Abstract

Multiparameter families of possibly lacunary polynomials in $x$ are constructed that have remarkably explicit (or “almost” explicit) expansions about both $x = 0$ and $x = 1$. They are placed in the framework of questions about mixed $q$-analogues with fewest terms. The properties of the most tractable of these surprisingly tractable polynomials are established with the aid of solutions to partial differential equations of a type much studied by Truesdell.

Keywords: Analogues

1. Introduction

The ordinary binomial theorem essentially tells us that $(x - 1)^n$ is a polynomial whose expansion about both $x = 0$ and $x = 1$ can be described in a simple explicit way. Our main result, Theorem 1, is that there are certain special multiparameter families of polynomials $P(x) = P(m, n, r_1, \ldots, r_m; x)$, in general lacunary, for which the same can “almost” be done. For example, see formulas (5.1) and (5.2). However, we prefer to view the result from the point of view of the following question about mixed $q$-analogues.
Set \( [k]_q = (1 - q^k)/(1 - q) \). Then the usual \( q \)-analogue of \( n! \) is
\[
\prod_{k=1}^{n} [k]_q.
\]
We regard
\[
m! \prod_{k=m+1}^{n} [k]_q
\]
as an example of a mixed \( q \)-analogue of \( n! \). We also regard
\[
2 \cdot 4 \cdot 6 \cdot 8 \cdot [3]_q \cdot [5]_q \cdot [7]_q
\]
as a mixed \( q \)-analogue of \( 8! \) and so forth. We shall not give a formal definition of “mixed \( q \)-analogue,” but we shall require the denominator to be a power of \((q - 1)\) and the limit as \( q \to 1 \) to reduce to the original expression. Also, we would allow expressions such as
\[
\frac{1 - q^{3k-4}}{1 - q},
\]
but not
\[
\frac{1 - q^{b_k}}{1 - q}.
\]
In other words, the exponent of any \( q \) shall be a polynomial of degree at most 1 in at most one indeterminate.

Possible \( q \)-analogues of \( abc \) include
\[
abc, \quad \frac{q^a - 1}{1 - q}, \quad \frac{(q^a - 1)(q^b - 1)}{(q - 1)(q - 1)}c \quad \text{and} \quad \frac{(q^a - 1)(q^b - 1)(q^c - 1)}{(q - 1)^3}.
\]
We refer to these respectively as \( q \)-analogues with coefficients of degree 3, 2, 1 and 0, since the numerators, viewed as polynomials in \( q \), have (homogeneous) polynomial coefficients of these degrees. We also observe that expansion of the numerators in \((*)\) yields 1, 2, 4 and 8 terms, respectively.

**Question 1.** Of all mixed \( q \)-analogues with coefficient degree \( d \) of a product of linear forms, which has the fewest terms?

The above question has not been formulated precisely (it may be premature to do so), but special cases of it lead to mixed \( q \)-analogues of interest. For the case of \( a_1a_2 \ldots a_N \), in which each linear form is an independent indeterminant, there are clearly \( q \)-analogues with coefficient degree \( d \) and \( 2^N - d \) terms. But what about cases in which the linear forms are not independent? Consider coefficient degree 2 \( q \)-analogues of \( trs(r - s) \). An obvious \( q \)-analogue is
\[
\frac{(q^t - 1)(q^r - 1)}{(q - 1)^2} s(r - s).
\]
Here the numerator has 4 terms. But this is not optimal. We have
\[
\lim_{q \to 1} \frac{2t(s(q^r - 1) - r(q^s - 1))}{(q - 1)^2} = trs(r - s)
\]
and since
\[
s(q^r - 1) - r(g^s - 1) = sq^r - rq^s + (r - s).
\]
The numerator has only 3 terms.

