}(Z) \geq 1$ iff $\dim^{\mathrm{cl}}(\theta(Z)) \geq 1$ for some θ projection from $\mathbb{K}^n$ onto a coordinate axis. The inductive step follows from the fact that

$$X^{n,\geq k} = \Big(\prod_{n+m-1}^{n+m}(X)\Big)^{n-1,\geq k} \cup \big(X^{n+m-1,\geq 1}\big)^{n-1,\geq k-1}.$$

$(1 \Rightarrow 3)$ Let $X \subseteq \mathbb{K}^{n+m}$ be definable with parameters from A. Then, $X^{n,k}$ is $\mathbb{M}$-definable, by assumption. Moreover, by Remark 3.43, $X^{n,k}$ is type-definable over A, and therefore invariant under automorphisms that fix A point-wise. Hence, by Beth's definability theorem, $X^{n,k}$ is definable over A. □

Corollary 3.58. *If T expands $T_{R|0}$, then* $\dim^{\mathrm{cl}}$ *is definable.*

Proof. By Lemmas 3.47 and 3.57(2). □

See Remark 14.5 for examples when $\dim^{\mathrm{cl}}$ is not definable.

Examples 3.59. 1. Let λ and η be ordinal numbers, such that λ is a power of ω (e.g., $\lambda = 1, \lambda = \omega, \dots$). Let $\mathbb{K}$ be a monster model, and assume that

- either $\mathbb{K}$ is superstable of Lascar U-rank η;
- or $\mathbb{K}$ is supersimple of SU-rank η;
- or $\mathbb{K}$ is superrosy of þ-rank η (see [11] for definitions).

Denote by R be corresponding rank in the various cases (U, SU, $\mathrm{U}^{\text{þ}}$). Assume that $\eta < m \cdot \lambda$ for some $m \in \mathbb{N}$. For every $a \in \mathbb{K}$ and $B \subset \mathbb{K}$, define $a \in \mathrm{cl}_\lambda(B)$ if $\mathrm{R}(a/B) < \lambda$. It is easy to see that cl_λ is a closure operator on $\mathbb{K}$ satisfying Existence. Assume now that $\eta < 2\lambda$; then, cl_λ is a matroid. Moreover, cl_λ is nontrivial iff there exists a unary type p such that $\mathrm{R}(p) \geq \lambda$ (which, in general, is a stronger condition than $\mathrm{R}(\mathbb{K}) \geq \lambda$). Moreover, for every type q, $\mathrm{R}(q) = \mathrm{rk}^{\mathrm{cl}_\lambda}(q) \cdot \lambda + \rho$, where ρ is some (unique) ordinal such that $\rho < \lambda$. However, cl_λ might not be definable.

2. Let λ be as above, and let $\mathbb{G}$ be a monster model of a superstable group, such that $\mathrm{U}(\mathbb{G}) = \lambda$. Define cl_λ as in (1). Then, cl_λ is nontrivial, because there exists at least one generic type (i.e., a type of U-rank λ) [21, Corollary 5.2]. If X is a definable subset of $\mathbb{G}$, then $\dim^{\mathrm{cl}_\lambda}(X) = 1$ iff X is generic (that is, finitely many bilateral translates of X cover $\mathbb{G}$). By [21, Lemma 5.4], and Lemma 3.57(2), cl_λ is a definable (and thus existential) matroid, with definable dimension.
3. Let $\mathbb{K}$ be a monster differentially closed field, and $p \geq 0$ be its characteristic. If $p = 0$, then $\mathbb{K}$ is superstable, and $\mathrm{U}(\mathbb{K}) = \omega$; hence, by the previous example, there exists a (unique) existential matroid cl on $\mathbb{K}$. It is easy to see that, if A is a differential subfield of $\mathbb{K}$ and $b \in \mathbb{K}$, then $b \in \mathrm{cl}(A)$ iff b is differential-algebraic over A (that is, iff $b, \mathrm{d}b, \mathrm{d}^2 b, \dots$ are algebraically dependent over A); see [28,25, 2.25]. On the other hand, if $p > 0$, then there is no existential matroid on $\mathbb{K}$, because $\mathbb{K}$ is not perfect (Corollary 3.50).

3.3. Morley sequences

Most of the results of this subsection remain true for an arbitrary independence relation $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}$ instead of $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\mathrm{cl}}$.

Definition 3.60. Let $C \subseteq B$, $p(\bar{x}) \in S_n(B)$, and let $\langle I, \leq \rangle$ be a linear order. A **Morley sequence** over C indexed by I in p is a sequence $(\bar{a}_i : i \in I)$ of tuples in $\mathbb{M}^n$, such that $(\bar{a}_i : i \in I)$ are order-indiscernible over B and independent over C, and every $\bar{a}_i$ realises $p(\bar{x})$.

A Morley sequence over C is a Morley sequence over C in some $p \in S_n(C)$. A Morley sequence in p is a Morley sequence over B in p.

Lemma 3.61. *Let $\langle I, \leq \rangle$ be a linear order, with $|I| < \kappa$. Let $p(\bar{x}) \in S_n(C)$. Then, there exists a Morley sequence in $p(\bar{x})$ indexed by I. If, moreover, $\bar{b} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\mathrm{cl}}_C \bar{d}$, then there exists a Morley sequence $(\bar{a}_i : i \in I)$ over C indexed by I in $p(\bar{x})$, such that $(\bar{b}\bar{a}_i : i \in I)$ are order-indiscernible over $C\bar{d}$ and, for every $i \in I$, $\bar{b}\bar{a}_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\mathrm{cl}}_C \bar{d}(\bar{a}_j : i \neq j \in I)$.*

Proof. Let $(\bar{x}_i : i \in I)$ be a sequence of n-tuples of variables. Consider the following set of C-formulae:

$$\Gamma_1(\bar{x}_i : i \in I) := \bigwedge_{i \in I} p(\bar{x}_i) \,\&\, \bigwedge_{i \in I} \bar{x}_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\mathrm{cl}}_C (\bar{x}_j : j < i).$$

First, notice that, by Remark 3.39, Γ_1 *is* a set of formulae. Consider the following set of C-formulae:

$$\Gamma_2(\bar{x}_i : i \in I) := \Gamma_1(\bar{x}_i : i \in I) \,\&\, (\bar{x}_i : i \in I) \text{ are order-indiscernible over } C.$$

By [1, 1.12], Γ_2 is consistent.

We give an alternative proof of the above fact, which does not use the Erdös–Rado theorem.

Claim 1. *Γ_2 is consistent.*

First, we prove that Γ_1 is finitely satisfiable; hence, w.l.o.g., $I = \{0, \dots, m\}$ is finite. Let $\bar{a}_0$ be any realisation of $p(\bar{x})$. Let $\bar{a}_1 \equiv_C \bar{a}_0$ be such that $\bar{a}_1 \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\mathrm{cl}}_C \bar{a}_0, \dots$, and let $\bar{a}_m \equiv_C \bar{a}_0$ be such that $\bar{a}_m \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\mathrm{cl}}_C \bar{a}_0 \dots \bar{a}_{m-1}$. Therefore, Γ_1 is consistent, and thus, by Ramsey's theorem, Γ_2 is also consistent.

Since $|I| < \kappa$, there exists a realisation $(\bar{a}_i : i \in I)$ of Γ_2. Then, by Lemma 3.12, $(\bar{a}_i : i \in I)$ is a Morley sequence in $p(\bar{x})$ over C.

If, moreover, $\bar{b}$ and $\bar{d}$ satisfy $\bar{b} \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\mathrm{cl}}_C \bar{d}$, let $q(\bar{x}, \bar{y}, \bar{z})$ be the extension of $p(\bar{x})$ to $S(C\bar{b}\bar{d})$ satisfying $\bar{y} = \bar{b}$ and $\bar{z} = \bar{d}$. Let $(\bar{a}_i\bar{b}\bar{d} : i \in I)$ be a Morley sequence in $q(\bar{x}, \bar{y}, \bar{z})$. By Lemma 3.11, for every $i \in I$, we have $\bar{b}\bar{a}_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}^{\mathrm{cl}}_C \bar{d}(\bar{a}_j : i \neq j \in I)$. □

Definition 3.62. A type $p \in S_n(A)$ is **stationary** if, for every $B \supseteq A$, there exists a unique $q \in S_n(B)$ such that q is a nonforking extension of p.

Remark 3.63. Let $p \in S_n(A)$. If $\dim^{\mathrm{cl}}(p) = 0$, then p is stationary iff p is realised in $\mathrm{dcl}(A)$.

Hence, unlike the stable case, if $\mathrm{cl} \neq \mathrm{acl}$, then there are types over models which are nonstationary.

Lemma 3.64. *Let $C \supseteq B$ and $q \in S_n(C)$ be such that $q \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_B C$. Let $(\bar{a}_i : i \in I)$ be a sequence of realisations of q independent over C. Then, $(\bar{a}_i : i \in I)$ is also independent over B. If, moreover, q is stationary, then the following hold.*

1. *$(\bar{a}_i : i \in I)$ is a totally indiscernible set over C, and in particular it is a Morley sequence for q over B.*
2. *If $(\bar{a}' : i \in I)$ is another sequence of realisations of q independent over C, then $(\bar{a}_i : i \in I) \equiv_C (\bar{a}'_i : i \in I)$.*

Proof. Standard proof. More precisely, for every $i \in I$, let $\bar{d}_i := (a_j : i \neq j \in I)$. By assumption, $\bar{a}_i \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_C \bar{d}_i$, and, since $q \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_B C$, $\bar{a}_i \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_B C$, and therefore $\bar{a}_i \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_B \bar{d}_i$, proving that $(\bar{a}_i : i \in I)$ is independent over B.

Let us prove Statement (2). By compactness, w.l.o.g., $I = \{1, \dots, m\}$ is finite. Assume, for contradiction, that $(\bar{a} : i \leq m) \not\equiv_C (\bar{a}' : i \leq m)$; by induction on m, we can assume that $(\bar{a}_i : i \leq m-1) \equiv_C (\bar{a}'_i : i \leq m-1)$, and therefore, w.l.o.g., that $\bar{a}_i = \bar{a}'_i$ for $i = 1, \dots, m-1$. However, since q is stationary, $\bar{a}_m \equiv_C \bar{a}'_m$, $\bar{a}_m \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_C (\bar{a}_i : i \leq m-1)$, and $\bar{a}'_m \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_C (\bar{a}_i : i \leq m-1)$, we have that $\bar{a}_m \equiv_{C(\bar{a}_i : i \leq m-1)} \bar{a}'_m$, which is absurd.

Finally, it remains to prove that the set $(\bar{a}_i : i \in I)$ is totally indiscernible over C. If σ is any permutation of I, then $(\bar{a}_{\sigma(i)} : i \in I)$ is also a sequence of realisations of q independent over C, and therefore, by Statement (2), $(\bar{a}_{\sigma(i)} : i \in I) \equiv_C (\bar{a}_i : i \in I)$. □

Corollary 3.65. *Assume that there is a definable linear ordering on $\mathbb{M}$. Then, $p \in S_n(A)$ is stationary iff p is realised in $\mathrm{dcl}(A)$. Hence, if $\dim^{\mathrm{cl}}(p) > 0$, every nonforking extension of p is nonstationary.*

Proof. Assume that p is stationary, but, for contradiction, that $\dim^{\mathrm{cl}}(p) > 0$. Then, there is a Morley sequence in p with at least two elements $\bar{a}_0$ and $\bar{a}_1$. Since $\dim^{\mathrm{cl}}(p) > 0$, $\bar{a}_0 \neq \bar{a}_1$. By Lemma 3.64, $\mathrm{tp}(\bar{a}_0\bar{a}_1/A) = \mathrm{tp}(\bar{a}_1\bar{a}_0/A)$, which is absurd. □

Contrast the above corollary to the case of stable theories, where instead every type has at least one stationary nonforking extension.

Corollary 3.66. *Let $B \subseteq C$ and $q \in S_n(C)$. Then, t.f.a.e.:*

1. *$q \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_B C$;*
2. *there exists an infinite sequence of realisations of q which are independent over B;*
3. *every sequence $(\bar{a}_i : i \in I)$ of realisations of q which are independent over C are independent also over B;*
4. *there exists an infinite Morley sequence in q over B.*

Proof. Cf. [1, 1.12–13].

(1 ⇒ 3) Let $(\bar{a}_i : i \in I)$ be a sequence of realisations of q independent over C. For every $i \in I$, let $\bar{d}_i := (\bar{a}_j : i \neq j \in I)$. Since $\bar{a}_i \mathop{\smile\!\!\!\!\!|}_C \bar{d}_i$ and $\bar{a}_i \mathop{\smile\!\!\!\!\!|}_B C$, we have $\bar{a}_i \mathop{\smile\!\!\!\!\!|}_B \bar{d}_i$.

(3 ⇒ 4) Let $(\bar{a}_i : i \in I)$ be an infinite Morley sequence in q over C; such a sequence exists by Lemma 3.61 (or by [1, 1.12]). Then, $(\bar{a}_i : i \in I)$ is independent also over B, and hence is a Morley sequence for q over B.

(4 ⇒ 2) is obvious.

(2 ⇒ 1) Choose $\lambda < \kappa$ a regular cardinal large enough. Let $(\bar{a}'_i : i < \omega)$ be a sequence of realisations of q independent over B. By saturation, there exists a sequence $(\bar{a}_i : i < \lambda)$ of realisations of q independent over B. By Local Character, and since λ is regular, there exists $\alpha < \lambda$ such that $\bar{a}_\alpha \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_{B\bar{d}} C$, where $\bar{d} := (\bar{a}_i : i < \alpha)$. Since, moreover, $\bar{a}_\alpha \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_B \bar{d}$, we have $\bar{a}_\alpha \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_B C$, and therefore $q \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_B C$. □

3.4. Local properties of dimension

In this subsection, we will show that the dimension of a set can be checked locally; what this means precisely will be clear in Section 9, where the results given here will be applied to a "concrete" situation.

Definition 3.67. A quasi-ordered set $\langle I, \leq \rangle$ is a **directed set** if every pair of elements of I has an upper bound.

Lemma 3.68. *Let $\langle I, \leq \rangle$ be a directed set, definable in $\mathbb{M}$ with parameters $\bar{c}$. Then, for every $\bar{a} \in I$ and $\bar{d} \subset \mathbb{M}$ there exists $\bar{b} \in I$ such that $\bar{b} \geq \bar{a}$ and $\bar{d}\bar{a} \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_{\bar{c}} \bar{b}$.*

Proof. Fix $\bar{a} \in I$ and $\bar{d} \subset \mathbb{M}$, and assume, for contradiction, that every $\bar{b} \geq \bar{a}$ satisfies $\bar{d}\bar{a} \not\mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}}_{\bar{c}} \bar{b}$.

W.l.o.g., $\bar{c} = \emptyset$. Let λ be a large enough cardinal; at the price of increasing κ if necessary, we may assume that $\lambda < \kappa$. By Lemma 3.61, there exists a Morley sequence $(\bar{d}'_i\bar{a}'_i : i < \lambda)$ in $\mathrm{tp}(\bar{d}\bar{a}/\emptyset)$. Consider the following set of formulae over $\{\bar{a}'_i : i < \lambda\}$:

$$\Lambda(\bar{x}) := \left\{\bar{x} \in I, \bar{x} \geq \bar{a}'_i : i < \lambda\right\}.$$

Since $\langle I, \leq \rangle$ is a directed set, Λ is consistent; let $\bar{b} \in I$ be a realisation of Λ. By the Erdös–Rado theorem, there exists a Morley sequence $(\bar{d}_i\bar{a}_i : i < \omega)$ in $\mathrm{tp}(\bar{d}\bar{a}/\emptyset)$, such that all the $\bar{d}_i\bar{a}_i$ satisfy the same type $q(\bar{x}, \bar{y})$ over $\bar{b}$, and $\bar{a}_i \leq \bar{b}$ for every $i < \omega$. Therefore, by Corollary 3.66, $q \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}} \bar{b}$, and in particular $\bar{a}_0\bar{d}_0 \mathop{\smile\!\!\!\!\!|}^{\mathrm{cl}} \bar{b}$. Since $\bar{a}_0\bar{d}_0 \equiv \bar{a}\bar{d}$, there exists $\bar{b}' \geq \bar{a}$ such that $\bar{a}_0\bar{d}_0\bar{b} \equiv \bar{a}\bar{d}\bar{b}'$, so $\bar{b}' \mathop{\smile\!\!\!\!\!|} \bar{d}\bar{a}$ and $\bar{b}' \geq \bar{a}$, a contradiction. □

Lemma 3.69. *Let $X \subseteq \mathbb{M}^n$ be definable with parameters $\bar{c}$, and let $\left(U_{\bar{t}}\right)_{\bar{t} \in I}$ be a family of subsets of $\mathbb{M}^n$, such that each $U_{\bar{t}}$ is definable with parameters $\bar{t}\bar{c}$. Let $d \leq n$, and assume that, for every $\bar{a} \in X$, there exists $\bar{b} \in I$ such that $\bar{a} \in U_{\bar{b}}$, $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{\bar{c}} \bar{b}$, and $\dim^{\mathrm{cl}}(X \cap U_{\bar{b}}) \leq d$. Then, $\dim^{\mathrm{cl}}(X) \leq d$.*

Proof. Assume, for contradiction, that $\dim^{\mathrm{cl}}(X) > d$; let $\bar{a} \in X$ be such that $\mathrm{rk}^{\mathrm{cl}}(\bar{a}/\bar{c}) > d$. Choose $\bar{b}$ as in the hypothesis of the lemma; then, $\mathrm{rk}^{\mathrm{cl}}(\bar{a}/\bar{b}\bar{c}) > d$, which is absurd. □

Lemma 3.70. *Let $I \subseteq \mathbb{M}^n$ be definable and let $<$ be a definable linear ordering on I. Let $\left(X_{\bar{b}}\right)_{\bar{b} \in I}$ be a definable increasing family of subsets of $\mathbb{K}^m$ and $X := \bigcup_{\bar{b} \in I} X_{\bar{b}}$. Let $d \leq m$, and assume that, for every $\bar{b} \in I$, $\dim^{\mathrm{cl}}(X_{\bar{b}}) \leq d$. Then, $\dim^{\mathrm{cl}}(X) \leq d$.*

Proof. Let $\bar{c}$ be the parameters used to define I, $<$, and $\left(X_{\bar{b}}\right)_{\bar{b} \in I}$. Let $\bar{a} \in X$ be such that $\mathrm{rk}^{\mathrm{cl}}(\bar{a}/\bar{c}) = \dim^{\mathrm{cl}}(X)$. Let $\bar{b} \in I$ be such that $\bar{a} \in X_{\bar{b}}$. Choose $\bar{a}', \bar{b}' \subset \mathbb{M}$ such that $\bar{a}'\bar{b}' \equiv_{\bar{c}} \bar{a}\bar{b}$ and $\bar{a}'\bar{b}' \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}_{\bar{c}} \bar{a}\bar{b}$. W.l.o.g., $\bar{b}' \geq \bar{b}$; hence, $\bar{a} \in X_{\bar{b}'}$ and

$$d \geq \dim^{\mathrm{cl}}(X_{\bar{b}'}) \geq \mathrm{rk}^{\mathrm{cl}}(\bar{a}/\bar{c}\bar{b}') = \mathrm{rk}^{\mathrm{cl}}(\bar{a}/\bar{c}) = \dim^{\mathrm{cl}}(X). \quad \square$$

We can extend the above lemma to directed families.

Lemma 3.71. *Let $\langle I, \leq \rangle$ be a definable directed set. Let $\left(X_{\bar{b}}\right)_{\bar{b} \in I}$ be a definable increasing family of subsets of $\mathbb{M}^m$ and let $X := \bigcup_{\bar{b} \in I} X_{\bar{b}}$. Let $d \leq m$, and assume that, for every $\bar{b} \in I$, $\dim^{\mathrm{cl}}(X_{\bar{b}}) \leq d$. Then, $\dim^{\mathrm{cl}}(X) \leq d$.*

Proof. W.l.o.g., $\langle I, \leq \rangle$ and the family $\left(X_{\bar{b}}\right)_{\bar{b} \in I}$ are definable without parameters. Let $\bar{a} \in X$ be such that $\mathrm{rk}^{\mathrm{cl}}(\bar{a}) = \dim^{\mathrm{cl}}(X)$, and let $\bar{b}_0 \in I$ be such that $a \in X_{\bar{b}_0}$. By Lemma 3.68, there exists $\bar{b} \in I$ such that $\bar{b} \geq \bar{b}_0$ and $\bar{a}\bar{b}_0 \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}} \bar{b}$. Hence, $\bar{a} \in X_{\bar{b}}$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}} \bar{b}$, and therefore

$$d \geq \dim^{\mathrm{cl}}(X_{\bar{b}}) \geq \mathrm{rk}(\bar{a}/\bar{b}) = \mathrm{rk}(\bar{a}) = \dim^{\mathrm{cl}}(X). \quad \square$$

Remark 3.72. The above lemma is not true if $\left(X_{\bar{b}}\right)_{\bar{b} \in I}$ is a definable *decreasing* family of subsets of $\mathbb{M}^m$, instead of increasing. For instance, let $\mathbb{K}$ be a real closed field, $\mathrm{cl} = \mathrm{acl}$, $I := (\mathbb{K}^{<0} \times \mathbb{K}) \cup \{\langle 0, 0 \rangle\}$; define $\langle x, y \rangle \leq \langle x', y' \rangle$ if $x \leq x'$ and $y = y'$, or $x = 0$. Let $I_{b_1, b_2} := \{\langle x, y \rangle \in I : \langle x, y \rangle \geq \langle b_1, b_2 \rangle\}$. Then, $\langle I, \leq \rangle$ is a directed set, $\dim^{\mathrm{acl}}(I) = 2$, but $\dim^{\mathrm{acl}}(I_{\bar{b}}) \leq 1$ for every $\bar{b} \in I$.

4. Matroids from dimensions

In [25], van den Dries gave a definition of dimension for definable sets; we will show that his approach is almost equivalent to ours. Let $\mathbb{K}$ be a first-order structure.

Definition 4.1. A **dimension function** on $\mathbb{K}$ is a function d from $\mathbb{K}$-definable sets to $\{-\infty\} \cup \mathbb{N}$, such that, for all $m \in \mathbb{N}$ and S, S_1 and S_2 definable subsets of $\mathbb{K}^m$, we have the following.

(Dim 1) $d(S) = -\infty$ iff $S = \emptyset$, $d(\{a\}) = 0$ for every $a \in \mathbb{K}$, $d(\mathbb{K}) = 1$.
(Dim 2) $d(S_1 \cup S_2) = \max\big(d(S_1), d(S_2)\big)$.
(Dim 3) $d(S^\sigma) = d(S)$ for every permutation σ of the coordinates of $\mathbb{K}^m$.
(Dim 4) Let U be a definable subset of $\mathbb{K}^{m+1}$, and, for $i = 0, 1$, let $U(i) := \{x \in \mathbb{K}^m : d(U_x) = i\}$. Then, $U(i)$ is definable *with the same parameters as* U, and $d(U \cap \pi^{-1}(U(i))) = d(U(i)) + i$, $i = 0, 1$, where $\pi := \Pi^{m+1}_m$.

Notice that the axiom (Dim 4) is slightly stronger that the original axiom in [25]; however, after expanding $\mathbb{K}$ by at most $|T|$ many constants, the situation in [25] can be reduced to ours.

Definition 4.2. Given a dimension function d on $\mathbb{K}$, for every $A \subset \mathbb{K}$ and $b \in \mathbb{K}$ we define $b \in \mathrm{cl}^d(A)$ iff there exists $X \subseteq \mathbb{K}$ definable with parameters in A, such that $d(X) = 0$ and $b \in X$.

Theorem 4.3. *The operator cl^d (more precisely, the extension of cl^d to a monster model) is an existential matroid with definable dimension. The dimension induced by cl^d is precisely d.*

Conversely, if cl is an existential matroid with definable dimension, then $\dim^{\mathrm{cl}}$ is a dimension function, and $\mathrm{cl}^{\dim^{\mathrm{cl}}} = \mathrm{cl}$.

Proof. The only nontrivial facts are that, if d is a dimension function, then cl^d is definable and satisfies the EP and the Existence axiom.

(Definability) Let $a \in \mathrm{cl}(B)$. Let $X \subseteq \mathbb{K}$ be B-definable such that $d(X) = 0$ and $a \in X$. Let $\phi(x, \bar{b})$ be the B-formula defining X. By (Dim 4), w.l.o.g., $d(\phi(\mathbb{K}, \bar{y}) \leq 0$ for every $\bar{y}$.[3] Hence, $\phi(x, \bar{y})$ is an x-narrow formula.

(EP) Let $a \in \mathrm{cl}(Bc) \setminus \mathrm{cl}(B)$. Assume, for contradiction, that $c \notin \mathrm{cl}(Ba)$. Let $X \subseteq \mathbb{K}^2$ be B-definable, such that $a \in X_c$ and $d(X_c) = 0$. Let $X' := X \cap \pi^{-1}(X(0))$, where $\pi := \Pi^2_1$. By assumption, $\langle c, a \rangle \in X'$ and, by (Dim 4), $\dim(X') \leq 1$; w.l.o.g., $X = X'$.

Let $Z := \{u \in \mathbb{K} : d(X^u) = 1\}$. Since $c \in X^a$ and $c \notin \mathrm{cl}(Ba)$, $a \in Z$. Since $a \notin \mathrm{cl}(B)$, $d(Z) = 1$. Hence, by (Dim 4) and (Dim 3), $d(X) = 2$, which is absurd.

(Existence) Immediate from Lemma 3.21(5). □

[3] Here it is important that in (Dim 4) we asked that the parameters of $U(i)$ are the same as the parameters of U.

5. Expansions

Remember that $\mathbb{M}$ is a monster model of a complete $\mathcal{L}$-theory T. We are interested in the behaviour of definable matroids under expansions of $\mathbb{M}$. In this section, we assume that $\mathrm{cl} = \mathrm{cl}^{\mathbb{M}}$ is a closure operator on the monster model $\mathbb{M}$.

Definition 5.1. Given $X \subseteq \mathbb{M}$, let the **restriction** $\mathrm{cl}^X : \mathcal{P}(X) \to \mathcal{P}(X)$ and the **relativisation** $\mathrm{cl}_X : \mathcal{P}(\mathbb{M}) \to \mathcal{P}(\mathbb{M})$ of $\mathrm{cl}^{\mathbb{M}}$ be defined as $\mathrm{cl}^X(Y) := \mathrm{cl}^{\mathbb{M}}(Y) \cap X$ and $\mathrm{cl}_X(Y) := \mathrm{cl}^{\mathbb{M}}(XY)$.

Notice that when $\mathbb{M}' \preceq \mathbb{M}$ we have already introduced in Remark 3.27 the notation $\mathrm{cl}^{\mathbb{M}'}$ for the "extension" of $\mathrm{cl}^{\mathbb{M}}$ to $\mathbb{M}'$; this is not problematic, because the two notions coincide for existential matroids.

Remark 5.2. Given $X \subseteq \mathbb{M}$, cl^X is a closure operator on X and cl_X is a closure operator on $\mathbb{M}$. If, moreover, cl is a matroid, then both cl^X and cl_X are matroids, $A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{cl}_X}_{B} C$ iff $A \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{cl}}_{XB} C$, and $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{cl}^X}$ is the restriction of $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{cl}}$ to the subsets of X.

In particular, for every $X \subseteq \mathbb{M}$, the rank and the notion of independence coincide for $\mathrm{cl}^{\mathbb{M}}$ and cl^X (but they are quite different from the corresponding notions for cl_X!), and therefore we do not need to specify for example if the rank is taken w.r.t. $\mathrm{cl}^{\mathbb{M}}$ or w.r.t. cl^X.

Remark 5.3. Given $B \subset \mathbb{M}$ (with $|B| < \kappa$), let $\mathbb{M}_B$ be the expansion of $\mathbb{M}$ with all constants from B, and consider cl_B as a matroid on $\mathbb{M}_B$.

1. If $\mathrm{cl}^{\mathbb{M}}$ is definable, then cl_B is also definable (see Remark 3.28).
2. If $\mathrm{cl}^{\mathbb{M}}$ is a matroid, then cl_B is also a matroid.
3. If $\mathrm{cl}^{\mathbb{M}}$ is definable and satisfies Existence, then cl_B satisfies Existence too.
4. If $\mathrm{cl}^{\mathbb{M}}$ is an existential matroid, then cl_B is also an existential matroid, and $\dim^{\mathrm{cl}^{\mathbb{M}}}$ and $\dim^{\mathrm{cl}_B}$ coincide (the definable sets of $\mathbb{M}$ and of $\mathbb{M}_B$ are the same).

Example 5.4. In the above remark, it is not true that, if $\mathrm{cl}^{\mathbb{M}}$ is a definable matroid, and cl_B satisfies Existence, then $\mathrm{cl}^{\mathbb{M}}$ satisfies Existence. For instance, let B be any nonempty subset of $\mathbb{M}$ (of cardinality less than κ), and $\mathrm{cl}^{\mathbb{M}} = \mathrm{cl}^1$ (see Example 3.26); then, $\mathrm{cl}_B = \mathrm{cl}^0$ satisfies Existence, but $\mathrm{cl}^{\mathbb{M}}$ does not.

Lemma 5.5. *Let $X \subseteq \mathbb{M}$. Let $\mathbb{M}'$ be the expansion of $\mathbb{M}$ with a predicate P for X. Assume that $\mathbb{M}'$ is a monster model, and denote by cl'_X the closure operator $\mathrm{cl}'_X(Y) := \mathrm{cl}^{\mathbb{M}}(XY)$ on $\mathbb{M}'$ (cl'_X coincides with cl_X).*

1. *If $\mathrm{cl}^{\mathbb{M}}$ is definable, then cl'_X is definable on $\mathbb{M}'$.*
2. *If $\mathrm{cl}^{\mathbb{M}}$ is a matroid, then cl'_X is a matroid.*

Proof. Let $D \subseteq X$ be such that $|D| < \kappa$ and $\mathrm{cl}^{\mathbb{M}}(X) = \mathrm{cl}^{\mathbb{M}}(D)$.

1. $b \in \mathrm{cl}'_X(A)$ iff $b \in \mathrm{cl}^{\mathbb{M}}(AX)$ iff $\mathbb{M} \models \phi(b, \bar{a}, \bar{c})$ for some x-narrow formula $\phi(x, \bar{y}, \bar{z})$, some $\bar{a} \subseteq A$ and some $\bar{c} \in X^n$. Define $\psi(x, \bar{y}) := \exists \bar{z}\, \big(P(\bar{z})\ \&\ \phi(x, \bar{y}, \bar{z})\big)$. Notice that ψ is an $\mathcal{L}(P)$-formula, and that, for every $\bar{a}' \subset \mathbb{M}$, $\psi(\mathbb{M}', \bar{a}') \subseteq \mathrm{cl}'_X(\bar{a}')$.
2. Trivial. □

Remark 5.6. Let $\mathbb{M}$, X and $\mathbb{M}'$ be as in the above lemma. Let $\langle \mathbb{B}, Y\rangle \prec \langle \mathbb{M}, X\rangle$; assume, moreover, that $\mathrm{cl}^{\mathbb{M}}$ is a definable closure operator on $\mathbb{M}$. Then, $(\mathrm{cl}^{\mathbb{B}})_Y = (\mathrm{cl}_X)^{\mathbb{B}}$; that is, for every $A \subseteq \mathbb{B}$, $\mathbb{B} \cap \mathrm{cl}_Y(A) = \mathbb{B} \cap \mathrm{cl}_X(A)$.

Hence, in the above situation, inside $\mathbb{B}$ we do not need to distinguish between cl_X and cl_Y.

Remark 5.7. Let cl be a definable matroid (not necessarily existential), and let X, Y, X^*, and Y^* be elementary substructures of $\mathbb{M}$, such that $X \subseteq X^* \cap Y$ and $X^* \cup Y \subseteq Y^*$. Let $\mathcal{L}^2$ be the expansion of $\mathcal{L}$ with a new unary predicate P, and consider $\langle Y, X\rangle$ and $\langle Y^*, X^*\rangle$ as $\mathcal{L}^2$-structures. Assume that $(Y, X) \preceq (Y^*, X^*)$. Then, $X^* \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{cl}}_{X} Y$.

Proof. Let $\bar{x}^* \subset X^*$; it suffices to prove that $\bar{x}^* \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{cl}}_{X} Y$. However, $\mathrm{tp}_{\mathcal{L}}(x^*/Y)$ is finitely satisfied in X, and we are done. □

Assume that $\mathbb{M}$ expands a ring without zero divisors. Let $\mathbb{M}'$ be an expansion of $\mathbb{M}$ to a larger language $\mathcal{L}'$; assume that $\mathbb{M}'$ is also a monster model and that cl' is an existential matroid on $\mathbb{M}'$. We have seen that in this case cl' is the unique existential matroid on $\mathbb{M}'$, and that, for every X definable subset of $\mathbb{M}'$, $\dim'(X) = 0$ iff $F(X^4) \neq \mathbb{M}'$ (where $\dim'$ is the dimension induced by cl'). It is clear that cl', in general, is not definable in $\mathbb{M}$. However, the dimension function $\dim'$ *is* definable in $\mathbb{M}$; hence, we can restrict the dimension function $\dim'$ to the sets definable in $\mathbb{M}$ (with parameters), and get a function dim.

Remark 5.8. Let $\mathbb{M}$, $\mathbb{M}'$, $\dim'$, and dim be as above. Then, dim is a dimension function on $\mathbb{M}$ (i.e., it satisfies the axioms in Definition 3.29). The matroid cl induces by dim is characterised by the following.

For every A and b, we have $b \in \mathrm{cl}(A)$ iff there exists $X \subseteq \mathbb{M}$, definable in $\mathbb{M}$ with parameters from A, such that $F(X^4) \neq \mathbb{M}$ and $b \in X$.

Corollary 5.9. *Assume that $\mathbb{M}$ expands a ring without zero divisors. Let $\mathbb{M}'$ be an expansion of $\mathbb{M}$. If $\mathbb{M}'$ is geometric, then $\mathbb{M}$ is also geometric.*

Compare the above corollary with [1, Corollary 2.38 and Example 2.40].

6. Extension to imaginary elements

Again, $\mathbb{M}$ is a monster model of a complete theory T, and cl is an existential matroid on $\mathbb{M}$. Let $\mathbb{M}^{eq}$ be the set of imaginary elements, and let T^{eq} be the theory of $\mathbb{M}^{eq}$. Our aim is to extend the matroid cl to a closure operator cl^{eq} on $\mathbb{M}^{eq}$.

We will start with the definition of $a \in \mathrm{cl}^{eq}(B)$ when a is real and B is imaginary.

Definition 6.1. Let B be a set of imaginary elements (of cardinality less than κ), and let a be a real element. We say that $a \in \mathrm{cl}^{eq}(B)$ iff $\Xi(a/B)$ has finite $\mathrm{rk}^{\mathrm{cl}}$.

It is relatively easy to prove the following fact.

Remark 6.2 (*Exchange Principle [13, 3.1]*)**.** The operator cl^{eq} satisfies the Exchange Principle for real points over imaginary parameters. That is, for a and b real elements and C imaginary, if $a \in \mathrm{cl}^{eq}(bC) \setminus \mathrm{cl}^{eq}(C)$, then $b \in \mathrm{cl}^{eq}(aC)$.

Recall that $\mathbb{M}$ has geometric elimination of imaginaries if every for imaginary element a there exists a real tuple $\bar{b}$ such that a and $\bar{b}$ are interalgebraic. If $\mathbb{M}$ had geometric elimination of imaginaries, we could define $a \in \mathrm{cl}^{eq}(B)$ iff there exists a real tuple $\bar{c}$ such that $a \in \mathrm{acl}^{eq}(\bar{c})$ and $\bar{c} \subset \mathrm{cl}^{eq}(B)$. Without geometric elimination of imaginaries, the definition is substantially more complicated; however, one can proceed from Remark 6.2 as in [13, Section 3] to define the desired extension cl^{eq} (notice that [13] uses dim for what we would call $\mathrm{rk}^{\mathrm{cl}}$).

If cl has definable dimension $\dim^{\mathrm{cl}}$, then the definition of cl^{eq} is much simpler, and proceeds as follows. Let $X \subset \mathbb{M}^n$ be definable, and let E be a definable equivalence relation on X. If the dimension of each equivalence class is constant e, we define the dimension of the imaginary set X/E as $\dim^{\mathrm{cl}^{eq}}(X/E) := \dim^{\mathrm{cl}}(X) - e$. In the general case, let $X_i := \left\{x \in X : \dim^{\mathrm{cl}}(Ex) = i\right\}$ (where Ex is the equivalence class of x); then each X_i is definable, and $X = X_0 \sqcup \cdots \sqcup X_n$; thus, we define $\dim^{\mathrm{cl}^{eq}}(X/E) := \max_i\left(\dim^{\mathrm{cl}^{eq}}(X_i/E)\right)$. It is easy to verify that $\dim^{\mathrm{cl}^{eq}}$ is the dimension function associated to cl^{eq}, and therefore we can define cl^{eq} as

$$\mathrm{cl}^{eq}(A) = \left\{c \in \mathbb{M}^{eq} : \exists X \subset \mathbb{M}^{eq} A\text{-definable s.t. } c \in X \ \&\ \dim^{\mathrm{cl}^{eq}}(X) = 0\right\}.$$

In general, we can use cl^{eq} (or, better, the associated rank $\mathrm{rk}^{\mathrm{cl}^{eq}}$) to extend the independence relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$ to imaginary elements, setting $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}^{eq}}_C B$ iff, for every finite subset A' of A, $\mathrm{rk}^{\mathrm{cl}^{eq}}(A'/BC) = \mathrm{rk}^{\mathrm{cl}^{eq}}(A'/C)$; it is then easy to verify that $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}^{eq}}$ is an independence relation on $\mathbb{M}^{eq}$ extending $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}}$, and that the corresponding version of antireflexivity holds for it (cf. Remark 3.7). When no danger of confusion arises, we will freely use cl to denote also cl^{eq}, and similarly for the related notions $\dim^{\mathrm{cl}^{eq}}$, $\mathrm{rk}^{\mathrm{cl}^{eq}}$, and $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{cl}^{eq}}$.

Notice that acl^{eq} is a closure operator on $\mathbb{M}^{eq}$ extending acl; however, even when cl $=$ acl, in general $\mathrm{cl}^{eq} \neq \mathrm{acl}^{eq}$; hence, when cl $=$ acl , we will have to pay attention not to confuse the two possible extensions of cl to $\mathbb{M}^{eq}$ (cf. the next remark). On the other hand, by dcl^{eq} we will always denote the usual extension of dcl to an imaginary element: $a \in \mathrm{dcl}(b)$ if $\Xi(a/B) = \{a\}$.

Remark 6.3. Assume that $\mathbb{M}$ is a pregeometric structure and that cl $=$ acl. Given $\bar{b}$ a real or imaginary tuple, we have $\mathrm{acl}^{eq}(\bar{b}) \subseteq \mathrm{cl}^{eq}(\bar{b})$ and $\mathrm{cl}^{eq}(\bar{b}) \cap \mathbb{M} = \mathrm{acl}^{eq}(\bar{b}) \cap \mathbb{M}$. However, it is not true in general that $\mathrm{cl}^{eq} = \mathrm{acl}^{eq}$; more precisely, $\mathrm{cl}^{eq} = \mathrm{acl}^{eq}$ iff $\mathbb{M}$ is surgical [13]. For instance, if $\mathbb{K}$ is either a p-adic field, or an algebraically closed valued field, then $\mathbb{K}$ is geometric but not surgical; its value group Γ has dimension 0 but it is infinite; therefore, there exists $\gamma \in \Gamma$ such that $\gamma \in \mathrm{cl}^{eq}(\emptyset) \setminus \mathrm{acl}^{eq}(\emptyset)$.

7. Density

Again, $\mathbb{M}$ is a monster model of a complete theory T, and $\mathrm{cl} = \mathrm{cl}^{\mathbb{M}}$ is an existential matroid on $\mathbb{M}$.