The significance of Theorem 1 from this point of view is that it illustrates how very far from optimal the “obvious” $q$-analogues are from the point of view of minimizing the number of terms. In particular, the obvious procedure shows that a product of $N = (m + 1)^2$ terms has a $q$-analogue of coefficient degree $m^2$ with at most
\[
2(m+1)^2 - m^2 = 2m+1 = (1/2)4\sqrt{N}
\]
terms. Theorem 1 (see Remark 2 following it) displays products of $N = (m + 1)^2$ terms that have a $q$-analogue of coefficient degree $m^2$ with only $2\sqrt{N}$ terms. Moreover, the polynomials these linear form products are related to may have even more special properties for certain values of the coefficient variables. This is the content of Theorem 2. The proof of Theorem 2 involves some Truesdellian partial differential equations.

We believe that the zeros of the polynomials introduced here, some of which exceed 1 in modulus, are of interest. Therefore, we shall revert to using $x$ (as in the first paragraph) rather than $q$ as the principle variable. Also, the literature on self-inversive polynomials usually designates the principle variable as $x$ or $z$.

2. A self-inversive polynomial

Let $n$ be a positive integer and let $r_1, r_2, \ldots, r_m$ be indeterminates. We set $r_0 = 0$ for notational convenience. At times we shall assign positive integer values to the other $r_i$ so that
\[
0 = r_0 < r_1 < r_2 < \cdots < r_m.
\]
Define numerator polynomials by
\[
N_1 = \prod_{j=0}^{m} (2n - 2r_j), \quad N_2 = \prod_{0 \leq i < j \leq m} (r_j - r_i)
\]
and
\[
N_3 = \prod_{0 \leq i < j \leq m} (2n - r_i - r_j).
\]
Define denominator polynomials by
\[
D_1(k) = 2n - 2r_k, \quad D_2(k) = \prod_{j=0}^{k-1} (r_k - r_j) \prod_{p=k+1}^{m} (r_p - r_k)
\]
and
\[ D_3(k) = \prod_{j=0 \atop (j \neq k)}^{m} (2n - r_k - r_j). \]

Define coefficients
\[ c_k = (-1)^k \frac{N_1}{D_1(k)} \frac{N_2}{D_2(k)} \frac{N_3}{D_3(k)}, \quad 0 \leq k \leq m, \]

and a polynomial
\[ P(x) = \sum_{k=0}^{m} c_k x^{r_k} - \sum_{k=0}^{m} c_{m-k} x^{2n-r_m-k}. \]

Note that
\[ P(x) = P(m, n, r_1, \ldots, r_m; x) \]

has \(2(m + 1)\) terms, that \(P(x)\) has the self-inversive property
\[ x^{2n} P \left( \frac{1}{x} \right) = -P(x), \]

and that when \(m = 0\) we have
\[ P(x) = 1 - x^{2n}. \]

Moreover, every coefficient of \(P(x)\) is a homogeneous polynomial of degree
\[ m + 1 + \left( \frac{m + 1}{2} \right) + \left( \frac{m + 1}{2} \right) - 1 - m - m = m^2. \]

3. The main result

Define
\[ H = H(m, n, r_1, \ldots, r_m) \]
\[ = n \prod_{i=1}^{m} r_i \prod_{i=1}^{m} (n - r_i) \prod_{i=1 \atop 1 \leq i < j \leq m} (2n - r_i - r_j). \]

Since
\[ 1 + m + m + m + \left( \frac{m}{2} \right) + \left( \frac{m}{2} \right) = (m + 1)^2, \]

the product \(H\) has \((m + 1)^2\) variable factors. Let \(D = d/dx\).

Theorem 1. For all integers \(\alpha \geq 0\) the polynomial
\[ D^\alpha P(x)\big|_{x=1} \]

is divisible by \(H\). Moreover, it is identically zero for \(0 \leq \alpha \leq 2m\), and
\[ D^{2m+1} P(x)\big|_{x=1} = -2^{m+1} H. \]
Remark 1. This is clear for \( m = 0 \).