Definition 7.1. Let $\mathbb{K} \preceq \mathbb{M}$, and let $X \subseteq \mathbb{K}$. We say that X is **dense** in $\mathbb{K}$ if, for every $\mathbb{K}$-definable subset U of $\mathbb{K}$, if $\dim^{\mathrm{cl}}(U) = 1$, then $U \cap X \neq \emptyset$. Recall that $\mathrm{cl}^{\mathbb{K}}(X) := \mathrm{cl}^{\mathbb{M}}(X) \cap \mathbb{K}$; we say that X is cl-**closed** in $\mathbb{K}$ if $\mathrm{cl}^{\mathbb{K}}(X) = X$.

Examples 7.2. 1. If $\mathbb{K}$ is geometric, then X is dense in $\mathbb{K}$ iff X intersects every infinite definable subset of $\mathbb{K}$; in that case, our definition of density coincides with the one in [16, Section 1].
2. If $\mathbb{K}$ is strongly minimal, then X is dense in $\mathbb{K}$ iff X is infinite.
3. If $\mathbb{K}$ is o-minimal and densely ordered, or if $\mathbb{K}$ is the field of p-adic numbers, then X is dense in $\mathbb{K}$ in the sense of the above definition iff X is topologically dense in $\mathbb{K}$ (this is the motivation here and in [16] for the choice of the term "dense"). See also Section 9 for a generalisation of this example.

Remark 7.3. If $X \subset \mathbb{K}$ is dense (in $\mathbb{K}$), and $a \in X$, then $X \setminus \{a\}$ is also dense.

Proof. If $U \subseteq \mathbb{K}$ is definable and of dimension 1, then $U \setminus \{a\}$ is also definable and of dimension 1. □

Lemma 7.4. *Let $X \subseteq \mathbb{K} \preceq \mathbb{M}$. If X is* cl*-closed and dense in $\mathbb{K}$, then $X \preceq \mathbb{K}$.*

Proof. Tarski–Vaught test. Let $A \subseteq \mathbb{K}$ be definable, with parameters from A; we must show that $A \cap X \neq \emptyset$. If $\dim^{\mathrm{cl}}(A) = 1$, this is true because X is dense in $\mathbb{K}$. If $\dim^{\mathrm{cl}}(A) = 0$, this is true because X is cl-closed in $\mathbb{K}$. $\square$

Lemma 7.5. *Let $\mathbb{K} \preceq \mathbb{M}$ be a saturated model of cardinality $\lambda > |T|$. Then, there exists $X \subset \mathbb{K}$ such that X is a* cl-*basis of $\mathbb{K}$ and X is dense in $\mathbb{K}$. Moreover, there exists $\mathbb{F} \prec \mathbb{K}$ such that $\mathbb{F}$ is* cl-*closed and dense in $\mathbb{K}$ and $\mathbb{F}$ is not equal to $\mathbb{K}$.*

Proof. Let $(A_i)_{i<\lambda}$ be an enumeration of all subsets of $\mathbb{K}$ which are definable (with parameters from $\mathbb{K}$) and of dimension 1. Build a cl-independent sequence $(a_i)_{i<\lambda}$ inductively: for every $\mu < \lambda$, we make sure that $(a_i)_{i<\mu}$ is cl-independent, and that, for every $i < \mu$, there exists $j < \mu$ such that $a_j \in A_i$. Fix $\mu < \lambda$, and assume that we have already defined a_i for every $i < \mu$; we have to define a_μ.

Claim 1. *There exists $a_\mu \in A_\mu$ such that a_μ is* cl-*independent from $(a_i)_{i<\mu}$.*

Otherwise, $\mathrm{rk}^{\mathrm{cl}}(A_\mu) < \lambda$, which is absurd.

Define a_μ as in the above claim. By construction, $X' := \{a_i : i < \lambda\}$ is cl-independent and dense in $\mathbb{K}$; we can complete it to a cl-basis X, which is also dense.

Choose $a \in X$, let $Y := X \setminus \{a\}$, and let $\mathbb{F} := \mathrm{cl}(Y)$. Since X is dense, Y is also dense, and therefore $\mathbb{F}$ is dense in $\mathbb{K}$. Moreover, since X is a cl-basis, $a \notin \mathbb{F}$. Finally, by Lemma 7.4, $\mathbb{F} \prec \mathbb{K}$. $\square$

The proof of the above lemma shows the following stronger results.

Corollary 7.6. *Let $\mathbb{K}$ be as in Lemma 7.5. Let $c \in \mathbb{K} \setminus \mathrm{cl}(\emptyset)$. Then, there exists $\mathbb{F} \prec \mathbb{K}$* cl-*closed and dense in $\mathbb{K}$, such that $c \notin \mathbb{F}$.*

Given $\mathbb{K} \models T$, and X, Y subsets of $\mathbb{K}$, we say that X is dense in $\mathbb{K}$ w.r.t. Y if, for every subset U of $\mathbb{K}$ definable with parameters from Y, if $\dim^{\mathrm{cl}}(U) = 1$, then $U \cap X \neq \emptyset$.

Lemma 7.7. *There exist $\mathbb{F}$ and $\mathbb{K}$ models of T, such that $\mathbb{F} \prec \mathbb{K}$ and $\mathbb{F}$ is a proper dense and* cl-*closed subset of $\mathbb{K}$.*

Proof. Notice that, if T has a saturated model of cardinality $> |T|$, we can apply Lemma 7.5. Otherwise, let $\mathbb{K}_0 \prec \mathbb{K}_1 \prec \cdots$ be an elementary chain of models of T, such that, for every $n \in \mathbb{N}$, $\mathbb{K}_{n+1}$ is $\big(|\mathbb{K}_n| + |T|\big)^+$-saturated, and let $\mathbb{K} := \bigcup_{n\in\mathbb{N}} \mathbb{K}_n$. Proceeding as in the proof of Lemma 7.5, for every $n \in \mathbb{N}$ we build a cl-independent set $\mathcal{A}_n$ of elements in $\mathbb{K}_{n+1}$, such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ and $\mathcal{A}_n$ is dense in $\mathbb{K}_{n+1}$ w.r.t. $\mathbb{K}_n$. Let $\mathcal{A} := \bigcup_n \mathcal{A}_n$. Then, $\mathcal{A}$ is a cl-independent set of elements in $\mathbb{K}$, which is also dense in $\mathbb{K}$. Conclude as in Lemma 7.5. $\square$

8. Dense pairs

Let $\mathbb{B}$ be a real closed field and $\mathbb{A}$ be a *proper* dense subfield of $\mathbb{A}$, such that $\mathbb{A}$ is also real closed. We call $\langle \mathbb{B}, \mathbb{A} \rangle$ a dense pair of real closed fields, and we consider its theory, in the language of ordered fields expanded with a predicate for a (dense) subfield. Robinson [22] proved that the theory of dense pairs of real closed fields is complete. In [26], van den Dries extended Robinson's theorem to o-minimal theories: if T is a complete o-minimal theory expanding the theory of (densely) ordered Abelian groups, then the theory of dense elementary pairs of models of T is complete. Macintyre [16] introduced an abstract notion of density, in the context of geometric theories, which for o-minimal theories specialises to the usual topological notion, and proved various results; more recent work has been done in the context of so-called "lovely pairs" either of geometric structures (see for instance [4,6]) or of simple structures (see [2], which extends Poizat's work on "beautiful pairs" of stable structures [19]).

In Section 7, we also proposed an abstract notion of density, which for geometric theories specialises to the one given by Macintyre. However, it is not true in general that the theory of dense pairs of models of T is complete (unless T is geometric and expands the theory of integral domains); the main result of this section is that if T expands the theory of integral domains, and we add the additional condition that $\mathbb{A}$ is cl-closed in $\mathbb{B}$, we obtain a complete theory, which we denote by T^d (if T is geometric, the additional condition is trivially true). We will also show that T^d admits an existential matroid (the small closure: Section 8.4), which will allow us to iterate the procedure, by considering dense pairs of models of T^d itself, and so on; see Section 13. For the exposition we will follow [26], using, however, some ideas from [6,2].

We assume that the structure $\mathbb{M}$ is a monster model of a complete theory T, and that $\mathrm{cl} = \mathrm{cl}^{\mathbb{M}}$ is an existential matroid on $\mathbb{M}$. For this section, we will write dim instead of $\dim^{\mathrm{cl}}$, rk instead of $\mathrm{rk}^{\mathrm{cl}}$, and $\mathop{\smile\!\!\!\!\!\mid}$ instead of $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}$.

Definition 8.1. Let $\mathcal{L}^2$ be the expansion of $\mathcal{L}$ by a new unary predicate P. Let T^2 be the $\mathcal{L}^2$-expansion of T, whose models are the pairs $\langle \mathbb{K}, \mathbb{F} \rangle$, with $\mathbb{F} \prec \mathbb{K}$, $\mathbb{F} \neq \mathbb{K}$, and $\mathbb{F}$ cl-closed in $\mathbb{K}$.

Assume that dim is definable. Let T^d be the $\mathcal{L}^2$-expansion of T saying that $\mathbb{F}$ is a proper, cl-closed and dense subset of $\mathbb{K}$ (we need definability of dim to express in a first-order way that $\mathbb{F}$ is dense in $\mathbb{K}$).

Notice that, by Lemma 7.4, T^d extends T^2. Notice that, if $\mathrm{cl} = \mathrm{acl}$, then T^2 is the theory of pairs $\langle \mathbb{K}, \mathbb{F} \rangle$, with $\mathbb{F} \prec \mathbb{K} \models T$; however, if $\mathrm{cl} \neq \mathrm{acl}$, then there exists $\mathbb{F} \prec \mathbb{M}$ with $\mathbb{F}$ not cl-closed in $\mathbb{M}$ (take any $\mathbb{F} \prec \mathbb{M}$ such that $|\mathbb{F}| < \kappa$).

Remark 8.2. The theory T^d is consistent.

Proof. By Lemma 7.7. $\square$

Proviso. *For the remainder of this section, we assume that T expands the theory of integral domains* (*and therefore* dim *is definable*)*, and that* $\langle \mathbb{K}, \mathbb{F}\rangle \models T^d$.

Theorem 8.3. *The theory T^d is complete.*

Definition 8.4. An $\mathcal{L}^2$-formula $\phi(\bar{x})$ is **basic** if it is of the form

$$\exists \bar{y}\big(P(\bar{y}) \,\&\, \psi(\bar{x}, \bar{y})\big),$$

where ψ is an $\mathcal{L}$-formula.[4]

Theorem 8.5. *Each $\mathcal{L}^2$-formula $\psi(\bar{x})$ is equivalent, modulo T^d, to a Boolean combination of basic formulae, with the same parameters as ψ.*

Theorems 8.3 and 8.5 will be proved in Section 8.2.

8.1. Small sets

In this subsection, we will assume that $\langle \mathbb{K}, \mathbb{A}\rangle \models T^2$.

Definition 8.6. A subset X of $\mathbb{K}$ is $\mathbb{A}$**-small** if $X \subseteq f(\mathbb{A}^n)$, for some Z-application $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ which is definable in $\mathbb{K}$ (cf. Definition 3.54).

Definition 8.7. Let $X \subseteq \mathbb{K}^n$. We say that X is **weakly dense** in $\mathbb{K}^n$ if, for every definable $U \subseteq \mathbb{K}^n$, if $X \subseteq U$, then $\dim(U) = n$.

For instance, if $\mathrm{cl} = \mathrm{acl}$, then X is a weakly dense subset of $\mathbb{K}$ iff X is infinite.

Remark 8.8. If X is a weakly dense subset of $\mathbb{K}$, then X^n is a weakly dense subset of $\mathbb{K}^n$.

Lemma 8.9. *If $\mathbb{K} \models T$ and $\mathbb{K}' \preceq \mathbb{K}$, then $\mathbb{K}'$ is weakly dense in $\mathbb{K}$.*

Proof. W.l.o.g., the pair $\langle \mathbb{K}, \mathbb{K}'\rangle$ is ω-saturated. Assume, for contradiction, that $U \subset \mathbb{K}$ is definable, with parameters $\bar{b} \in \mathbb{K}^n$, $\dim(U) = 0$, and $\mathbb{K}' \subseteq U$. By saturation, $\mathrm{rk}(\mathbb{K}')$ is infinite; let $\bar{c} \in \mathbb{K}'^{n+1}$ be independent elements. However, $\bar{c} \in U$, and therefore $\bar{c} \subset \mathrm{cl}(\bar{b})$, which is absurd. □

The following result is the most delicate one; the use of Z-applications will allow us to mimic van den Dries' proof.

Lemma 8.10 (*[26, 1.1]*)**.** *Let $f : \mathbb{K}^{n+1} \rightsquigarrow \mathbb{K}$ be a Z-application $\mathbb{A}$-definable in $\mathbb{K}$, and let $b_0 \in \mathbb{K} \setminus \mathbb{A}$. For every $x \in \mathbb{K}$ and $\bar{y} = \langle y_0, \dots, y_n\rangle \in \mathbb{K}^{n+1}$, let $p(\bar{y}, x) := y_0 + y_1 x + \cdots + y_n x^n$. Then, there exists $\bar{a} \in \mathbb{A}^{n+1}$ such that*

$$p(\bar{a}, b_0) \notin f(\mathbb{A}^n \times \{b_0\}).$$

Proof. Otherwise, there is, for each $\bar{a} \in \mathbb{A}^{n+1}$, a tuple $\bar{c} \in \mathbb{A}^n$ such that $p(\bar{a}, b_0) \in f(\bar{c}, b_0)$. W.l.o.g., f is definable without parameters. For each $\bar{y} \in \mathbb{K}^{n+1}$ and $\bar{z} \in \mathbb{K}^n$, let $D(\bar{y}, \bar{z}) := \{x \in \mathbb{K} : p(\bar{y}, x) \in f(\bar{z}, x)\}$. Define $W := \{\langle \bar{y}, \bar{z}\rangle := \dim(D(\bar{y}, \bar{z})) = 1\}$, and $Y := \Pi^{2n+1}_{n+1}(W)$. Since $b_0 \notin \mathbb{A}$ and $\mathbb{A}$ is cl-closed in $\mathbb{K}$, we have $\mathbb{A}^{n+1} \subseteq Y$. Since Y is definable, Remark 8.8 and Lemma 8.9 imply that $\dim(Y) = n+1$; therefore, $\dim(W) \geq n+1$. Let $Z := \big\{\bar{z} \in \mathbb{K}^n : \dim(W^{\bar{z}}) \geq 1\big\}$. Since $\dim(W) \geq n+1$ and $\dim(\mathbb{K}^n) = n$, we have that $\dim(Z) \geq n$, and hence Z is nonempty. Choose $\bar{c} \in Z$. Let $\bar{a} \in \mathbb{K}^{n+1}$ be such that $\langle \bar{a}, \bar{c}\rangle \in W$ and $\mathrm{rk}(\bar{a}/\bar{c}) \geq 1$. By definition of W, $\dim(D(\bar{a}, \bar{c})) = 1$; choose $b \in D(\bar{a}, \bar{c})$ such that $\mathrm{rk}(b/\bar{c}\bar{a}) = 1$. Define $d := p(\bar{a}, b)$; remember that $d \in f(\bar{c}, b)$, and therefore $d \in \mathrm{cl}(\bar{c}b)$. Let $\bar{a}' \in \mathbb{K}^{n+1}$ be such that $\bar{a}' \equiv_{\bar{c}bd} \bar{a}$ and $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{\bar{c}bd} \bar{a}$. Since $d \in \mathrm{cl}(\bar{c}, b)$, we have $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{\bar{c}b} \bar{a}$. Moreover, $p(\bar{a}', b) = d$; therefore, $p(\bar{a} - \bar{a}', b) = 0$.

If $\bar{a} \neq \bar{a}'$, this implies that b is algebraic over $\bar{a} - \bar{a}'$, and therefore $b \in \mathrm{cl}(\bar{a}\bar{a}')$, contradicting the fact that $b \notin \mathrm{cl}(\bar{a}\bar{c})$ and $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{\bar{c}b} \bar{a}$.

If instead $\bar{a} = \bar{a}'$, then $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{\bar{c}b} \bar{a}$ implies that $\bar{a} \subset \mathrm{cl}(\bar{c}b)$, contradicting the facts that $b \notin \mathrm{cl}(\bar{c}\bar{a})$ and $\mathrm{rk}(\bar{a}/\bar{c}) \geq 1$. □

Notice that the hypothesis of the above lemma can be weakened to the following.

$\mathbb{K} \models T$ and $\mathbb{A}$ is a proper cl-closed and weakly dense subset of $\mathbb{K}$.

Remark 8.11 (*[26, 1.3]*)**.** Each $\mathbb{A}$-small subset of $\mathbb{K}$ is a proper subset of $\mathbb{K}$.

Proof. The same as [26, Corollary 1.3]. □

Remark 8.12. A finite union of $\mathbb{A}$-small subsets of $\mathbb{K}$ is also $\mathbb{A}$-small.

Lemma 8.13. *Let $B \subseteq \mathbb{K}$ be a proper* cl*-closed subset. Then, B is co-dense in $\mathbb{K}$; that is, $\mathbb{K} \setminus B$ is dense in $\mathbb{K}$.*

Proof. Since B is cl-closed in $\mathbb{K}$, $F(B^4) \subseteq B$ (cf. Definition 3.46). Assume, for contradiction, that there exists U definable in $\mathbb{K}$, such that $\dim(U) = 1$ and $U \subseteq B$. Then, $F(U^4) = \mathbb{K}$, and therefore $F(B^4) = \mathbb{K}$, contradicting the assumption that $B \neq \mathbb{K}$. □

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4 Basic formulae were called "special" in [26].

Lemma 8.14 *([26, Lemma 1.5]). If the pair $\langle \mathbb{K}, \mathbb{A}\rangle$ is λ-saturated, where λ is an infinite cardinal with $|T| < \lambda$, then* $\dim(\mathbb{K}/\mathbb{A}) \geq \lambda$. *Hence, if $|X| < \lambda$, then $\mathrm{cl}^{\mathbb{K}}(\mathbb{A}X)$ is co-dense in $\mathbb{K}$.*

Proof. The same as [26, Lemma 1.5]. Let E be a generating set for $\mathbb{K}/\mathbb{A}$, and suppose that $|E| < \lambda$. Let $\Gamma(v)$ be the set of $\mathcal{L}^2$-formulae of the form

$$\forall y_1 \dots \forall y_n \big(P(\bar{y}) \to v \notin f(\bar{y}, e_1, \dots, e_p)\big),$$

where $f(\bar{y}, \bar{z})$ is a Z-application $\emptyset$-definable in $\mathbb{K}$, and $e_1, \dots, e_p$ are in E. By Remarks 8.11 and 8.12, $\Gamma(v)$ is a consistent set of formulae, with fewer than λ many parameters. By saturation, there exists $b \in \mathbb{K}$ realising the partial type $\Gamma(v)$. Thus $b \notin \mathrm{cl}^{\mathbb{K}}(\mathbb{A}E)$, which is absurd. $\square$

Notice that, in the original [26, Lemma 1.5], if T expands RCF, then van den Dries' assumption that $\mathcal{A}$ is dense in $\mathcal{B}$ is superfluous; density is used if, however, T expands only the theory of ordered Abelian groups.

8.2. Proof of Theorems 8.3 and 8.5

The proof is similar to the ones in [6,2]; the following definition is a variant of the ones they use.

Definition 8.15. Let $\langle \mathbb{B}, \mathbb{A}\rangle \models T^2$ and $C \subseteq \mathbb{B}$. Let $\bar{c}$ be a tuple of elements from $\mathbb{B}^{\mathrm{eq}}$; the P-type of $\bar{c}$, denoted by P-tp$(\bar{c})$, is the information which tells us which members of $\bar{c}$ are in $\mathbb{A}$ (notice that the elements in $\bar{c}$ are real or imaginary, but only real elements can be in $\mathbb{A}$). We say that $\bar{c}$ is P-independent if $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{\mathbb{A}\cap\bar{c}} \mathbb{A}$ (where, again, only the real elements of $\bar{c}$ can be in $\mathbb{A} \cap \bar{c}$).

Notation 8.16. We will use a superscript 1 to denote model-theoretic notions for $\mathcal{L}$, and a superscript 2 to denote those notions for $\mathcal{L}^2$; for instance, we will write $a \equiv^1_C a'$ if the $\mathcal{L}$-types of a and a' over C are the same, and $a \equiv^2_C a'$ if the $\mathcal{L}^2$-types of a and a' over C are the same; similarly, acl^2 will denote the T^2-algebraic closure.