Remark 2. The Taylor series expansion of \( P(x) \) about \( x = 1 \) together with Theorem 1 tells us that

\[
\lim_{x \to 1} \frac{P(m, n, r_1, \ldots, r_m; x)}{(x - 1)^{2m + 1}} = -2^{m+1} H(m, n, r_1, \ldots, r_m).
\]

Hence we have found an \( x \)-analogue of a product of \((m + 1)^2\) factors that has only 2\((m + 1)\) terms.

The proof of Theorem 1 given in Section 4 hinges upon the fact that since every coefficient of \( P(x) \) has degree \( m^2 \), every coefficient of its \( \alpha \)th derivative has degree \( m^2 + \alpha \). But \( m^2 + \alpha \leq (m + 1)^2 \) for \( \alpha \leq 2m \), so the mere fact of divisibility by \( H \) implies the vanishing of the \( \alpha \)th derivatives at \( x = 1 \) for \( \alpha \leq 2m \), and that the \((2m + 1)\)th derivative at \( x = 1 \) has the form \( C_m H \), where \( C_m \) depends only upon \( m \). We establish divisibility by \( H \) by establishing divisibility by each individual factor of \( H \). In fact, we shall show that when any such factor vanishes, the individual terms of

\[
D^\alpha P(x)|_{x=1}
\]

can be grouped into \( m + 1 \) pairs such that the elements of each pair are negatives of each other. In fact, in “most” cases, both elements will be zero.

4. The proof

As indicated above, we shall show that the vanishing of any factor of \( H \) causes pairwise cancellation among the terms of \( D^\alpha P(x) \), at least when \( x = 1 \). Since \( P(1) = 0 \), we may assume \( \alpha \geq 1 \).

The case \( n = 0 \) is simple. No denominator factor can cancel the 2\( n - 0 \) factor of \( N_1 \), except for the \( D_1(k) \) factor, and then only when \( k = 0 \). But differentiation removes the \( c_0 x^0 \) term, and causes the \(-c_0 x^{2n-\alpha} \) term to have a factor of \( n \).

For the case \( r_k = n \) note that if \( t \neq k \) the factor \( D_1(t) \) cannot remove the \( n - r_k \) factor from the numerator factor \( N_1 \). Hence the only possible nonzero terms are \( c_k x^k \) and \(-c_k x^{2n-r_k} \). But these two, no matter how often differentiated, will cancel each other.

For the \( r_k = 0 \) case we first claim that \( c_k = -c_0 \). The idea is to examine \( D_1(k) \), \( D_2(k) \) and \( D_3(k) \) when \( r_k \) is replaced by 0, and to compare them with \( D_1(0) \), \( D_2(0) \) and \( D_3(0) \). First, however, remove the \( r_k \) factor from both \( D_2(k) \) and \( D_2(0) \) (think of it in each case as having canceled with the corresponding factor in the numerator of \( c_k \), respectively, \( c_0 \)) and work with

\[
D^*_1(k) = D_2(k)/r_k \quad \text{and} \quad D^*_2(0) = D_2(0)/r_k.
\]

We have

\[
D_1(k) = 2n,
\]
\[ D^*_2(k) = \prod_{j=1}^{k-1} (-r_j) \prod_{p=k+1}^m r_p = (-1)^{k-1} \prod_{p=1}^m r_p, \]

\[ D_3(k) = 2n \prod_{j=1}^{k-1} (2n - r_j) \prod_{p=k+1}^m (2n - r_p), \]

and

\[ D_1(0) = 2n, \]

\[ D^*_2(0) = \prod_{p=1}^m r_p, \]

\[ D_3(0) = \prod_{p=1}^m (2n - r_p) = 2n \prod_{p=1}^m (2n - r_p) \prod_{p=k+1}^m (2n - r_p). \]

Thanks to the factor of \((-1)^k\) in the definition of \(c_k\), we now have that \(c_k = -c_0\). Next, we see from the definition of \(P(x)\) that in \(D^a P(x)|_{x=1}\) the terms stemming from \(c_k x^{2n-r_k}\) and \(c_0 x^{2n}\) must always cancel when \(r_k = 0\), since \(c_k = -c_0\) and each differentiation brings equal factors into \(c_k\) and \(c_0\). Also, the \(c_0 x^0\) term vanishes upon differentiation while the \(c_k x^t\) term acquires a vanishing factor of \(r_k\) upon differentiation. We conclude this case by observing that all other terms have a factor of \(r_k\) in the numerator that is not canceled by any corresponding factor in the denominator.

Perhaps the most intricate case is that of \(2n - r_k - r_t = 0\) for \(k \neq t\). Here we claim that \(c_k = c_t\). We shall first show that \(|c_k| = |c_t|\) by showing that their denominators have the same absolute values. We then examine their signs. The idea is to examine the \(D_1(k)\) when \(r_k\) is replaced by \(2n - r_t\), and to compare them with the \(D_1(t)\). We shall first, however, remove the \(2n - r_k - r_t\) factors from the \(D_3\) products (think of them as having canceled with the corresponding factors in the numerators of \(c_k\) and \(c_t\)) and work with

\[ D^*_3(k) = D_3(k)/(2n - r_k - r_t) \quad \text{and} \quad D^*_3(t) = D_3(t)/(2n - r_k - r_t). \]

We have

\[ D_1(k) = r_t - r_k, \]

\[ D_2(k) = \prod_{j=0}^{k-1} (2n - r_j - r_t) \prod_{p=k+1}^m (r_p - 2n + r_t) \quad (m - 1 \text{ factors}), \]

\[ D^*_3(k) = \prod_{j=0}^{k-1} (r_t - r_j) \prod_{p=k+1}^m (r_t - r_p) \quad (m - 2 \text{ factors}), \]

and
\[
D_1(t) = 2(n - r_t),
\]
\[
D_2(t) = \prod_{j=0}^{t-1} (r_t - r_j) \prod_{p=t+1}^{m} (r_p - r_t) \quad (m - 1 \text{ factors}),
\]
\[
D_3^*(t) = \prod_{j=0}^{t-1} (2n - r_j - r_t) \prod_{p=t+1}^{m} (2n - r_t - r_p) \quad (m - 2 \text{ factors}).
\]

Observe that exactly one of the factors in \(D_2(k)\) has a form different from the others, namely \(2(r_t - n)\). Thus
\[
|D_2(k)| = |D_1(t)D_3^*(t)|.
\]

Also note that \(D_1(k)\) has the form of a typical factor in \(D_3^*(k)\). In fact, it is easily seen that
\[
|D_1(k)D_3^*(k)| = |D_2(t)|.
\]

Upon multiplying corresponding sides of the last two equations together we find that \(|c_k| = |c_t|\). Next, we determine the number of minus signs required to put each factor of \(D_1(k), D_2(k), \ldots, D_3^*(t)\) into a certain standard form. We shall say a factor is in standard form if it has the form \(r_t - (\cdot), 2n - (\cdot) - (\cdot), \text{ or } n - r_t\). For the \(D_1(k), D_2(k)\) and \(D_3^*(k)\) factors we require \((-1)^{m-k}\) (only \(D_2(k)\) needed adjusting) and for the \(D_1(t), D_2(t)\) and \(D_3^*(t)\) factors we require \((-1)^{m-t}\) (only \(D_2(t)\) needed adjusting). Thus (recall the definition of \(c_j\))
\[
(-1)^{m-k}\left(\frac{c_k}{(-1)^k}\right) = (-1)^{m-t}\left(\frac{c_t}{(-1)^t}\right)
\]
and the claim that \(c_k = c_t\) is verified.