Both theorems are immediate consequences of the following proposition.

Proposition 8.17. *Let $\langle \mathbb{B}, \mathbb{A}\rangle$ and $\langle \mathbb{B}', \mathbb{A}'\rangle$ be models of T^d. Let $\bar{c}$ be a (possibly infinite) P-independent tuple in $\mathbb{B}^{\mathrm{eq}}$, and let $\bar{c}'$ be a P-independent tuple in $(\mathbb{B}')^{\mathrm{eq}}$ of the same length and the same sorts. If $\bar{c} \equiv^1 \bar{c}'$ and P-tp$(\bar{c})$ = P-tp$(\bar{c}')$, then $\bar{c} \equiv^2 \bar{c}'$.*

Proof. Back-and-forth argument. Let λ be a cardinal such that $|T| + |\bar{c}| < \lambda < \kappa$. W.l.o.g., we can assume that both $\langle \mathbb{B}, \mathbb{A}\rangle$ and $\langle \mathbb{B}', \mathbb{A}'\rangle$ are λ-saturated. Let $\bar{e}$ (resp. $\bar{e}'$) be the subtuple of $\bar{c}$ (resp. of $\bar{c}'$) of nonreal elements. Let

$$\begin{aligned}\Gamma := \Big\{ f : \tilde{c} \to \tilde{c}' :\quad & \bar{c} \subset \tilde{c} \subset \mathbb{B}^{\mathrm{eq}}, \quad \bar{c}' \subset \tilde{c}' \subset (\mathbb{B}')^{\mathrm{eq}},\\ & \tilde{c}\ \&\ \tilde{c}' \text{ of the same length less than } \lambda \text{ and of the same sorts},\\ & \text{with all nonreal elements of } \tilde{c} \text{ in } \bar{e},\\ & f \text{ is a bijection},\\ & \tilde{c}\ \&\ \tilde{c}' \text{ are P-independent}, \quad \tilde{c} \equiv^1 \tilde{c}', \quad \text{P-tp}(\tilde{c}) = \text{P-tp}(\tilde{c}')\Big\}.\end{aligned}$$

We want to prove that Γ has the back-and-forth property. So, let $f : \tilde{c} \to \tilde{c}'$ be in Γ, and let $d \in \mathbb{B} \setminus \bar{c}$; we want to find $g \in \Gamma$ such that g extends f and d is in the domain of g. W.l.o.g., $\tilde{c} = \bar{c}$ and $\tilde{c}' = \bar{c}'$. Let $\bar{a} := \bar{c} \cap \mathbb{A}$, and let $\bar{a}' := \bar{c}' \cap \mathbb{A}'$. Notice that $f(\bar{a}) = \bar{a}'$ and that $\mathbb{A} \cap \mathrm{cl}(\bar{c}) = \mathbb{A} \cap \mathrm{cl}(\bar{a}) =: \mathrm{cl}^{\mathbb{A}}(\bar{a})$, and similarly for $\bar{c}'$. We distinguish some cases.

Case 1:: $d \in \mathbb{A} \cap \mathrm{cl}^{\mathbb{B}}(\bar{c}) = \mathrm{cl}^{\mathbb{A}}(\bar{a})$. Notice that $\bar{c}d \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}d} \mathbb{A}$, and therefore $\bar{c}d$ is P-independent. There is a x-narrow formula $\phi(x, \bar{y})$ such that $\mathbb{B} \models \phi(d, \bar{a})$. Choose $d' \in \mathbb{A}'$ such that $\bar{c}d \equiv^1 \bar{c}'d'$; therefore, $\mathbb{B}' \models \phi(d', \bar{a}')$; hence, $d' \in \mathrm{cl}^{\mathbb{B}'}(\bar{a}') \subset \mathbb{A}'$, and thus $\bar{c}'d'$ is also P-independent and has the same P-type as $\bar{c}d$. Thus, we can extend f to $\bar{c}d$ setting $g(d) := d'$.

Case 2:: $d \in \mathbb{A} \setminus \mathrm{cl}^{\mathbb{B}}(\bar{c}) = \mathbb{A} \setminus \mathrm{cl}^{\mathbb{A}}(\bar{a})$. Since $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}} \mathbb{A}$ and $\bar{c} \subset \mathbb{A}$, we have $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}d} \mathbb{A}$, and therefore $\bar{c}d$ is P-independent. Let $q(x) := \mathrm{tp}^1(d/\bar{c})$, and let $q' := f(q) \in S^1_1(\bar{c}')$. Notice that $q \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}} \bar{c}$ (because $d \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}} \bar{c}$), and therefore $q' \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}'} \bar{c}'$. Since $\mathbb{A}'$ is *dense* in $\mathbb{B}'$ and $\langle \mathbb{B}', \mathbb{A}'\rangle$ is λ-saturated, there exists $d' \in \mathbb{A}'$ realising q'. It is now easy to see that $\bar{c}'d'$ is P-independent, and that we can extend f to $\bar{c}d$ by setting $g(d) := d'$.

Case 3:: $d \in \mathrm{cl}^{\mathbb{B}}(\bar{c}\mathbb{A}) \setminus \mathbb{A}$. Let $\bar{a}_0 \in \mathbb{A}^n$ be such that $d \in \mathrm{cl}^{\mathbb{B}}(\bar{b}\bar{a}_0)$ ($\bar{a}_0$ exists because cl is finitary). By applying n times the cases 1 or 2, we can extend f to $f' \in \Gamma$ such that $\bar{a}_0$ is a subset of the domain of f'. By substituting f with f', we are reduced to the case that $d \in \mathrm{cl}^{\mathbb{B}}(\bar{c}) \setminus \mathbb{A}$. Since $\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}} \mathbb{A}$ and $d \in \mathrm{cl}^{\mathbb{B}}(\bar{c})$, we have $\bar{c}d \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}} \mathbb{A}$, and hence $\bar{c}d$ is P-independent. Let $d' \in \mathbb{B}'$ be such that $d'\bar{c}' \equiv^1 d\bar{c}$. For the same reason as above, $\bar{c}'d'$ is also P-independent. It remains to show that $\bar{c}d$ and $\bar{c}'d'$ have the same P-type, that is, that $d' \notin \mathbb{A}'$. If, for contradiction, $d' \in \mathbb{A}'$, then $d' \in \mathrm{cl}^{\mathbb{B}'}(\bar{c}') \cap \mathbb{A}' = \mathrm{cl}^{\mathbb{A}'}(\bar{a}')$; therefore, there would be a x-narrow-formula witnessing it, and thus $d \in \mathrm{cl}^{\mathbb{B}}(\bar{a}) \subseteq \mathbb{A}$, which is absurd.

Case 4:: $d \notin \mathrm{cl}^{\mathbb{B}}(\bar{c}\mathbb{A})$. Let $\bar{a}_0 \subset \mathbb{A}$ be of cardinality less than λ such that $d \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}_0\bar{a}} \mathbb{A}$ ($\bar{a}_0$ exists because $\mathop{\smile\!\!\!\!\!\!|}$ satisfies Local Character). By applying cases 1 and 2 sufficiently many times, we can extend f to $f' \in \Gamma$ such that $\bar{a}_0$ is contained in the domain of f'; thus, w.l.o.g., $d \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}} \mathbb{A}$. Let $d' \in \mathbb{A}'$ be such that $d'\bar{c}' \equiv^1 d\bar{c}$; moreover, by Lemma 8.14, we can also assume that $d' \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}'} \mathbb{A}'$. We need only to show that $d' \notin \mathbb{A}'$. Assume, for contradiction, that $d' \in \mathbb{A}'$ and $d' \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}'} \mathbb{A}'$; then, $d' \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}'} d'$, thus $d' \in \mathrm{cl}^{\mathbb{B}'}(\bar{a}')$, and hence $d \in \mathrm{cl}^{\mathbb{B}}(\bar{a})$, which is absurd. $\square$

8.3. Additional facts

Reasoning as in [26, 2.6–2.9], from Theorems 8.3 and 8.5, and Proposition 8.17, we can deduce the following facts. We are still assuming that T expands an integral domain, and we are still using Notation 8.16. To simplify the statements of various results, we also assume that T is **model-complete**.

Corollary 8.18 *([26, 2.6]). Let $\langle \mathbb{B}, \mathbb{A}\rangle$ be a model of T^d. Suppose that $Y \subseteq \mathbb{B}^n$ is A_0-definable in $\langle \mathbb{B}, \mathbb{A}\rangle$, for some $A_0 \subset \mathbb{A}$. Then $Y \cap \mathbb{A}^n$ is A_0-definable in $\mathbb{A}$.*

Corollary 8.19 *([26, 2.7]). Let $\langle \mathbb{B}, \mathbb{A}\rangle$ and $\langle \mathbb{B}', \mathbb{A}'\rangle$ be models of T^d, such that $\langle \mathbb{B}', \mathbb{A}'\rangle \subseteq \langle \mathbb{B}, \mathbb{A}\rangle$ and $\mathbb{B}'$ and $\mathbb{A}$ are* cl*-independent over $\mathbb{A}'$. Then, $\langle \mathbb{B}', \mathbb{A}'\rangle \preceq \langle \mathbb{B}, \mathbb{A}\rangle$. In particular, if $\mathbb{A} \prec \mathbb{B}' \preceq \mathbb{B}$, with $\mathbb{A} \neq \mathbb{B}'$, then $\langle \mathbb{B}', \mathbb{A}\rangle \preceq \langle \mathbb{B}, \mathbb{A}\rangle$.*

Corollary 8.20 *([26, 2.8]). Let $A \subseteq B \subset \mathbb{M}$ be substructures. Assume that $\langle B, A\rangle$ have extensions $\langle \mathbb{B}_1, \mathbb{A}_1\rangle \models T^d$ and $\langle \mathbb{B}_2, \mathbb{A}_2\rangle \models T^d$, such that $B \mathop{\smile\!\!\!\!\!\!|}_A \mathbb{A}_k$ and $B \cap \mathbb{A}_k = A$, $k = 1, 2$. Then, $\langle \mathbb{B}_1, \mathbb{A}_1\rangle \equiv^2_B \langle \mathbb{B}_2, \mathbb{A}_2\rangle$. More generally, for every $\bar a_1 \in (\mathbb{A}_1)^n$ and $\bar a_2 \in (\mathbb{A}_2)^n$, if $\bar a_1 \equiv^1_B \bar a_2$, then $\bar a_1 \equiv^2_B \bar a_2$.*

Notice that the hypothesis of the above corollary implies that A is cl-closed (but not necessarily dense) in B.

Proof. Let $\bar c_k := B\bar a_k$. Notice that $\bar c_1$ and $\bar c_2$ have the same P-type, they are both P-independent, and $\bar c_1 \equiv^1 \bar c_2$; the conclusion follows from Proposition 8.17. □

Corollary 8.21 *([26, 2.9]). Let $\langle \mathbb{B}_1, \mathbb{A}_1\rangle \models T^d$ and $\langle \mathbb{B}_2, \mathbb{A}_2\rangle \models T^d$, and let A be a common subset of $\mathbb{A}_1$ and $\mathbb{A}_2$. Suppose that $b_1 \in \mathbb{B}_1 \setminus \mathbb{A}_1$ and $b_2 \in \mathbb{B}_2 \setminus \mathbb{A}_2$ satisfy $b_1 \equiv^1_A b_2$. Then, $b_1 \equiv^2_A b_2$.*

Proof. Let $\bar c_i := b_i\mathbb{A}_i$, $i = 1, 2$. By assumption, $\bar c_1 \equiv^1 \bar c_2$, they have the same P-type, and they are both P-independent. The conclusion follows from Proposition 8.17. □

For the remainder of this section, we will assume that $\langle \mathbb{B}, \mathbb{A}\rangle$ is a model of T^d, and that λ is a cardinal number such that $\kappa > \lambda > |T| + |\mathbb{B}|$.

Lemma 8.22 *([26, Theorem 2]). Let $\bar b \subset \mathbb{B}$ be P-independent. Given a set $Y \subset \mathbb{A}^n$, t.f.a.e.:*

1. *Y is T^2-definable over $\bar b$;*
2. *$Y = Z \cap \mathbb{A}^n$ for some set $Z \subseteq \mathbb{B}^n$ that is T-definable over $\bar b$.*

Proof. ($1 \Rightarrow 2$) follows from compactness and the fact that the $\mathcal{L}^2$-type over $\bar b$ of elements from $\mathbb{A}$ is determined by their P-type (cf. the proof of [26, Theorem 2]). ($2 \Rightarrow 1$) is obvious. □

Lemma 8.23 *([26, 3.1]). The structure $\mathbb{A}$ is T^2-algebraically closed in $\langle \mathbb{B}, \mathbb{A}\rangle$.*

Proof. As in [26, 3.1]. Let $b \in \mathbb{B} \setminus \mathbb{A}$. Let $\langle \mathbb{B}^*, \mathbb{A}^*\rangle \succeq \langle \mathbb{B}, \mathbb{A}\rangle$ be a monster model, and let $\mathrm{cl}^{\mathbb{B}^*}$ be the extension of cl to $\mathbb{B}^*$. Since $\mathrm{cl}^{\mathbb{B}^*}$ is existential, and $b \notin \mathrm{cl}^{\mathbb{B}^*}(\mathbb{A})$, there exist infinitely many distinct $b' \in \mathbb{B}^*$ such that $b \equiv^1_{\mathbb{A}} b'$. By Corollary 8.21, $b \equiv^2_{\mathbb{A}} b'$. Thus, b is not T^2-$\mathbb{A}$-algebraic in $\langle \mathbb{B}^*, \mathbb{A}^*\rangle$, and therefore not T^2-$\mathbb{A}$-algebraic in $\langle \mathbb{B}, \mathbb{A}\rangle$. □

Lemma 8.24 *([26, 3.2]). Let $A_0 \subseteq \mathbb{A}$ be T-algebraically closed (resp., T-definably closed). Then A_0 is T^2-algebraically closed (T^2-definably closed).*

Proof. Assume that A_0 is T-algebraically closed. Let $c \in \mathrm{acl}^2(A_0)$, and let $C := \{c_1, \dots, c_n\}$ be the set of $\mathcal{L}^2$-conjugates of c/A_0. By definition, C is A_0-definable in $\langle \mathbb{B}, \mathbb{A}\rangle$, and, by the above Lemma, $C \subset \mathbb{A}$. Hence, by Corollary 8.18, C is A_0-definable in $\mathbb{A}$. The case when A_0 is T-definably closed is similar. □

Lemma 8.25. *Assume that $\langle \mathbb{B}, \mathbb{A}\rangle$ is a λ-saturated model of T^d. Let $D \subset \mathbb{B}$ be such that $|D| < \lambda$, and let $c \in \mathbb{B} \setminus \mathrm{cl}(D)$. Define $C := \left\{c' \in \mathbb{B} : c' \equiv^1_D c\right\} \cap \mathbb{A}$. Then, $|C| \geq \lambda$.*

Proof. For every $\mu < \lambda$, consider the following partial $\mathcal{L}^2$-type over D:

$$p(x_i : i < \mu) := \Big(\bigwedge_i x_i \equiv^1_D c\Big) \;\&\; \Big(\bigwedge_i P(x_i)\Big) \;\&\; \Big(\bigwedge_{i<j} x_i \neq x_j\Big).$$

Claim 1. *The type p is consistent.*

If not, there exist $\bar d \subset D$, $\bar b \subset \mathbb{B}$, and $\phi(x, \bar d) \in \mathrm{tp}^1(c/D)$, such that $\phi(\mathbb{B}, \bar d) \setminus \mathbb{A} = \bar b$. Let $X := \phi(\mathbb{B}, \bar d) \setminus \bar b$; notice that X is definable in $\mathbb{B}$, and that $X \subseteq \mathbb{A}$. Hence, since $\mathbb{A}$ is co-dense in $\mathbb{B}$, we conclude that $\dim(X) \leq 0$, and therefore $\dim(\phi(\mathbb{B}, \bar d)) \leq 0$. Thus, $c \in \mathrm{cl}^{\mathbb{B}}(\bar d) \subseteq \mathrm{cl}^{\mathbb{B}}(D)$, which is absurd.

Thus, p is satisfied in $\langle \mathbb{B}, \mathbb{A}\rangle$, and the conclusion follows. □

Proposition 8.26 *([26, 3.3]). Let $\bar b \subset \mathbb{B}$ be P-independent. Then, $\mathrm{dcl}^2(\bar b) = \mathrm{dcl}^1(\bar b)$, and similarly for the algebraic closure. Let $c \in \mathbb{B}^{\mathrm{eq}}$ (i.e., c is an imaginary element for the structure $\mathbb{B}$). Then, $c \in \mathrm{dcl}^2(\bar b)$ iff $c \in \mathrm{dcl}^1(\bar b)$, and similarly for the algebraic closure.*

Sketch of Proof. W.l.o.g., we can assume that $\langle \mathbb{B}, \mathbb{A}\rangle$ is ω-saturated and that $\bar{b}$ has finite length. Let $c \in \mathbb{B}$ be such that $c \in \operatorname{acl}^2(\bar{b})$. We want to prove that $c \in \operatorname{acl}^1(\bar{b})$.

If $\bar{b} \subseteq \mathbb{A}$, the conclusion follows from Lemma 8.24. Otherwise, let $\mathbb{B}_1 := \operatorname{cl}^{\mathbb{B}}(\mathbb{A}\bar{b})$; by Corollary 8.19, $\langle \mathbb{B}_1, \mathbb{A}\rangle \preceq \langle \mathbb{B}, \mathbb{A}\rangle$, and in particular $\mathbb{B}_2$ is T^2-algebraically closed in $\langle \mathbb{B}, \mathbb{A}\rangle$, and therefore $c \in {\mathbb{B}_1}^{\mathrm{eq}}$. Let $n \geq 0$ be minimal such that there exist $\bar{a} \in \mathbb{A}^n$ with $c \in \operatorname{cl}^{\mathbb{B}}(\bar{b}\bar{a})$.

Claim 1. *$c \in \operatorname{cl}^{\mathbb{B}}(\bar{b})$, i.e. $n = 0$.*

If $n > 0$, by substituting $\bar{b}$ with $\bar{b}a_1 \dots a_{n-1}$, and proceeding by induction on n, we can reduce to the case $n = 1$; let $a := a_1$. Consider the following partial $\mathcal{L}$-type over $\bar{b}a$:

$$q(x) := (x \equiv^1_{\bar{b}} a) \;\&\; (x \mathop{\smile\!\!\!\!\!\mid}_{\bar{b}} a).$$

Since $\mathop{\smile\!\!\!\!\!\mid}$ satisfies Existence, q is consistent. Let $d \in \mathbb{B}$ be any realisation of q. Since $d \mathop{\smile\!\!\!\!\!\mid}_{\bar{b}} a$, we conclude that either $d \notin \operatorname{cl}^{\mathbb{B}}(\bar{b}a)$ or $d \in \operatorname{cl}^{\mathbb{B}}(\bar{b})$. However, the latter cannot happen, since $d \equiv^1_{\bar{b}} a \notin \operatorname{cl}^{\mathbb{B}}(\bar{b})$; thus, $d \notin \operatorname{cl}^{\mathbb{B}}(\bar{b}a)$, and therefore $\dim(q) = 1$. Hence, since $\mathbb{A}$ is dense in $\mathbb{B}$ and $\langle \mathbb{B}, \mathbb{A}\rangle$ is ω-saturated, there exists $a' \in \mathbb{A}$ satisfying q. Reasoning in the same way, we can show that there exists a Morley sequence $(a'_2, a'_3, a'_4, \dots)$ in q contained in $\mathbb{A}$. By Corollary 8.20, $a'_i \equiv^2_{\bar{b}} a$ for every i. Let $c_1, c_2, \dots, c_m$ be all the $\mathcal{L}^2$-conjugates of c over $\bar{b}$ (there are finitely many of them), and let $\phi(x, y, \bar{z})$ be an x-narrow $\mathcal{L}$-formula without parameters such that $\mathbb{B} \models \phi(c, a, \bar{b})$.