Now recall that
\[
P(x) = \sum_{j=0}^{m} c_j x^j = \sum_{j=0}^{m} c_{m-j} x^{2n - r_{m-j}}.
\]

The terms stemming from the \(r_t\)th derivatives of \(c_kx^{r_t}\) and \(c_jx^{2n - r_t}\) must always cancel at \(x = 1\) when \(2n - r_k - r_t = 0\) since here the \(c_t\) appears after a minus sign in the definition of \(P(x)\), and each differentiation augments \(c_k\) and \(c_t\) with equal factors. Moreover, the same occurs with the terms stemming from \(c_t x^{r_t}\) and \(c_k x^{2n - r_t}\). On the other hand, a term not stemming from one of these four will have a factor of \(2n - r_k - r_t\) in its numerator. That concludes this case.

In the case of \(r_k - r_t = 0\) the claim is that \(c_k = -c_t\) and that the terms stemming from \(c_k x^{r_t}\) and \(c_t x^{2n - r_t}\) cancel, as well as those stemming from \(c_k x^{2n - r_k}\) and \(c_t x^{2n - r_t}\). In the case of \(2n - r_k = 0\) the claim is that \(c_k = c_0\) and that the terms stemming from \(c_k x^{r_t}\) and \(c_0 x^{2n}\) cancel (those stemming from \(c_0 x^0\) and \(c_k x^{2n - r_k}\) will clearly be zero). By proceeding in the same way as before (the details are no worse than those for \(2n - r_k - r_t\)) we conclude that \(D^n P(x)\) is always divisible by \(H\). It only remains to evaluate the constant \(C_m\), depending only on \(m\), in the formula for the \((2m + 1)\)th derivative (see Section 3).
Consider the case \( r_i = i \) and \( n = m + 1 \). We have

\[
P(x) = \sum_{k=0}^{m} c_k x^k - \sum_{k=0}^{m} c_{m-k} x^{m+2+k}.
\]

In the expansion of \( D^{2m+1} P(x) \) only the last two terms, namely \(-c_0 x^{2m+2} - c_1 x^{2m+1}\), can give a nonzero contribution. Since

\[
D^{2m+1} x^{2m+1} \big|_{x=1} = (2m+1)! \quad \text{and} \quad D^{2m+1} x^{2m+2} \big|_{x=1} = (2m+2)!
\]

it is

\[
(2m+1)! \left[ -c_0 (2m+2) - c_1 \right] = A_1 A_2,
\]

where

\[
A_1 = (2m+1)! 2^{m+1} (m+1)! m! \prod_{1 \leq i < j \leq m} (j-i) \frac{(2m+1)!}{(m+1)!} \prod_{1 \leq i < j \leq m} (2n-i-j)
\]

and

\[
A_2 = \left[ \frac{-2m+2}{(2m+2)m!} \frac{(2m+1)!}{(m+1)!} + \frac{1}{(2m)(m-1)! (2m+1)} \frac{(2m-1)!}{m!} \right] = -\frac{1}{(2m+1)!}.
\]

On the other hand,

\[
H = (m+1)m! m! \prod_{1 \leq i < j \leq m} (j-i) \frac{(2m+1)!}{(m+1)!} \prod_{1 \leq i < j \leq m} (2n-i-j).
\]

It follows that \( C_2 = -2^{m+1} \), and this completes the proof. \( \square \)

5. Special cases

Theorem 1 tells us much about the expansion of \( P(x) \) about \( x = 1 \), but in general does not give us complete information. However, essentially complete information is available in a few remarkable special cases in which the polynomial coefficients of all the powers of \( x - 1 \) factor completely. To determine ones for which this is true we need to make some Diophantine observations, and to introduce some auxiliary functions satisfying what we shall call Truesdellian partial differential equations. These are partial differential equations of the form

\[
\frac{\partial U(m, x)}{\partial x} = \beta(m) U(m + 1, x),
\]

and variations thereof. (In [4] Truesdell used such equations to unify a large body of special function identities. It is curious that the presently definitive treatise on special functions, namely [1], does not list [4] among its references.)