The $\mathcal{L}$-formula (in y, with parameters in $\bar{b}c_1 \dots c_m$) $\bigvee_i \phi(c_i, y, \bar{b})$ is equivalent to an $\mathcal{L}^2$-formula in y *with parameters* $\bar{b}$; hence, every a'_i satisfies it (because $a'_i \equiv^2_{\bar{b}} a$). Hence, w.l.o.g., $c_1 \in \operatorname{cl}^{\mathbb{B}}(\bar{b}a'_2) \cap \operatorname{cl}^{\mathbb{B}}(\bar{b}a'_3) = \operatorname{cl}^{\mathbb{B}}(\bar{b})$ (because $a'_2 \mathop{\smile\!\!\!\!\!\mid}_{\bar{b}} a'_3$). Therefore, $c \in \operatorname{cl}^{\mathbb{B}}(\bar{b})$.

It remains to show that $c \in \operatorname{acl}^1(\bar{b})$. Let $c_2 \in \mathbb{B}^{\mathrm{eq}}$ be such that $c_2 \equiv^1_{\bar{b}} c$. Since $\mathbb{B}$ is ω-saturated, it suffices to prove that there are only finitely many such c_2. Since $c \in \operatorname{acl}^2(\bar{b})$, it suffices to prove that $c_2 \equiv^2_{\bar{b}} c$. Let $\bar{b}_1 := \bar{b}c$, $\bar{b}_2 := \bar{b}c_2$, and $\bar{d} := \bar{b} \cap \mathbb{A}$. By assumption, $\bar{b}_1 \equiv^1 \bar{b}_2$. By Claim 1, we have $\bar{b}_1 \subseteq \operatorname{cl}^{\mathbb{B}}(\bar{b})$, and therefore, since $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_{\bar{d}} \mathbb{A}$, $\bar{b}_1$ is P-independent. Claim 1 also implies that $\bar{b}_2 \subseteq \operatorname{cl}^{\mathbb{B}}(\bar{b})$, and hence $\bar{b}_2$ is also P-independent. It remains to show that $\bar{b}_1$ and $\bar{b}_2$ have the same P-type. Assume for example that $c \in \mathbb{A}$. Since $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_{\bar{d}} \mathbb{A}$, we have that $c \in \operatorname{cl}^{\mathbb{B}}(\bar{d})$, and therefore $c_2 \in \operatorname{cl}^{\mathbb{B}}(\bar{d}) = \operatorname{cl}^{\mathbb{A}}(\bar{d}) \subseteq \mathbb{A}$.

The other assertions are proved in a similar way. $\square$

8.4. The small closure

We will are still assuming that T expands an integral domain. Let $\mathbb{M}^* := \langle \mathbb{B}^*, \mathbb{A}^*\rangle$ be a κ-saturated and strongly κ-homogeneous monster model of T^d, and let $\langle \mathbb{B}, \mathbb{A}\rangle \prec \mathbb{M}^*$, with $|\mathbb{B}| < \kappa$. Let $\operatorname{cl}^{\mathbb{B}^*}$ be the extension of cl to $\mathbb{B}^*$, and denote by rk the corresponding rank. Notice that $\operatorname{rk}(\mathbb{B}^*/\mathbb{A}^*) \geq \kappa$.

Definition 8.27. For every $X \subseteq \mathbb{B}^*$ we define the **small closure** of X as

$$\operatorname{Scl}(X) := \operatorname{cl}^{\mathbb{B}^*}(X\mathbb{A}^*).$$

For lovely pairs of geometric structures (e.g., dense pairs of o-minimal structures), the small closure was already defined in [4, Def. 4.5].

Remark 8.28. The matroid Scl is a definable matroid (on $\mathbb{M}^*$).

Proof. Notice that Scl coincides with the operator $(\operatorname{cl}^{\mathbb{B}^*})_{\mathbb{A}^*}$ in Lemma 5.5. $\square$

Notice that we can apply Remark 5.6, an obtain that $\operatorname{Scl}^{\mathbb{B}} = (\operatorname{cl}^{\mathbb{B}})_{\mathbb{A}}$; that is, we can "compute" the small closure of a subset of $\mathbb{B}$ inside $\mathbb{B}$ by using $\mathbb{A}$ instead of $\mathbb{A}^*$.

We want to prove that Scl is existential; we will need a preliminary lemma.

Lemma 8.29. *Let $b \in \mathbb{B}^* \setminus \mathbb{A}^*$. Define $\mathbb{M}^*_b$ the expansion of $\mathbb{M}^*$ with a constant for b, and $\operatorname{Scl}_b(X) := \operatorname{Scl}(bX) = \operatorname{cl}^{\mathbb{B}^*}(X\mathbb{A}^*b)$. Then, Scl_b is an existential matroid on $\mathbb{M}^*_b$.*

Proof. That Scl_b is a definable matroid follows from Lemma 5.5, applied to Scl. Let $X \subseteq \mathbb{M}^*$, and let $Y := \operatorname{Scl}_b(X)$.

Claim 1. *$Y \prec \mathbb{M}^*$ (as an $\mathcal{L}^2$-structure).*

By Lemma 7.4, Y is an elementary $\mathcal{L}$-substructure of $\mathbb{B}^*$. Corollary 8.19 applied to $\mathbb{B}' := Y$ implies the claim.

The lemma then follows from the above claim and Lemma 3.23; nontriviality follows from the fact that $\operatorname{rk}(\mathbb{B}^*/\mathbb{A}^*) \geq \kappa$. $\square$

Lemma 8.30. *The matroid* Scl *is an existential matroid* (*on* $\mathbb{M}^*$).

Proof. The only thing that needs proving is Existence. Define $\Xi^2(a/C)$ as the set of conjugates of a over C in $\mathbb{M}^*$. Assume that $\Xi^2(a/C) \subseteq \mathrm{Scl}(CD)$. We want to prove that $a \in \mathrm{Scl}(C)$. By Lemma 8.14, we can choose b and $b' \in \mathbb{B}^*$ which are $\mathrm{cl}^{\mathbb{B}^*}$-independent over $\mathbb{A}^*C$. By applying the previous lemma to Scl_b and $\mathrm{Scl}_{b'}$, we see that

$$a \in \mathrm{Scl}_b(C) \cap \mathrm{Scl}_{b'}(C) = \mathrm{cl}^{\mathbb{B}^*}(\mathbb{A}^*Cb) \cap \mathrm{cl}^{\mathbb{B}^*}(\mathbb{A}^*Cb') = \mathrm{cl}^{\mathbb{B}^*}(\mathbb{A}^*C) = \mathrm{Scl}(C). \quad \square$$

Hence, we can define the dimension induced by Scl, and denote it by Sdim.

Notice that, by Theorem 3.48, Scl is the only existential matroid on T^d.

Lemma 8.31. *Let $X \subseteq \mathbb{B}^n$ be definable in $\mathbb{B}$. Then* $\mathrm{Sdim}(X) = \dim(X)$.

Proof. From $\mathrm{cl}^{\mathbb{B}^*} \subseteq \mathrm{Scl}$ it follows immediately that $\mathrm{Sdim}(X) \leq \dim(X)$. For the opposite inequality, we proceed by induction on $k := \dim(X)$. Assume, for contradiction, that $\mathrm{Sdim}(X) < k$. W.l.o.g., $\dim(\Pi^n_k(X)) = k$; therefore, w.l.o.g., $k = n$. If $k = 1$, then $\mathrm{Sdim}(X) = 0$, and therefore $F(X^4) \neq \mathbb{B}$, contradicting $\dim(X) = 1$. For the inductive step, assume that $k = n > 1$, and let $U := \{a \in \mathbb{B}^n : \dim(X_a) = 1\}$. Notice that U is definable in $\mathbb{B}$, and therefore, by inductive hypothesis, $\mathrm{Sdim}(U) = \dim(U) = n - 1$. By the case $k = 1$, for every $a \in \mathbb{B}^{n-1}$, $\dim(X_a) = \mathrm{Sdim}(X_a)$, and therefore $\mathrm{Sdim}(X_a) = 1$ for every $a \in U$. Thus, $\mathrm{Sdim}(X) = n$. $\square$

Definition 8.32. Let $X \subseteq (\mathbb{B}^*)^n$ be definable in $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$. We say that X is **small** if $\mathrm{Sdim}(X) = 0$. Let $Y \subseteq \mathbb{B}^n$ be definable in $\langle \mathbb{B}, \mathbb{A} \rangle$. We say that Y is small if $\mathrm{Sdim}(Y^*) = 0$, where Y^* is the interpretation of Y inside $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$.

Notice that, if $X \subset \mathbb{B}^n$ is $\mathbb{A}$-small (in the sense of Definition 8.6), then X is also small in the above sense. The next lemma shows that the converse is also true.

Lemma 8.33. *Let $\langle \mathbb{B}, \mathbb{A} \rangle \preceq \langle \mathbb{B}^*, \mathbb{A}^* \rangle$ and $X \subseteq \mathbb{B}^n$ be definable in $\langle \mathbb{B}, \mathbb{A} \rangle$. Let X^* be the interpretation of X inside $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$. Let $\bar{c} \in \mathbb{B}^k$ be the parameters of definition of X. T.f.a.e.:*

1. *X is small;*
2. *X^* is small;*
3. *$X^* \subseteq \mathrm{Scl}(\bar{b})$ for some finite tuple $\bar{b} \subset \mathbb{B}^*$;*
4. *$X^* \subseteq \mathrm{Scl}(\bar{c})$;*
5. *$X^* \subseteq \mathrm{cl}^{\mathbb{B}^*}(\bar{c}\mathbb{A}^*)$;*
6. *X^* is $\mathbb{A}^*$-small; that is, there exists a Z-application $f^* : \mathbb{B}^{*m} \rightsquigarrow \mathbb{B}^{*n}$, definable in $\mathbb{B}^*$, such that $f^*(\mathbb{A}^{*m}) \supseteq X^*$;*
7. *X is $\mathbb{A}$-small; that is, there exists a Z-application $f : \mathbb{B}^m \rightsquigarrow \mathbb{B}^n$, definable in $\mathbb{B}$ (with parameters $\bar{c}$), such that $f(\mathbb{A}^m) \supseteq X$;*
8. *there exists a Z-application $g^* : \mathbb{B}^{*m+k} \rightsquigarrow \mathbb{B}^{*n}$, definable in $\mathbb{B}^*$ without parameters, such that $g^*\big(\mathbb{A}^{*m} \times \{\bar{c}\}\big) \supseteq X^*$;*
9. *there exists a Z-application $g : \mathbb{B}^{m+k} \rightsquigarrow \mathbb{B}^n$, definable in $\mathbb{B}$ without parameters, such that $f(\mathbb{A}^m \times \{\bar{c}\}) \supseteq X$.*

Proof. The only nontrivial implication is ($5 \Rightarrow 7$), which is proved by a compactness argument using Remark 3.55. $\square$

Conjecture 8.34 ([26, 3.6]). *Let $f : \mathbb{A}^n \to \mathbb{A}$ be T^2-definable with parameters $\bar{b}$. Let $\bar{a} \in \mathbb{A}^m$ be such that $\bar{b} \mathop{\smile\!\!\!\!|}_{\bar{a}} \mathbb{A}$ and $\mathrm{dcl}^1(\bar{b}\bar{a}) \cap \mathbb{A} = \mathrm{dcl}^1(\bar{a})$. Then, f is given piecewise by functions definable in $\mathbb{A}$ with parameters $\bar{a}$.*

Lemma 8.35 ([6, 6.1.3]). *Let $f : \mathbb{A}^n \to \mathbb{B}$ be T^2-definable with parameters $\bar{b}$. Assume that $\bar{b}$ is P-independent. Then, there exists $g : \mathbb{B}^n \to \mathbb{B}$ which is T-definable with parameters $\bar{b}$, and such that $f = g \restriction \mathbb{A}^n$.*

Proof. Let $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$ be an elementary extension of $\langle \mathbb{B}, \mathbb{A} \rangle$ and $a^* \in (\mathbb{A}^*)^n$. By Proposition 8.26, there exists a function $g_i : \mathbb{B}^n \to \mathbb{B}$ which is T-definable with parameters $\bar{b}$, such that $f(a) = g_i(a)$. By compactness, finitely many g_i will suffice. The conclusion then follows from Lemma 8.22. $\square$

Proposition 8.36 ([26, 3.5]). *Let $\bar{b} \in \mathbb{B}^k$ and $\bar{a} \in \mathbb{B}^{k'}$ be such that $\bar{b} \mathop{\smile\!\!\!\!|}_{\bar{a}} \mathbb{A}$ and $\bar{b} \cap \mathbb{A} \subseteq \bar{a}$. Let $X \subseteq \mathbb{B}^{\mathrm{eq}}$ be T-definable with parameters $\bar{b}$, such that $\dim(X) = d$. Let $Y \subseteq X$ be T^2-definable, with parameters $\bar{b}$. Then, there exist $S \subset X$ which is T^2-definable with parameters $\bar{b}$, and $Z \subseteq X$ which is T-definable with parameters $\bar{b}\bar{a}$, such that $Z \mathbin{\Delta} Y \subseteq S$ and $\mathrm{Sdim}(S) < d$.*

In particular, if $\dim(X) = 0$, then every T^2-definable subset of X is already T-definable.

Proof. The proof is a variant of Beth's definability theorem, using Proposition 8.17. W.l.o.g., $\langle \mathbb{B}, \mathbb{A} \rangle$ is λ-saturated, for some cardinal λ such that $|T| < \lambda < \kappa$.

Let $W := \big\{p \in S^2_X(\bar{a}\bar{b}) : \mathrm{Sdim}(p) = d\big\}$. Notice that W is a closed subset of $S^2_X(\bar{a}\bar{b})$ (the Stone space of T^2-types over $\bar{a}\bar{b}$ containing the formula "$\bar{x} \in X$"). Let $\theta : S^2_X(\bar{a}\bar{b}) \to S^1_X(\bar{a}\bar{b})$ be the restriction map; notice that θ is continuous, and therefore $V := \theta(W)$ is compact and hence closed in $S^1_X(\bar{a}\bar{b})$. Let $\rho := \theta \restriction W$.

Claim 1. *The map ρ is injective (and therefore ρ is a homeomorphism between W and V).*

We have to prove that, for every $\bar{c}$ and $\bar{c}' \in X$, if $\mathrm{Srk}(\bar{c}/\bar{a}\bar{b}) = \mathrm{Srk}(\bar{c}'/\bar{a}\bar{b}) = d$ and $\bar{c} \equiv^1_{\bar{a}\bar{b}} \bar{c}'$, then $\bar{c} \equiv^2_{\bar{a}\bar{b}} \bar{c}'$. Let $\bar{d} := \bar{a}\bar{b}\bar{c}$ and $\bar{d}' := \bar{a}\bar{b}\bar{c}'$. By Proposition 8.17, it suffices to prove that $\bar{d}$ and $\bar{d}'$ are both P-independent and have the same P-type. Since $\mathrm{Srk}(\bar{c}/\bar{a}\bar{b}) = d$ and $\bar{c} \in X$, we have that $\mathrm{Srk}(\bar{c}/\bar{a}\bar{b}) = \mathrm{rk}(\bar{c}/\bar{a}\bar{b})$, which is equivalent to $\bar{c} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}\bar{b}} \mathbb{A}$, and hence (since $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}} \mathbb{A}$) $\bar{d} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}} \mathbb{A}$, that is $\bar{d}$ is P-independent, and similarly for $\bar{d}'$. It remains to show that $\bar{d}$ and $\bar{d}'$ have the same P-type. Let $d_i \in \mathbb{A}$; we have to prove that $d_i' \in \mathbb{A}$. Since $\bar{d} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}} \mathbb{A}$, we have $d_i \in \mathrm{cl}^{\mathbb{B}^*}(\bar{a})$, and hence $d_i' \in \mathrm{cl}^{\mathbb{B}}(\bar{a}') \subseteq \mathbb{A}$.

Let $U := S^2_Y(\bar{a}\bar{b}) \cap W$; since Y is definable, U is clopen in W, and since ρ is a homeomorphism, $\rho(U)$ is clopen in V. Hence, there exists Z subset of X, such that Z is T-definable over $\bar{a}\bar{b}$ and $V \cap S^1_Z(\bar{a}\bar{b}) = \rho(U)$.

Claim 2. *There exists $S \subset X$ which is T^2-definable over $\bar{b}$, such that $\mathrm{Sdim}(S) < d$ and $Y \mathbin{\Delta} Z \subseteq S$.*

Assume not. Then, the following partial type over $\bar{a}\bar{b}$ is consistent:

$$\Phi(\bar{x}) := \bar{x} \in X \mathbin{\&} \bar{x} \in Y \mathbin{\Delta} Z \mathbin{\&} \bar{x} \notin S,$$

where S varies among the subsets of X which are T^2-definable over $\bar{b}$, with $\mathrm{Sdim}(S) < d$. Let $\bar{c} \in X$ be a realisation of Φ and $p := \mathrm{tp}^2(\bar{c}/\bar{a}\bar{b}) \in S^2_X(\bar{a}\bar{b})$. By assumption, $\mathrm{Sdim}(\bar{c}/\bar{a}\bar{b}) = d$, and therefore $p \in W$. Hence, $\rho(p) = \mathrm{tp}^1(\bar{c}/\bar{a}\bar{b}) \in V$. Since ρ is injective, we have

$$\rho(p) \in \rho\big(S^2_Y(\bar{a}\bar{b}) \cap W\big) \mathbin{\Delta} \rho\big(S^2_Z(\bar{a}\bar{b}) \cap W\big) \subseteq S^1_Z(\bar{a}\bar{b}) \mathbin{\Delta} S^1_Z(\bar{a}\bar{b}) = \emptyset,$$

which is absurd. $\square$

In general, given $\bar{b} \in \mathbb{B}^n$, it is always possible to find $\bar{a} \in \mathbb{A}^{n'}$ such that $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}} \mathbb{A}$. However, [4, Example 6.13] shows that it can happen that $\mathbb{B}$ is o-minimal, but $\bar{a}$ cannot be found inside $\mathrm{dcl}^2(\bar{b})$.

Corollary 8.37 (*[26, 3.4]*)**.** *Let $\bar{b}$ and $\bar{a}$ be as in the above proposition. Let Γ be a T-definable set (possibly, in some imaginary sort) over $\bar{b}$, and let the function $f : \mathbb{B}^n \to \Gamma$ be T^2-definable with parameters $\bar{b}$. Then, there exist $S \subseteq \mathbb{B}^n$, which is T^2-definable over $\bar{b}$ and with $\mathrm{Sdim}(S) < n$, and $\hat{f} : \mathbb{B}^n \to \Gamma$, which is T-definable over $\bar{b}\bar{a}$, such that f agrees with $\hat{f}$ outside S.*

Proof. W.l.o.g., $\langle \mathbb{B}, \mathbb{A}\rangle$ is ω-saturated. Let $\mathcal{G}$ be the set of functions $g : \mathbb{B}^n \to \Gamma$ that are T-definable with parameters $\bar{b}\bar{a}$.

Claim 1. *There exist a set $S \subset \mathbb{B}^n$ which is T^2-definable with parameters $\bar{b}$, with $\mathrm{Sdim}(S) < n$, and finitely many functions $g_1, \ldots, g_k$ in $\mathcal{G}$, such that f agree outside S with some of the g_i.*

Assume that the claim does not hold. Hence, for every S as in the claim and every $g \in \mathcal{G}$, there exists $\bar{c} \in \mathbb{B}^n$ such that $\bar{c} \notin S$ and $f(\bar{c}) \neq g(\bar{c})$. Thus, the following partial $\mathcal{L}^2$-type over $\bar{b}\bar{a}$ is consistent:

$$p(\bar{x}) := \big\{\bar{x} \in \mathbb{B}^n \setminus \mathrm{Scl}(\bar{b})\big\} \cup \{f(\bar{x}) \neq g(\bar{x}) : g \in \mathcal{G}\}.$$

Let $\bar{c}$ be a realisation of p. Notice that the choice of $\bar{a}$ and the fact that $\mathrm{Srk}(\bar{c}/\bar{a}\bar{b}) = n$ imply that $\bar{c}\bar{b}\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{\bar{a}} \mathbb{A}$. Hence, by Proposition 8.26, $f(\bar{c}) \in \mathrm{dcl}^1(\bar{c}\bar{b}\bar{a})$. Thus, $f(\bar{c}) = g(\bar{c})$ for some function $g : \mathbb{B}^n \to \mathbb{B}$ which is T-definable with parameters $\bar{b}\bar{a}$, which is absurd.