We shall restrict our present considerations to the case \( m = 2 \). Let
\[ R(x) = R(n, r, s; x) = (1/4)P(2, n, r, s; x) \]
\[ = (n-r)(n-s)(s-r)(2n-r-s) - n(n-s)(2n-s)x \]
\[ + n(n-r)(2n-r)rx - n(n-r)(2n-r)rx^{2n-s} \]
\[ + n(n-s)(2n-s)sx^{2n-r} - (n-r)(n-s)(s-r)(2n-r-s)x^{2n}. \quad (5.1) \]

Here Theorem 1 tells us that
\[ \lim_{x \to 1} \frac{R(n, r, s; x)}{(x-1)^5} \]
\[ = -\frac{2}{5!}nrs(s-r)(n-r)(n-s)(2n-r)(2n-s)(2n-r-s). \quad (5.2) \]

**Theorem 2.** Let \( r < s \) be fixed positive integers. Then
\[ D^m R(x) |_{x=1} \]

is a product of rational linear factors in \( n \) for all \( m \) if and only if \( (r, s) \) is one of \((1, 2), (1, 3), (1, 4), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\).

Before proving this, we employ computer algebra to ascertain the form of \( D^6 R \) and \( D^7 R \). We first find that
\[ D^6 R(x) |_{x=1} = 6n(2n-5)(n-r)(2n-r)r(n-s)(2n-r-s)(r-s). \]
The factor of \( (2n-5) \) is not surprising since for \( n = 5/2 \) we obtain a polynomial that need have no term of higher order than fifth. Next, we obtain
\[ D^7 R(x) |_{x=1} = 2n(n-r)(2n-r)r(n-s)(2n-r-s)(r-s)s \]
\[ \cdot [175 - 126n + 24n^2 - 2nr + r^2 - 2ns + s^2]. \]

**Lemma.** If \( 1 \leq r < s \) and
\[ 175 - 126n + 24n^2 - 2nr + r^2 - 2ns + s^2 = 24n^2 - 2n(63 + r + s) + (175 + r^2 + s^2) \]
is a product of rational linear factors in \( n \), then \((r, s)\) is one of \((1, 2), (1, 3), (1, 4), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\).

**Proof.** The discriminant must be a perfect square so
\[ 4(63 + r + s)^2 - 4 \cdot 24 \cdot (175 + r^2 + s^2) = y^2. \quad (5.3) \]
Thus
\[ (63 + r + s)^2 \geq 24 \left( 175 + \frac{(r + s)^2}{2} \right) \]
since \( r^2 + s^2 > 2((r + s)/2)^2 \). Let \( w = r + s \). Then
\[ 11w^2 - 126w + 231 \leq 0 \]
and $2.29 \leq w \leq 9.16$, so $3 \leq r + s \leq 9$. The proof of the lemma is finished by tabulating all cases, and eliminating those for which the left side of (5.3) is not a perfect square.

We may now proceed to the proof of the theorem.

6. Proof of Theorem 2

We begin by using computer algebra to eliminate most of the possibilities not excluded by the previous lemma. For each case we list below the smallest value of $m$ for which factorization into linear factors does not take place, and the corresponding irreducible factor (there will be only one) of degree at least 2,

- $(1, 4):$ $9; 381 - 215n + 30n^2$,
- $(2, 4):$ $9; 321 - 160n + 20n^2$,
- $(2, 5):$ $9; 813 - 440n + 60n^2$,
- $(3, 4):$ $9; -1829 - 1398n - 355n^2 + 30n^3$,
- $(3, 5):$ $8; 104 - 65n + 10n^2$,
- $(4, 5):$ $9; 183 - 85n + 10n^2$.