The above claim plus Proposition 8.36 imply the conclusion. $\square$

The above corollary gives a way to find the parameters of the definition of $\hat{f}$ (and of S) starting from the parameters $\bar{b}$ of f.

Example 8.38. In general, $\hat{f}$ cannot be defined using only $\bar{b}$ as parameters. Consider a_1 and a_2 in $\mathbb{A}$ which are independent over the empty set, $b_1 \in \mathbb{B} \setminus \mathbb{A}$, and $b_2 := a_1 + b_1 \cdot a_2 \in \mathbb{B} \setminus \mathbb{A}$. Let $\bar{a} := \langle a_1, a_2\rangle$ and $\bar{b} := \langle b_1, b_2\rangle$. Notice that $\mathrm{rk}(\bar{a}\bar{b}) = 3$, while $\mathrm{Srk}(\bar{a}\bar{b}) = 1$. Let f be the constant function a_1. Then, f is T^2-definable over $\bar{b}$, but is not T-definable over $\bar{b}$.

Question 8.39. Assume that T is d-minimal (see Section 9). Is it true that, for every $X \subseteq \mathbb{B}^*$, $\mathrm{Scl}(X) = \mathrm{acl}^1(\mathbb{A}^*X)$ (cf. Proposition 8.26)?

Conjecture 8.40 (*J. Ramakrishnan*)**.** *Assume that T is o-minimal. Then, for every $X \subset \mathbb{B}$,*

$$\mathrm{acl}^2(X) = \mathrm{acl}^1\big(X \cup (\mathrm{acl}^2(X) \cap \mathbb{A})\big).$$

8.5. Elimination of imaginaries

Let cl be an existential matroid on $\mathbb{M}$ and $\mathrm{cl}^{\mathrm{eq}}$ be the extension of cl to $\mathbb{M}^{\mathrm{eq}}$ defined in Section 6. Remember that element $e \in \mathbb{M}^{\mathrm{eq}}$ is an equivalence class $X \subseteq \mathbb{M}^n$ for some $\emptyset$-definable equivalence relation E on $\mathbb{M}^n$. If $\bar{c} \in X$, we say that $\bar{c}$ **represents** e.

Definition 8.41. We say that $\mathbb{M}$ has cl-**elimination of imaginaries** if, for every $e \in \mathbb{M}^{eq}$, there exists $\bar{c}$ representing e, such that $\bar{c} \in \mathrm{cl}^{eq}(e)$. Given $\bar{b} \subset \mathbb{M}$, we say that $\mathbb{M}$ has cl-elimination of imaginaries **modulo** $\bar{b}$ if, for every $e \in \mathbb{M}^{eq}$, there exists $\bar{c}$ representing e, such that $\bar{c} \in \mathrm{cl}^{eq}(e\bar{b})$.

If $\mathbb{K} \preceq \mathbb{M}$, we say that $\mathbb{K}$ has cl-elimination of imaginaries (modulo some $\bar{b} \subset \mathbb{K}$) if $\mathbb{M}$ has it.

Compare the above notion with weak elimination of imaginaries (see [8]).

Remark 8.42. $\mathbb{M}$ has cl-elimination of imaginaries iff, for every $\mathbb{M}$-definable set X, we have $X \cap \mathrm{cl}^{eq}(\ulcorner X \urcorner)$ is nonempty, where $\ulcorner X \urcorner \in \mathbb{M}^{eq}$ is the canonical parameter of X.

We will prove the next proposition later.

Proposition 8.43. *Let $\bar{b} \subset \mathbb{M}$. Assume that* $\mathrm{cl}(\bar{b})$ *is dense in* $\mathbb{M}$. *Then,* $\mathbb{M}$ *has* cl*-elimination of imaginaries modulo* $\bar{b}$.

Corollary 8.44. *Let* $\mathbb{M}$ *be geometric. Assume that* $\mathrm{acl}(\emptyset)$ *is* acl*-dense in* $\mathbb{M}$ *(e.g.,* $\mathbb{M}$ *is a pure algebraically closed field). Then,* $\mathbb{M}$ *has weak elimination of imaginaries. If, moreover,* $\mathbb{M}$ *expands a field, then* $\mathbb{M}$ *has elimination of imaginaries.*

Corollary 8.45. *Assume that* $\mathbb{M}$ *expands an integral domain. Let* $\langle \mathbb{B}, \mathbb{A} \rangle \models T^d$*. Let* $b \in \mathbb{B} \setminus \mathbb{A}$*. Then,* $\langle \mathbb{B}, \mathbb{A} \rangle$ *has* Scl*-elimination of imaginaries modulo* b*.*

Proof. For every $b \in \mathbb{B} \setminus \mathbb{A}$, we have that $\mathrm{Scl}^{\mathbb{B}}(b)$ is Scl-dense in $\langle \mathbb{B}, \mathbb{A} \rangle$. $\square$

In the situation of the above corollary, it is not true that $\langle \mathbb{B}, \mathbb{A} \rangle$ has Scl-elimination of imaginaries (modulo $\emptyset$). For instance, let $X := \mathbb{B} \setminus \mathbb{A}$. Then, $X \cap \mathrm{Scl}^{eq}(\ulcorner X \urcorner) = \emptyset$.

Before proving the Proposition 8.43, we need some preliminaries. Let $X \subseteq \mathbb{M}^n$ be a subset definable with parameters $\bar{b}$. Let $\mathbb{M}'$ be the expansion of $\mathbb{M}$ with a new predicate denoting X. Notice that $\mathbb{M}$ and $\mathbb{M}'$ have the same definable sets. However, cl is no longer an existential matroid on $\mathbb{M}'$; for instance, if $X = \{b\}$ is a singleton, and $b \notin \mathrm{cl}(\emptyset)$, then $b \in \mathrm{acl}'(\emptyset) \setminus \mathrm{cl}(\emptyset)$, where acl' is the algebraic closure in $\mathbb{M}'$, and therefore cl is not existential on $\mathbb{M}'$. However, notice that $\mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{cl}}$ satisfies all the axioms of a symmetric independence relation on $\mathbb{M}'$, except possibly the Extension axiom.

Let $e := \ulcorner X \urcorner \in \mathbb{M}^{eq}$ be the canonical parameter for X. For every $Z \subseteq \mathbb{M}$, define $\mathrm{cl}_e(Z) := \mathrm{cl}^{eq}(eZ) \cap \mathbb{M}$ (notice that, if $e = \emptyset$, then $\mathrm{cl}_e = \mathrm{cl}$).

Lemma 8.46. *The matroid* cl_e *is an existential matroid on* $\mathbb{M}'$*.*

Proof. We only need to check that cl_e satisfies Existence. Let B and C be subsets of $\mathbb{M}$ such that $a \notin \mathrm{cl}_e(B)$; that is, $a \notin \mathrm{cl}^{eq}(eB)$. Let $a' \equiv^{\mathbb{M}}_{eB} a$ be such that $a' \mathop{\smile\!\!\!\!\!\!\,|}^{\mathrm{cl}}_{e} BC$. Then, $a' \equiv^{\mathbb{M}'}_{B} a$ and $a' \notin \mathrm{cl}^{eq}(eBC) = \mathrm{cl}_e(BC)$. $\square$

Proof of Proposition 8.43. W.l.o.g., $\bar{b} = \emptyset$. Let X be an $\mathbb{M}$-definable set and $e := \ulcorner X \urcorner$; by Remark 8.42, we need to show that $X \cap \mathrm{cl}^{eq}(e) \neq \emptyset$. Let cl_e be defined as above. Since $\mathrm{cl}(\emptyset)$ is dense in $\mathbb{M}$ and $\mathrm{cl} \subseteq \mathrm{cl}_e$, we have that $\mathbb{K} := \mathrm{cl}_e(\emptyset)$ is also dense in $\mathbb{M}'$. Hence, by Lemma 7.4, $\mathbb{K} \preceq \mathbb{M}'$. Thus, since X is $\emptyset$-definable in $\mathbb{M}'$, there exists $\bar{c} \in X \cap \mathbb{K}$. $\square$

Other results on elimination of imaginaries for dense pairs of geometric structures were proved in [6].

9. D-minimal topological structures

In this section, we will introduce d-minimal structures. They are topological structures whose definable sets are particularly simple from the topological point of view; they generalise o-minimal structures. We will show that for d-minimal structures the topology induces a canonical existential matroid, which we denote by Zcl. Moreover, the abstract notion of density introduced in Section 7 coincides with the usual topological notion. Finally, if T is a complete d-minimal theory expanding the theory of fields, then in T^d the condition that the smaller structure is cl-closed is superfluous. Our definition of d-minimality extends an older definition by Miller [17], which applied only to linearly ordered structures.

Let $\mathbb{K}$ be a first-order topological structure in the sense of [18]. That is, $\mathbb{K}$ is a structure with a topology, such that a basis of the topology is given by $\{\Phi(\mathbb{K}, \bar{a}) : \bar{a} \in \mathbb{K}^m\}$ for a certain formula without parameters $\Phi(x, \bar{y})$; fix such a formula $\Phi(x, \bar{y})$, and denote $B_{\bar{a}} := \Phi(\mathbb{K}, \bar{a})$. Examples of topological structures are valued fields, or ordered structures. On $\mathbb{K}^n$ we put the product topology. Let $\mathbb{M} \succeq \mathbb{K}$ be a monster model. Given $X \subseteq \mathbb{K}^n$, we will denote by $\overline{X}$ and $\mathring{X}$, respectively, the topological closure and the interior of X inside $\mathbb{K}^n$.

Definition 9.1. The structure $\mathbb{K}$ is d-minimal if

1. it is T_1 (i.e., its points are closed);
2. it has no isolated points;
3. for every $X \subseteq \mathbb{M}$ definable subset (with parameters in $\mathbb{M}$), if X has empty interior, then X is a finite union of discrete sets;
4. for every $X \subset \mathbb{K}^n$ definable and discrete, $\Pi^n_1(X)$ has empty interior;
5. given $X \subseteq \mathbb{K}^2$ and $U \subseteq \Pi^2_1(X)$ definable sets, if U is open and nonempty, and X_a has nonempty interior for every $a \in U$, then X has nonempty interior.

Notice that (4) implies (2). [3, Section 4] introduces the notion of "geometric structures" (distinct from the one we used in this article) which, more or less, are d-minimal structures where every definable discrete set is finite, plus some additional conditions (such as definable Skolem functions), and proves for those theories the analogue of Corollary 9.17.

Examples 9.2. 1. p-adic fields and algebraically closed valued fields are d-minimal;
2. densely ordered o-minimal structures are d-minimal.

In both cases, a definable set is discrete iff it is finite.

Example 9.3. A structure $\mathbb{K}$ is **definably complete** if it expands a linear order $\langle K, <\rangle$, and every $\mathbb{K}$-definable subset of K has a supremum in $K \sqcup \{\pm\infty\}$. Miller defines a d-minimal structure as a definably complete structure $\mathbb{K}$ such that, given $\mathbb{K}'$ an $\aleph_0$-saturated elementary extension of $\mathbb{K}$, every $\mathbb{K}'$-definable subset of $\mathbb{K}'$ is the union of an open set and finitely many discrete sets. In particular, o-minimal structures and ultra-products of o-minimal structures are d-minimal in Miller's sense. If $\mathbb{K}$ expands a field and is a d-minimal structures in the sense of Miller, then $\mathbb{K}$ is d-minimal in our sense [12, Section 10]. Conversely, any definably complete structure which is d-minimal in our sense is also d-minimal in Miller's sense.

Proviso. *For the remainder of this section, we assume that $\mathbb{K}$ is d-minimal, and that T is the theory of $\mathbb{K}$.*

Remark 9.4. 1. Let $X \subset \mathbb{K}^n$ be discrete. Since $\mathbb{K}$ has no isolated points, X is nowhere dense; that is, $\mathring{\overline{X}} = \emptyset$.
2. Let $X_1, \dots, X_r$ be nowhere dense subsets of $\mathbb{K}^n$. Then $X_1 \cup \dots \cup X_r$ is also nowhere dense; this remains true if $\mathbb{K}$ is any topological space.
3. Hence, if $X_1, \dots, X_r$ are discrete subsets of $\mathbb{K}^n$, then $X_1 \cup \dots \cup X_r$ is nowhere dense (but no longer discrete, in general).
4. Let $X \subseteq \mathbb{K}$ be definable. Then, X has empty interior iff X is nowhere dense.
5. If X_1 and X_2 are definable subsets of $\mathbb{K}$ with empty interior, then $X_1 \cup X_2$ has empty interior. Hence, for every $X \subseteq \mathbb{K}$ definable, $\overline{X} \setminus \mathring{X}$ has empty interior.

Lemma 9.5. *Let $Z \subset \mathbb{K}^2$ be definable, such that $\Pi^2_1(Z)$ has empty interior, and Z_x has empty interior for every $x \in \mathbb{K}$. Then, $\theta(Z)$ has empty interior, where θ is the projection onto the second coordinate.*

Proof. By assumption, w.l.o.g., $\Pi^2_1(Z)$ is discrete and, for every $x \in \mathbb{K}$, Z_x is also discrete. Therefore, Z is discrete, and hence $\theta(Z)$ has empty interior. $\square$

Definition 9.6. Given $A \subset \mathbb{M}$ and $b \in \mathbb{M}$, we say that $b \in \operatorname{Zcl}(A)$ if there exists $X \subset \mathbb{M}$ A-definable such that $b \in X$ and X has empty interior (or, equivalently, X is discrete).

Lemma 9.7. *If $c \notin \operatorname{Zcl}(A)$, then $\Xi(c/A)$ has nonempty interior.*

Proof. Let $X \subseteq \mathbb{M}$ be any A-definable set containing c. Since $c \notin \operatorname{Zcl}(A)$, $c \in \mathring{X}$. Consider the partial type over cA

$$\Gamma(\bar{y}) := \left\{c \in B_{\bar{y}} \subseteq X : X \subseteq \mathbb{M} \text{ is } A\text{-definable and } c \in X\right\}.$$

By the above consideration, Γ is consistent; let $\bar{b} \subset \mathbb{M}$ be a realisation of Γ.

Claim 1. *$c \in B_{\bar{b}} \subseteq \Xi(c/A)$.*

Clearly, $c \in B_{\bar{b}}$. Let $c' \in B_{\bar{b}}$ and let $X \subseteq \mathbb{M}$ be A-definable and containing c. By our choice of $\bar{b}$, we have $c' \in X$, and therefore c' satisfies all the A-formulae satisfied by c. $\square$

Theorem 9.8. *The operator* Zcl *is an existential matroid.*

Proof. Finite character, extension and monotonicity are obvious.
The fact that Zcl is definable is also obvious.

(Idempotency) Let $\bar{b} := \langle b_1, \dots, b_n\rangle$, $a \in \operatorname{Zcl}(\bar{b}\bar{c})$ and $\bar{b} \subset \operatorname{Zcl}(\bar{c})$. We must prove that $a \in \operatorname{Zcl}(\bar{c})$. Let $\phi(x, \bar{y}, \bar{z})$ and $\psi_i(y, \bar{z})$ be formulae, $i = 1, \dots, n$, such that $\phi(\mathbb{M}, \bar{y}, \bar{z})$ and $\psi_i(\mathbb{M}, \bar{z})$ are discrete for every $\bar{y}$ and $\bar{z}$, and $\mathbb{M} \models \phi(a, \bar{b}, \bar{c})$ and $\mathbb{M} \models \psi_i(b_i, \bar{c})$, $i = 1, \dots, n$. Let

$$Z := \left\{\langle x, \bar{y}\rangle : \mathbb{M} \models \phi(x, \bar{y}, \bar{c}) \;\&\; \bigwedge_{i=1}^{n} \psi_i(y_i, \bar{c})\right\},$$

and let $W := \Pi^{n+1}_1 Z$. By hypothesis, Z is a discrete subset of $\mathbb{M}^{n+1}$, and therefore, by Assumption (4), W has empty interior. Moreover, W is $\bar{c}$-definable and $a \in W$, and hence $a \in \operatorname{Zcl}(\bar{c})$.

(EP) Let $a \in \operatorname{Zcl}(b\bar{c}) \setminus \operatorname{Zcl}(\bar{c})$. We must prove that $b \in \operatorname{Zcl}(a\bar{c})$. Assume not. Let $Z \subset \mathbb{M}^2$ be $\bar{c}$-definable, such that $\langle a, b\rangle \in Z$ and Z^y is discrete for every $y \in \mathbb{M}$. Since $b \in Z_a$ and $b \notin \operatorname{Zcl}(a\bar{c})$, $b \in \operatorname{int}(Z_a)$; hence, w.l.o.g., Z_x is open for every $x \in \mathbb{M}$. Let $U := \Pi^2_1(Z)$. Since $a \in U$ and $a \notin \operatorname{Zcl}(\bar{c})$, $a \in \mathring{U}$. Hence, by Condition (5), Z has nonempty interior; but this contradicts the fact Z^y is discrete for every $y \in \mathbb{M}$.

Existence follows from Lemma 9.7.

(Nontriviality) Consider the following partial type over the empty set:

$$\Lambda(x) := \{x \notin Y\},$$

where Y varies among the discrete $\emptyset$-definable sets. Since $\mathbb{M}$ has no isolated points, Λ is finitely satisfiable; if $a \in \mathbb{M}$ is a realisation of Λ, then $a \notin \operatorname{Zcl}(\emptyset)$. $\square$

We will denote by Zrk, $\mathop{\smile}\limits^{\vert}$, and dim the rank, independence relation, and dimension on $\mathbb{M}$ induced by Zcl.

Remark 9.9. Let $X \subseteq \mathbb{K}^n$ be definable. If X has nonempty interior, then $\dim(X) = n$. If $\Pi^n_d(X)$ has nonempty interior, then $\dim(X) \geq d$.

Conjecture 9.10. *Let $X \subseteq \mathbb{K}^n$ be definable. Then, $\dim(X) \geq d$ iff, after a permutation of variables, $\Pi^n_d(X)$ has nonempty interior.*

Conjecture 9.11. *For every $X \subseteq \mathbb{K}^n$ definable, $\dim(\overline{X}) = \dim X$.*

Example 9.12. It is not true that $\dim(\partial X) < \dim(X)$ if X is definable and nonempty. For instance, let $\mathbb{K} := \langle \mathbb{R}, +, \cdot, <, 2^{\mathbb{Z}} \rangle$ be the expansion of the real field by a predicate for the integer powers of 2. Then, $\mathbb{K}$ is d-minimal [24, Theorem II]. Let $X := 2^{\mathbb{Z}}$. Thus, $\partial X = \{0\}$, and hence $\dim(X) = 0 = \dim(\partial X)$.

Lemma 9.13. *The set X is Zcl-dense in $\mathbb{K}$ according to Definition 7.1 iff X is topologically dense in $\mathbb{K}$.*

Proof. Assume that X is dense in $\mathbb{K}$ according to Zcl. Let $A \subseteq \mathbb{K}$ be an open definable set; thus, $\dim(A) = 1$, and therefore $A \cap X \neq \emptyset$. Conversely, if X is topologically dense in $\mathbb{K}$, let $A \subseteq \mathbb{K}$ be definable and of dimension 1. Thus, A has nonempty interior, and therefore $A \cap X \neq \emptyset$. □

Lemma 9.14. *Let $d \in \mathbb{M}$, V be a definable neighbourhood of d, and let $C \subset \mathbb{M}$. Then, there exists $\bar{a} \in \mathbb{M}^m$ such that $\bar{a} \mathop{\smile}\limits^{\vert}_d C$ and $d \in B_{\bar{a}} \subseteq V$.*

Proof. Let $X := \{\bar{a} \in \mathbb{M}^n : d \in B_{\bar{a}}\}$. Let $\leq$ be the quasi-ordering on X given by reverse inclusion; that is, $\bar{a} \leq \bar{a}'$ if $B_{\bar{a}} \supseteq B_{\bar{a}'}$. Fix $\bar{b} \in X$ such that $B_{\bar{b}} \subseteq V$. Since $(X, \leq)$ is a directed set, by Lemma 3.68, there exists $\bar{a} \in X$ such that $\bar{a} \mathop{\smile}\limits^{\vert}_d C$ and $B_{\bar{a}} \subseteq B_{\bar{b}} \subseteq V$. □

Proviso 9.15. *For the remainder of this section, will assume that $\mathbb{K}$ is d-minimal and expands an integral domain, that $+$ and $-$ are continuous (and therefore $\langle \mathbb{K}, + \rangle$ is a topological group), and that T is the theory of $\mathbb{K}$. In the following, when $\mathbb{K}$ is a d-minimal expansion of an integral domain, we will always assume that $+$ and $-$ are continuous.*

Notice that an algebraically closed field with the Zariski topology is not a topological group, because $+$ is not continuous. Notice also that, since we required that points are closed, $\mathbb{K}$ is a regular topological space.