It now only remains to establish the positive results for $(1, 2)$ and $(1, 3)$. For $(r, s) = (1, 2)$ we define two auxiliary functions. Let

$$g_1(m, n, x) := 2(n - m)[(m - 1)(2m - 3) + 4(n - 2)(m - 1)(x - 1) + (n - 2)(2n - 3)(x - 1)^2]x^{2n - 2m - 3}$$

and

$$g_2(m, n, x) := [(m - 2)(2m - 3) + 2(n - 2)(2m - 3)(x - 1) + (n - 2)(2n - 3)(x - 1)^2]x^{2n - 2m - 2}.$$ 

Then

$$\frac{\partial^2}{\partial x^2} g_i(m, n, x) = 2(n - m)(2n - 2m - 1)g_i(m + 1, n, x)$$

for $i = 1, 2$. Observe that

$$\frac{\partial^3}{\partial x^3} R(n, 1, 2; x) = -2(n - 1)(2n - 1)g_1(1, n, x)$$

and

$$\frac{\partial^4}{\partial x^4} R(n, 1, 2; x) = -4(n - 1)^2n(2n - 1)(2n - 3)g_2(2, n, x).$$

Thus every derivative of $R$ with respect to $x$ is a product of factors of the desired form multiplied by a derivative of a $g_i$. Since at $x = 1$ we have

$$g_1(m, n, 1) = 2(n - m)(m - 1)(2m - 3)$$
and
\[ g_2(m, n, 1) = (m - 2)(2m - 3) \]
the result follows for \((r, s) = (1, 2)\).

**Remark.** It is also the case that
\[ \frac{\partial}{\partial x} g_2(m, n, x) = g_1(m, n, x) \]
and
\[ \frac{\partial}{\partial x} g_1(m, n, x) = 2(n-m)(2n-2m-1)g_2(m+1, n, x). \]

Next, for \((r, s) = (1, 3)\), we define
\[ s(m, n, x) = x^{2n-m-3} \left( (m - 4) \left( 3(m-3)n + \frac{1}{2} (24 - 7m - m^2) \right) 
+ 6(n - 3)(8 - 3m + (2m - 6)n)(x - 1) 
+ 3(n - 3)(4n^2 + 2(m - 9)n + 16 - 3m)(x - 1)^2 
+ 8(n - 1)(n - 2)(n - 3)(x - 1)^3 \right). \]
Observe that
\[ \frac{\partial^4}{\partial x^4} R(n, 1; 3; x) = -2n(n - 1)(2n - 1)(2n - 3)s(4, n, x) \]
and
\[ \frac{\partial}{\partial x} s(m, n, x) = (2n - m)s(m + 1, n, x). \]
Thus every derivative of \( R \) with respect to \( x \) is a product of factors of the desired form multiplied by a derivative of \( s \). Since at \( x = 1 \) we have
\[ s(m, n, 1) = (m - 4) \left( 3(m-3)n + \frac{1}{2} (24 - 7m - m^2) \right), \]
the proof of Theorem 2 is completed. □

### 7. Further questions

Is there an algorithm for finding the mixed \( q \)-analogue with coefficients of a given degree, for a given product of (possibly dependent) linear forms, that has the least number of terms? In particular, are the polynomials \( P(x) \) optimal in this respect? Much also remains to be done to understand the properties of the \( P(x) = P(m, n, r_1, \ldots, r_m; x) \) polynomials, even for \( m = 2 \).

**Conjecture.** All but \( 2(r - 1) \) of the zeros of \( P(2, n, r, s; x) \), \( 1 \leq r < s \), lie on the unit circle.
The author also has conjectures regarding discriminants and resultants of the $P(x)$ polynomials, but this is beyond the scope of the present paper.

We add that the problem of finding polynomials with a “large” order zero at $x = 1$ and whose expansion about 0 has coefficients that are “small” in absolute value has been studied with notable success by Kós. For an account of this see [3, pp. 137–139]. A further related topic is the determination of high degree solutions to the Prouhet–Tarry–Escott problem. For an account see [2, Chapter 11, pp. 85–95].

References