Remark 9.16. Let $X \subseteq \mathbb{K}$ be dense (but not necessarily definable). Then, for every $b \in \mathbb{K}$ and every V neighbourhood of 0, there exists $a \in X$ such that $b \in a + V$.

Proof. Since $-$ is continuous, there exists V' neighbourhood of 0 such that $V' = -V'$ and $V' \subseteq V$. Since X is dense, there exists $a \in X$ such that $a \in b + V'$. Hence, $b \in a - V' \subseteq a + V$. □

Corollary 9.17. *The theory T^d is complete. Besides, T^d is the theory of pairs $\langle \mathbb{K}, \mathbb{F} \rangle$ such that $\mathbb{F} \prec \mathbb{K} \models T$ and $\mathbb{F}$ is a (topologically) dense proper subset of $\mathbb{K}$.*

Proof. By Theorem 8.3, it suffices to show that, if $\mathbb{F} \preceq \mathbb{K}$ is dense in $\mathbb{K}$, then $\mathbb{F}$ is Zcl-closed in $\mathbb{K}$. W.l.o.g., the pair $\langle \mathbb{K}, \mathbb{F} \rangle$ is ω-saturated. Let $b \in \mathrm{Zcl}^{\mathbb{K}}(\mathbb{F})$; we must prove that $b \in \mathbb{F}$. Let $Z \subset \mathbb{K}$ be $\mathbb{F}$-definable and discrete, such that $b \in Z$. Let U' be a definable neighbourhood of b, such that $Z \cap U' = \{b\}$. Define $U := U' - b$; since $\mathbb{K}$ is a topological group, U is a neighbourhood of 0 in $\mathbb{K}$, and there exists V, an open neighbourhood of 0 definable in $\mathbb{K}$, such that $V = -V$ and $V + V \subseteq U$.

Claim 1. *There exists W, an $\mathbb{F}$-definable open neighbourhood of 0, such that $W \subseteq V$.*

Suppose that the claim is not true. Since $\mathbb{K}$ is a regular space, there exists X, a definable open neighbourhood of 0, such that $\overline{X} \subseteq V$. Let $X_{\mathbb{F}} := X \cap \mathbb{F}$. Since $X_{\mathbb{F}}$ is a neighbourhood of 0 in $\mathbb{F}$ and since the topology has a definable basis, there exists $W_{\mathbb{F}} \subseteq X_{\mathbb{F}}$ such that the set $W_{\mathbb{F}}$ is $\mathbb{F}$-definable and $W_{\mathbb{F}}$ is an open neighbourhood of 0. Let W be the interpretation of $W_{\mathbb{F}}$ in $\mathbb{K}$. Since W is open and $\mathbb{F}$ is dense in $\mathbb{K}$, $W_{\mathbb{F}}$ is dense in W; therefore, $W \subseteq \overline{W_{\mathbb{F}}} \subseteq \overline{X} \subseteq V$.

By Remark 9.16, there exists $a \in \mathbb{F}$ such that $b \in W'$, where $W' := a + W$.

Claim 2. $W' \subseteq U'$.

The claim is equivalent to $a + W \subseteq b + U$; that is, $W + (a - b) \subseteq U$. By assumption, $b \in a + W$, and therefore $a - b \in -W$. Hence, $W + (a - b) \subseteq W - W \subseteq V - V \subseteq U$.

Finally, W' is $\mathbb{F}$-definable, and $b \in W' \cap Z \subseteq V \cap Z = \{b\}$. Hence, b is $\mathbb{F}$-definable, and therefore $b \in \mathbb{F}$. □

Given $\bar{a} := \langle \bar{a}_1, \ldots, \bar{a}_n \rangle \in \mathbb{M}^{n \times m}$ and $\bar{b} \in \mathbb{M}^n$, denote

$$B_{\bar{a}} + \bar{b} := (B_{\bar{a}_1} + b_1) \times \cdots \times (B_{\bar{a}_n} + b_n) \subseteq \mathbb{M}^n.$$

Lemma 9.18. *Let $\bar{d} \in \mathbb{M}^n$, V be a definable neighbourhood of $\bar{d}$, and let $C \subset \mathbb{M}$. Then, there exist $\bar{a} \in \mathbb{M}^{m \times n}$ and $\bar{b} \in \mathbb{M}^n$ such that $\bar{d} \in B_{\bar{a}} + \bar{b} \subseteq V$ and $\bar{a}\bar{b} \mathop{\smile}\limits^{\vert} C\bar{d}$.*

Proof. Proceeding by induction on n, it suffices to treat the case $n = 1$. Let $V_0 := V - d$; it is a definable neighbourhood of 0. Since $\mathbb{M}$ is a topological group, there exists V_1 definable and open, such that $0 \in V_1$, $V_1 = -V_1$, and $V_1 + V_1 \subseteq V_0$. By Lemma 9.14, there exists $\bar{a} \in \mathbb{M}^m$ such that $\bar{a} \mathop{\smile}\limits^{\vert} Cd$ and $0 \in B_{\bar{a}} \subseteq V_1$. Let $W := d - B_{\bar{a}}$. Since $\dim(W) = 1$, there exists $b \in W$ such that $b \notin \mathrm{Zcl}(C\bar{a}d)$.

Claim 1. *$d \in B_{\bar{a}} + b$.*

In fact, $b \in -B_{\bar{a}} + d$, and therefore $d - b \in B_{\bar{a}}$.

Claim 2. *$\bar{a}b \mathop{\smile\!\!\!\!\!\!|}\, Cd$.*

By construction, $b \mathop{\smile\!\!\!\!\!\!|}\, C\bar{a}d$, and therefore $b \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}} Cd$, and hence $\bar{a}b \mathop{\smile\!\!\!\!\!\!|}_{\bar{a}} Cd$. Together with $\bar{a} \mathop{\smile\!\!\!\!\!\!|}\, Cd$, this implies the claim. □

Corollary 9.19. *Let $X \subseteq \mathbb{M}^n$ be a definable set, and let $k \in \mathbb{N}$. Assume that, for every $\bar{x} \in X$, there exists $V_{\bar{x}}$, a definable open neighbourhood of $\bar{x}$, such that $\dim(V_{\bar{x}} \cap X) \le k$. Then, $\dim(X) \le k$.*

Proof. Let C be the set of parameters of X. By Lemma 9.18, for every $\bar{x} \in X$ there exist $\bar{a} \in \mathbb{K}^{n \times m}$ and $\bar{b} \in \mathbb{K}^n$ such that $\bar{a}\bar{b} \mathop{\smile\!\!\!\!\!\!|}\, C\bar{x}$ and $\bar{x} \in B_{\bar{a}} + \bar{b} \subseteq V_{\bar{x}}$; notice that $\dim(X \cap (B_{\bar{a}} + \bar{b})) \le k$. Hence, by Lemma 3.69, $\dim(X) \le k$. □

We do not know if the above corollary remains true if we drop the assumption that $\mathbb{M}$ expands a group.

Corollary 9.20. *Let $C \subset \mathbb{M}$ and $p \in S_n(C)$. Then, p is stationary iff p is realised in $\mathrm{dcl}(C)$.*

Proof. Assume for contradiction, that p is stationary, but that $\dim(p) > 0$. Let $\bar{a}_0$ and $\bar{a}_1$ be realisations of p independent over C. Since $\dim(p) > 0$, $\bar{a}_0 \ne \bar{a}_1$. Since $\mathbb{M}$ is Hausdorff, Lemma 9.18 implies that there exists V, an open neighbourhood of $\bar{a}_0$, definable with parameters $\bar{b}$, such that $\bar{a}_1 \notin V$ and $\bar{b} \mathop{\smile\!\!\!\!\!\!|}\, C\bar{a}_0\bar{a}_1$. Hence, by Lemma 3.11, $\bar{a}_0 \mathop{\smile\!\!\!\!\!\!|}_{C\bar{b}} \bar{a}_1$. Since p is stationary, Lemma 3.64 implies that $\bar{a}_0 \equiv_{\bar{b}} \bar{a}_1$, contradicting the fact that $\bar{a}_0 \in V$, while $\bar{a}_1 \notin V$. □

10. Cl-minimal structures

Let $\mathbb{M}$ be a monster model, T be the theory of $\mathbb{M}$, and let cl be an existential matroid on $\mathbb{M}$. We denote by dim and rk the dimension and rank induced by cl.

Definition 10.1. A type $p \in S_n(A)$ is a **generic type** if $\dim(p) = n$. The structure $\mathbb{M}$ is cl-**minimal** if, for every $A \subset \mathbb{M}$, there exists only one generic 1-type over A.

Remark 10.2. For every $0 < n \in \mathbb{N}$ and $A \subset \mathbb{M}$, there exists at least one generic type in $S_n(A)$. If $\mathbb{M}$ is cl-minimal, then for every n and A there exists exactly one generic type in $S_n(A)$.

Lemma 10.3. *If $\mathbb{M}$ is cl-minimal, then dim is definable.*

Proof. Notice that, given $\bar{x} := \langle x_1, \dots, x_n \rangle$ and a formula $\phi(\bar{x}, \bar{y})$, the set $U^n_\phi := \{\bar{a} : \dim(\phi(\mathbb{K}, \bar{a})) = n\}$ is always type-definable (Remark 3.43). By the above remark, $\mathbb{K}^n \setminus U^n_\phi = U^n_{\neg\phi}$, and therefore U^n_ϕ is both type-definable and or-definable, and hence definable. □

Remark 10.4. The structure $\mathbb{M}$ is cl-minimal iff, for every $n > 0$ and every X definable subset of $\mathbb{K}^n$, exactly one among X and $\mathbb{K}^n \setminus X$ has dimension n.

Remark 10.5. If $\mathbb{K} \preceq \mathbb{M}$ and dim is definable, then $\mathbb{K}$ is cl-minimal iff, for every X definable subset of $\mathbb{K}$, either $\dim(X) \le 0$, or $\dim(\mathbb{K} \setminus X) \le 0$; that is, we can check cl-minimality inside $\mathbb{K}$.

Examples 10.6. 1. $\mathbb{M}$ is strongly minimal iff acl is a matroid and $\mathbb{M}$ is acl-minimal.
2. Consider Example 3.59(2). In that context, a type is generic in our sense iff it is generic in the sense of stable groups. Hence, $\mathbb{G}$ is cl-minimal iff it has only one generic type iff it is connected (in the sense of stable groups).

Lemma 10.7. *Assume that T is cl-minimal; let Scl be the small closure inside T^d. Then, T^d is Scl-minimal. Moreover, T^d coincides with T^2.*

Proof. Let $\langle \mathbb{B}^*, \mathbb{A}^* \rangle$ be a monster model of T^d. Let $C \subset \mathbb{B}^*$ with $|C| < \kappa$. Define $\mathbb{A} := \mathrm{cl}^{\mathbb{B}^*}(\mathbb{A}^* C)$, and $q_C(x)$ the partial $\mathcal{L}^2$-type over C given by

$$q_C(x) := x \notin \mathbb{A}.$$

It is clear that every generic 1-T^d-type over C expands q_C. Hence, it suffices to prove that q_C is complete. Let b and $b' \in \mathbb{B}^*$ satisfy q_C. By Corollary 8.19, $\langle \mathbb{B}^*, \mathbb{A}^* \rangle \preceq \langle \mathbb{B}^*, \mathbb{A} \rangle$. By assumption, b and b' are not in $\mathbb{A}$; hence, since T is cl-minimal, they satisfy the same generic 1-T-type $p_{\mathbb{A}}$; thus, by Corollary 8.21, $b \equiv^2_{\mathbb{A}} b'$. □

11. Connected groups

Let $\mathbb{M}$ be a monster model, and let cl be an existential matroid on it. Denote $\dim := \dim^{\mathrm{cl}}$, $\mathrm{rk} := \mathrm{rk}^{\mathrm{cl}}$, and $\mathop{\smile\!\!\!\!\!\mid} := \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{cl}}$.

Definition 11.1. Let $X \subseteq \mathbb{M}^n$ be definable (with parameters). Assume that $m := \dim(X) > 0$. We say that X is **connected** if, for every Y definable subset of X, either $\dim(Y) < n$, or $\dim(X \setminus Y) < n$.

For instance, if $\mathbb{M}$ is cl-minimal and $X = \mathbb{M}$, then X is connected.

Remark 11.2. If X is connected, then, for every $l \geq 0$, X^l is also connected.

Remark 11.3. Let $X \subseteq \mathbb{M}^n$ be definable and of dimension $m > 0$.

1. X is connected iff, for every $A \subset \mathbb{M}$ containing the parameters of definition of X, there exists exactly one n-type over A in X which is generic (i.e., of dimension m).
2. If X is connected and Y is a definable subset of X of dimension less than m (e.g., Y is finite), then $X \setminus Y$ is connected.

Lemma 11.4. *Let $G \subseteq \mathbb{M}^n$ be definable and connected. Assume that G is a semigroup with left cancellation. Assume, moreover, that G has either right cancellation or right identity. Then G is a group.*

Cf. [21, 1.1].

Proof. Assume not. Let $m := \dim(G)$. W.l.o.g., G is definable without parameters. For every $a \in G$, let $a \cdot G := \{a \cdot x : a \in G\}$. Since G has left cancellation, we have $\dim(a \cdot G) = m$.

Let $F := \{a \in G : a \cdot G = G\}$. Our aim is to prove that $F = G$. It is easy to see that F is multiplicatively closed.

First, assume that G has a right identity element 1. The reader can verify that following claim is true for any abstract semigroup with left cancellation and right identity.

Claim 1. *F is a group.*

Claim 2. $\dim(F) < m$.

Assume, for contradiction, that $\dim(F) = m$. Let $a \in G \setminus F$. Then, $F \cap (a \cdot F) \neq \emptyset$; let $u, v \in F$ be such that $u = a \cdot v$.

Since $u \in F$ and F is a group, there exists $w \in F$ such that $v \cdot w = 1$; hence, $u \cdot w = a \cdot 1 = a$, and therefore $a \in F$, which is absurd.

Choose $a, b \in G$ independent (over the empty set). Since $\dim(a \cdot G) = \dim(b \cdot G) = m$, we have $a \in b \cdot G$ and $b \in a \cdot G$. Let $u, v \in G$ be such that $b = a \cdot u$ and $a = b \cdot v$. Hence, $a = a \cdot u \cdot v$.

Since $a \cdot 1 = a \cdot u \cdot v$, we have $1 = u \cdot v$. Hence, both u and v are in F. However, since $\dim(F) < m$ and $b = a \cdot u$, we have $\mathrm{rk}(b/a) \leq \mathrm{rk}(u) < m$, which is absurd.

If instead G has right cancellation, it suffices, by symmetry, to show that G has a left identity. Reasoning as above, we can show that there exist a and b in G such that $a \cdot b = a$. We claim that b is a left identity. In fact, for every $c \in G$, we have $a \cdot b \cdot c = a \cdot c$, and therefore $b \cdot c = c$, and we are done. □

Proviso. *For the remainder of this section, $\langle G, \cdot \rangle$ is a definable connected group, of dimension $m > 0$, with identity 1.*

If G is Abelian, we will also use $+$ instead of $\cdot$ and 0 instead of 1.

Hence, if G expands a ring without zero divisors, then, by applying the above lemma to the multiplicative semigroup of G, we obtain that G is a division ring.

Remark 11.5. Let $X \subseteq G$ be definable, such that $X \cdot X \subseteq X$. Then, $\dim(X) = m$ iff $X = G$.

Proof. Assume that $\dim(X) = m$. Let $a \in G$. Then, $X \cap (a \cdot X^{-1}) \neq \emptyset$; choose $u, v \in X$ such that $u = a \cdot v^{-1}$. Hence, $a = u \cdot v \in X \cdot X = X$. □

Lemma 11.6. *Let $f : G \to G$ be a definable homomorphism. If $\dim(\ker f) = 0$, then f is surjective.*

Cf. [21, 1.7].

Proof. Let $H := f(G)$ and $K := \ker(f)$; notice that $H < G$ and $K < G$. Moreover, by additivity of dimension, $m = \dim(H) + \dim(K)$. Hence, if $\dim(K) = 0$, then $\dim(H) = m$; therefore $H = G$ and f is surjective. □

Example 11.7. The group $\langle \mathbb{Z}, + \rangle$ cannot be cl-minimal, because the homomorphism $x \mapsto 2x$ has trivial kernel but is not surjective.

Lemma 11.8. *Let $H < G$ be definable, with $\dim(H) = k < m$. Then, G/H is connected, and $\dim(G/H) = m - k$.*

Proof. That $\dim(G/H) = m - k$ is obvious. Let $X \subseteq G/H$ be definable of dimension $m - k$. We must prove that $\dim(G/H \setminus X) < m$. Let $\pi : G \to G/H$ be the canonical projection, and let $Y := \pi^{-1}(X)$. Then, $\dim(Y) = m$, and therefore $\dim(G \setminus Y) < m$. Thus, $\dim(G/H \setminus X) = \dim(\pi(G \setminus Y)) < m - k$. □

Conjecture 11.9. *If $m = 1$, then G is Abelian. Cf. Reineke's theorem [21, 3.10].*

Proceeding as in [21, 3.10], to prove the above conjecture it would be enough to consider the case when any two elements of G different from the identity are conjugate.

Lemma 11.10. *Assume that $m = 1$ and G is Abelian. Let p be a prime number. Then, either $pG = 0$, or G is divisible by p.*

Proof. Let $H := pG$ and $K := \{x \in G : px = 0\}$. If $\dim(H) = 1$, then $G = H$, and therefore G is p-divisible. If $\dim(H) = 0$, then $\dim(K) = 1$; thus $G = K$ and $pG = 0$. □

Notice that the above lemma needs the hypothesis that $m = 1$. For instance, let $\mathbb{M}$ be the algebraic closure of $\mathbb{F}_p$, and let $G := \mathbb{M} \times \mathbb{M}^*$ (where $\mathbb{M}^*$ is the multiplicative group of $\mathbb{M}$).

Theorem 11.11. *Assume that G expands an integral domain (and is connected). Then, G is an algebraically closed field.*

The proof if the above theorem is the same as that of Macintyre's theorem [21, 3.1 and 6.11] (cf. Corollary 3.53); notice also that the first step in the proof of Macintyre's theorem is showing that G is connected. Moreover, in the above theorem it is essential that G is connected; for instance, if $\mathbb{M}$ is a formally p-adic field, then $\mathbb{M}$ itself is a nonalgebraically closed field (of dimension 1).

Question 11.12. Can we weaken the hypothesis in the above theorem from "G expands an integral domain" to "G expands a ring without zero divisors"?

12. Ultraproducts

Let I be an infinite set, and let μ be an ultrafilter on I. For every $i \in I$, let $\langle \mathbb{K}_i, \mathrm{cl}_i \rangle$ be a pair given by a first-order $\mathcal{L}$-structure $\mathbb{K}_i$ and an existential matroid cl_i on $\mathbb{K}_i$. Let $\mathcal{K}$ be the family $\big(\langle \mathbb{K}_i, \mathrm{cl}_i \rangle\big)_{i \in I}$, and let $\mathbb{K} := \Pi_i \mathbb{K}_i/\mu$ be the corresponding ultraproduct.

We will give some sufficient condition on the family $\mathcal{K}$, such that there is an existential matroid on $\mathbb{K}$ induced by the family of cl_i. Denote by d_i the dimension induced by cl_i.

Definition 12.1. We say that the dimension is **uniformly definable** (for the family $\mathcal{K}$) if, for every formula $\phi(\bar{x}, \bar{y})$ without parameters, for every tuple $\bar{x} = \langle x_1, \dots, x_n \rangle$ and $\bar{y} = \langle y_1, \dots, y_m \rangle$, and for every $l \le n$, there is a formula $\psi(\bar{y})$, also without parameters, such that, for every $i \in I$,

$$\psi(\mathbb{K}_i) = \left\{\bar{y} \in \mathbb{K}_i^m : d_i\big(\phi(\mathbb{K}_i, \bar{y})\big) = l\right\}.$$

We denote by d^l_ϕ the formula ψ.

Remark 12.2. The dimension is uniformly definable if, for every formula $\phi(x, \bar{y})$ without parameters, $\bar{y} = \langle y_1, \dots, y_m \rangle$, there is a formula $\psi(\bar{y})$, also without parameters, such that, for every $i \in I$,

$$\psi(\mathbb{K}_i) = \left\{\bar{y} \in \mathbb{K}_i^m : d_i\big(\phi(\mathbb{K}_i, \bar{y})\big) = 1\right\}.$$

For instance, if every $\mathbb{K}_i$ expands a ring without zero divisors, then the dimension is uniformly definable; given $\psi(x, \bar{y})$, define $\psi(\bar{y})$ by

$$\forall z\ \exists x_1, \dots, x_4 \left(z = F(x_1, \dots, x_4)\ \&\ \bigwedge_{i=1}^{4} \phi(x_i, \bar{y})\right).$$

For the remainder of this section, we assume that the dimension is uniformly definable for $\mathcal{K}$.

Definition 12.3. Let d be the function from definable sets in $\mathbb{K}$ to $\{-\infty\} \cup \mathbb{N}$ defined in the following way.

Given a $\mathbb{K}$-definable set $X = \Pi_{i \in I} X_i/\mu$ and $l \in \mathbb{N}$, $d(X) = l$ if, for μ-almost every $i \in I$, $d_i(X_i) = l$.

The following result is the justification for Definitions 12.1 and 12.3.

Remark 12.4. The map d is a dimension function on $\mathbb{K}$. Let cl be the existential matroid induced by d. Then, $a \in \mathrm{cl}(\bar{b})$ implies that, for μ-almost every $i \in I$, $a_i \in \mathrm{cl}_i(\bar{b}_i)$, but the converse is *not* true.

Remark 12.5. Let $X \subseteq \mathbb{K}^n$ be definable with parameters $\bar{c}$; let $\phi(\bar{x}, \bar{c})$ be the formula defining X. Given $l \in N$, $d(X) = l$ iff, for μ-almost every $i \in I$, $\mathbb{K}_i \models d^l_\phi(\bar{c}_i)$.

Lemma 12.6. *If each $\mathbb{K}_i$ is* cl*-minimal, then $\mathbb{K}$ is also* cl*-minimal.*

Proof. By Remark 10.5. □

Example 12.7. The ultraproduct $\mathbb{K}$ of strongly minimal structures is not strongly minimal in general (it will not even be a pregeometric structure), but, if each structure expands a ring without zero divisors, then $\mathbb{K}$ will have a (unique) existential matroid, and will be cl-minimal.

In fact, let $\mathbb{F}$ be an algebraically closed field of finite characteristic. For every $n \in \mathbb{N}$, let P_n be a subset of $\mathbb{F}$ with n element. Let P be a new unary predicate, define $\mathbb{K}_n := (\mathbb{F}, P_n)$ in the language of fields expanded by P, and let $\mathbb{K} := \langle K, +, \cdot, P^* \rangle$ be a nonprincipal ultraproduct of the $\mathbb{K}_n$. Then, P^* will be an infinite definable subset of $\mathbb{K}$ of dimension 0, and therefore $\mathbb{K}$ will not be geometric. By taking instead for P_n suitable finite subsets of $\mathbb{F}^3$, we can also attain that any nonprincipal ultraproduct $\mathbb{K}$ of $\mathcal{K}$ is not geometric, does satisfy the Independence Property, and has an infinite definable subset with a definable linear ordering. Moreover, one can also impose that the trivial chain condition for uniformly definable subgroups of $\langle \mathbb{K}, + \rangle$ fails in $\mathbb{K}$ [21, 1.3].

However, $\mathbb{K}$ will satisfy the following conditions.

1. Every definable monoid with left cancellation is a group [21, 1.1].
2. Given G a definable group acting in a definable way on a definable set E, if A is a definable subset of E and $g \in G$ such that $g \cdot A \subseteq A$, then $g \cdot A = A$ [21, 1.2].

We do not know if conditions (1) and (2) in the above example are true for an arbitrary cl-minimal structure expanding a field.

Remark 12.8. Assume that each $\mathbb{K}_i$ is a first-order topological structure, and that the definable basis of the topology of each $\mathbb{K}_i$ is given by the same function $\Phi(x, \bar{y})$. Then, $\mathbb{K}$ is also a first-order topological structure, and $\Phi(x, \bar{y})$ defines a basis for the topology of $\mathbb{K}$. If each $\mathbb{K}_i$ is d-minimal, then $\mathbb{K}$ has an existential matroid, but it needs not be d-minimal.
Assume that each $\mathbb{K}_i$ is d-minimal and satisfies the additional condition.

(*) Every definable subset of $\mathbb{K}_i$ of dimension 0 is discrete.

Then, $\mathbb{K}$ is also d-minimal and satisfies condition (*).

Example 12.9. An ultraproduct of o-minimal structures is not necessarily o-minimal, but it is d-minimal, and satisfies condition (*).

13. Dense tuples of structures

In this section, we assume that T expands the theory of integral domains. We will extend the results of Section 8 to dense tuples of models of T.

Definition 13.1. Fix $n \geq 1$. Let $\mathcal{L}^n$ be the expansion of $\mathcal{L}$ by $(n-1)$ new unary predicates $P_1, \dots, P_{n-1}$. Let T^n be the $\mathcal{L}^n$-expansion of T, whose models are sequences $\mathbb{K}_1 \prec \cdots \prec \mathbb{K}_{n-1} \prec \mathbb{K}_n \models T$, where each $\mathbb{K}_i$ is a proper cl-closed elementary substructure of $\mathbb{K}_{i+1}$. Let T^{nd} be the expansion of T^{n+1} saying that $\mathbb{K}_1$ is dense in $\mathbb{K}_n$. We also define $T^{0d} := T$.

For instance, $T^1 = T$, T^2 is the theory we already defined in Section 8, and $T^{1d} = T^d$.

Lemma 13.2. *If T is* cl*-minimal, then T^n is complete for every $n \geq 1$ (and therefore coincides with $T^{(n-1)d}$). Moreover, T^n has a (unique) existential matroid* cl^n*; given $\langle \mathbb{K}_n, \dots, \mathbb{K}_1 \rangle \models T^n$, we have $b \in \mathrm{cl}^n(A)$ iff $b \in \mathrm{cl}^{\mathbb{K}_n}(A\mathbb{K}_{n-1})$. Finally, T^n is* cl^n*-minimal.*

Proof. By induction on n. Iterate n times Lemma 10.7. □

Corollary 13.3. *Assume that T is strongly minimal. Then, T^n is complete, and coincides with the theory of tuples $\mathbb{K}_1 \prec \cdots \prec \mathbb{K}_n \models T$.*

Proof. One can use either the above lemma, or reason as in [15], using Lemma 8.10. □

Remark 13.4. Let $\langle \mathbb{B}, \mathbb{A} \rangle$ be a λ-saturated model of T^d, for some cardinal λ. Let $U \subseteq \mathbb{B}$ be $\mathbb{B}$-definable and of dimension 1. Then, $\mathrm{rk}(U \cap \mathbb{A}) \geq \lambda$.

Theorem 13.5. *The theory T^{nd} is complete. There is a (unique) existential matroid on T^{nd}.*

Proof. By induction on n, we will prove that T^{nd} is $(\cdots(T^d)^d \cdots)^d$ iterated n times. This implies both that T^{nd} is complete, and that it has an existential matroid.

It suffices to treat the case $n = 2$. Notice that $\langle \mathbb{K}_2, \mathbb{K}_1 \rangle \prec \langle \mathbb{K}_3, \mathbb{K}_1 \rangle \models T^d$. It suffices to show that $\mathbb{K}_2$ is Scl-dense in $\langle \mathbb{K}_3, \mathbb{K}_1 \rangle$. W.l.o.g., we can assume that $\langle \mathbb{K}_3, \mathbb{K}_2, \mathbb{K}_1 \rangle$ is ω-saturated.

Let $X \subseteq \mathbb{K}_3$ be definable in $\langle \mathbb{K}_3, \mathbb{K}_1 \rangle$ (with parameters from $\mathbb{K}_3$), such that $\mathrm{Sdim}(X) = 1$. We need to show that X intersects $\mathbb{K}_2$. By Corollary 8.36, there exist U and S subsets of $\mathbb{K}_3$, such that U is definable in $\mathbb{K}_3$, S is definable in $\langle \mathbb{K}_3, \mathbb{K}_1 \rangle$ and small, and $X = U \mathbin{\Delta} S$. Therefore, $\dim(U) = 1$. If, by contradiction, $X \cap \mathbb{K}_2 = \emptyset$, then $\mathbb{K}_2 \cap U \subseteq S$; therefore, $\mathrm{Srk}(\mathbb{K}_2 \cap U) < \omega$ (where Srk is the rank induced by Scl), contradicting Remark 13.4. □

The above theorem has an analogue version for "beautiful tuples" of stable structures [5, Proposition 5].

Example 13.6. To clarify a possible source of confusion, consider the case when T is the theory of algebraically closed fields of characteristic 0. Then, T^2 is a complete theory, and therefore it coincides with both T^d and the theory of beautiful pairs for T. Hence, T^d is stable [19], and therefore we can consider in turn beautiful pairs of models of T^d. However, such a beautiful pair will not be a model of T^{2d}, because it will be of the form $\langle \mathbb{K}, \mathbb{F}_1, \mathbb{F}_2, \mathbb{L} \rangle$, where $\mathbb{L}$, $\mathbb{F}_1$, $\mathbb{F}_2$, and $\mathbb{K}$ are models of T, with $\mathbb{F}_1$ and $\mathbb{F}_2$ substructures of $\mathbb{K}$, $\mathbb{L} = \mathbb{K}_1 \cap \mathbb{K}_2$, and $\mathbb{K}_1 \mathop{\smile\!\!\!\!\!\!\,|}_{\mathbb{L}} \mathbb{K}_2$.

Corollary 13.7. *Assume that T is d-minimal (and that Proviso 9.15 holds). Then, T^{nd} coincides with the theory of $(n+1)$-tuples $\mathbb{K}_1 \prec \cdots \prec \mathbb{K}_n \prec \mathbb{K}_{n+1} \models T$, such that $\mathbb{K}_1$ is (topologically) dense in $\mathbb{K}_{n+1}$.*

Proof. Notice that, if $\langle \mathbb{K}_n, \ldots, \mathbb{K}_1 \rangle$ satisfy the assumption, then, by Corollary 9.17, each $\mathbb{K}_i$ is cl-closed in $\mathbb{K}_n$. $\square$

13.1. Dense tuples of topological structures

Assume that T expands the theory of integral domains. Assume that $\mathbb{M}$ has both an existential matroid cl and a definable topology (in the sense of [18]). We have two distinct notions of closure and of density on $\mathbb{M}$: the ones given by the topology and the ones given by the matroid; to distinguish them, we will speak about topological closure and cl-closure, respectively (and similarly for density).

Let $\Phi(x, \bar{y})$ be a formula such that the family of sets

$$B_{\bar{b}} := \Phi(\mathbb{M}, \bar{b}),$$

as $\bar{b}$ varies in $\mathbb{M}^k$, is a basis of the topology of $\mathbb{M}$. If $\bar{b} = \langle \bar{b}_1, \ldots, \bar{b}_m \rangle$, we denote by $B^n_{\bar{b}} := B_{\bar{b}_1} \times \cdots \times B_{\bar{b}_m} \subseteq \mathbb{M}^m$.

The first of the following two conditions is taken from [7].

Hypothesis. I. For every $m \in \mathbb{N}$, every U open subset of $\mathbb{M}^m$, and every $\bar{a} \in U$, the set $\left\{\bar{b} : \bar{a} \in B_{\bar{b}} \subseteq U\right\}$ has nonempty interior.

II. Every definable nonempty open subset of $\mathbb{M}$ has dimension 1.

Remark 13.8. Assumption II implies that a definable subset of $\mathbb{M}^m$ with nonempty interior has dimension m (but the converse is not true: there can be definable subsets of dimension m but with empty interior). Moreover, it implies that a cl-dense subset of $\mathbb{M}^m$ is also topologically dense (but, again, the converse is not true: see Theorem 13.11).

Examples 13.9. 1. If $\mathbb{M}$ is either a valued field (with the valuation topology) or a linearly ordered field (with the order topology), then it satisfies Assumption I.

2. If $\mathbb{M}$ is a d-minimal structure, then it satisfies Assumption II.
3. Let $\mathbb{M}$ be either a formally p-adic field, or an algebraically closed valued field, or a d-minimal expansion of a linearly ordered definably complete field (cf. Example 9.3). Then, $\mathbb{M}$ satisfies both assumptions.

Fact 13.10 *([7, Corollary 3.1]). Suppose that Assumption I is true. Let $\langle \mathbb{B}, \mathbb{A} \rangle \models T^2$ and $C \subseteq \mathbb{B}$. Assume that, for every $m \in \mathbb{N}$, there is a set $D_m \subseteq \mathbb{B}^m$ such that the following hold.*

1. *D_m is topologically dense in $\mathbb{B}^m$;*
2. *for every $\bar{a} \in D_m$ and every open set $U \subseteq \mathbb{B}^m$, if $\operatorname{tp}^1(\bar{a}/C)$ is realised in U, then $\operatorname{tp}^1(\bar{a}/C)$ is realised in $U \cap D_m$;*
3. *for every $\bar{d} \in D_m$, $\operatorname{tp}^2(\bar{d}/C)$ is implied by $\operatorname{tp}^1(\bar{d}/C)$ in conjunction with "$\bar{d} \in D_m$".*

Then, every open set T^2-definable over C is T-definable over C.

The following theorem, which is a generalisation of [7, Corollary 3.4], follows easily from the above fact.

Theorem 13.11. *Assume that the hypothesis holds. Let $\mathbb{C} := \langle \mathbb{B}, \mathbb{A}_{n-1}, \ldots, \mathbb{A}_1 \rangle \models T^{nd}$. Let $\bar{c} \subset \mathbb{B}$ be cl-independent over $\bar{c} \cap \mathbb{A}_{n-1}$. Let $U \subseteq \mathbb{B}^m$ be open and definable in $\mathbb{C}$, with parameters $\bar{c}$. Then, U is definable in $\mathbb{B}$, with parameters $\bar{c}$. Moreover, T^{nd} also satisfies the hypothesis.*

In the terminology of [10], the above theorem proves that $\mathbb{B}$ is the **open core** of $\mathbb{C}$.

Proof. By induction on n, it suffices to do the case when $n = 2$, i.e. when $\mathbb{C} = \langle \mathbb{B}, \mathbb{A} \rangle \models T^d$. W.l.o.g., $\mathbb{C}$ is λ-saturated and λ-homogeneous, for some $|T| < \lambda < \kappa$. Define $D_m := \left\{\bar{d} \in \mathbb{B}^m : \operatorname{Srk}(\bar{d}/\bar{c}) = m\right\}$. We want to verify that the hypothesis of Fact 13.10 is satisfied for the above D_m.

1. By Lemma 8.31, if $V \subseteq \mathbb{B}^m$ is $\mathbb{B}$-definable and of dimension m, then $V \cap D_m$ is nonempty; therefore, by Assumption II, D_m is topologically dense in $\mathbb{B}^m$.
2. Let $\bar{d} \in D_m$ and $U \subseteq \mathbb{M}^m$ be open, and assume that $p := \operatorname{tp}^1(\bar{d}/\bar{c})$ is realised in U. We have to show that p is realised in $U \cap D_m$. Let $\bar{d}' \in U$ be a realisation of p, and let $\bar{b} \subset \mathbb{B}$ be such that $\bar{d}' \in B_{\bar{b}} \subseteq U$. Since $\bar{d}' \equiv^1_{\bar{c}} \bar{d}$, we have that $\bar{d}'$ is cl-independent over $\bar{c}$. By changing $\bar{b}$ if necessary, we can also assume that $\bar{d}' \mathop{\smile\!\!\!\!\!\!\,|}\ \bar{b}\bar{c}$ (cf. the proof of Lemma 9.18), and thus $\bar{d}'$ is cl-independent over $\bar{b}\bar{c}$. Finally, since $\mathbb{A}$ is cl-dense in $\mathbb{B}$, there exists $\bar{d}'' \equiv^1_{\bar{b}\bar{c}} \bar{d}$ such that $\bar{d}''$ is cl-independent over $\bar{b}\bar{c}\mathbb{A}$, and therefore $\bar{d}'' \in B_{\bar{b}} \cap D_m \subseteq U \cap D_m$.
3. By Proposition 8.17.

Hence, we can apply Fact 13.10, and we are done. $\square$

14. The (pre)geometric case

Remember that $\mathbb{M}$ is a pregeometric structure if acl satisfies the EP. If, moreover, $\mathbb{M}$ eliminates the quantifier $\exists^\infty$, then $\mathbb{M}$ is geometric.

In this section, we gather various results about (pre)geometric structures, mainly in order to clarify and motivate the general case of structures with an existential matroid.

Remember that $\mathbb{M}$ has geometric elimination of imaginaries if every for imaginary tuple $\bar{a}$ there exists a real tuple $\bar{b}$ such that $\bar{a}$ and $\bar{b}$ are interalgebraic.

Remark 14.1. A theory T is pregeometric iff T is a real-rosy theory of real þ-rank 1. Moreover, if T is pregeometric and has geometric elimination of imaginaries, then $\mathop{\smile\!\!\!\!\!\!|}^{þ} = \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ and $\dim^{\mathrm{acl}}$ is equal to the þ-rank; see [11] for definitions and proofs.

Remark 14.2. The model-theoretic algebraic closure acl is a definable closure operator.

For the remainder of this section, $\mathbb{M}$ is pregeometric (and T is its theory).

Remark 14.3. The operator acl is an existential matroid on $\mathbb{M}$. The induced independence relation $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{acl}}$ coincides with the real þ-independence relation $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ and with the M-dividing notion $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{M}}$ of [1]. A formula is x-narrow (for acl) iff it is algebraic in x.

Remark 14.4. Let $X \subseteq \mathbb{M}^n$ be definable. We have that $\dim^{\mathrm{acl}}(X) \leq 0$ iff X is finite.

Remark 14.5. The structure $\mathbb{M}$ is geometric iff $\dim^{\mathrm{acl}}$ is definable.

Remark 14.6. The structure $\mathbb{M}$ is acl-minimal iff it is strongly minimal.

In Section 6, we defined an extension of acl to the imaginary sorts, which here we will denote by $\widetilde{\mathrm{acl}}$ (while will we use $\mathrm{acl}^{\mathrm{eq}}$ to denote the usual algebraic closure for imaginary elements).

Remark 14.7. If a is real and B is imaginary, then $a \in \widetilde{\mathrm{acl}}(B)$ iff $a \in \mathrm{acl}^{\mathrm{eq}}(B)$.

Remark 14.8. T.f.a.e.:

1. $\mathrm{acl}^{\mathrm{eq}}$ coincides with $\widetilde{\mathrm{acl}}$;
2. T is superrosy of þ-rank 1 [11];
3. T is surgical [13].

Remark 14.9. A set X is dense in $\mathbb{M}$ iff, for every U infinite definable subset of $\mathbb{M}$, $U \cap X \neq \emptyset$. If $\mathbb{F} \preceq \mathbb{K}$, then $\mathbb{F}$ is acl-closed in $\mathbb{K}$.

Remark 14.10. Assume that T is geometric. Then, T^2 is the theory of pairs $\langle \mathbb{K}, \mathbb{F} \rangle$, with $\mathbb{F} \prec \mathbb{K} \models T$, and T^d is the theory of pairs $\langle \mathbb{K}, \mathbb{F} \rangle \models T^2$, such that $\mathbb{F}$ is dense in $\mathbb{K}$. For every $X \subseteq \mathbb{K}$, $\mathrm{Scl}(X) = \mathrm{acl}^1(\mathbb{F}X) = \mathrm{acl}^2(\mathbb{F}X)$ (cf. Question 8.39).

For more on the theory T^d in the case when T is geometric, and in particular when T is o-minimal, see [6,4].

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