Positive versus compact support solutions
to a singular elliptic problem✩

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Received 18 April 2005
Available online 5 August 2005
Submitted by A. Cellina

Abstract

The semilinear elliptic problem

\[-\Delta u = K(x)(-u^q + \lambda u^p), \quad u \geq 0 \quad \text{in} \quad \Omega,\]
\[u = 0, \quad \text{on} \quad \partial \Omega,\]

is considered in this paper, where \(\Omega\) is a bounded domain of \(\mathbb{R}^N (N \geq 2)\) with \(C^2\) boundary, \(0 < q < p < 1\) and \(K(x) \to +\infty\) as \(x \to \partial \Omega\). We mainly study the effect of the blow-up rate of \(K(x)\) near \(\partial \Omega\) to the existence of positive and compact support solutions. Furthermore, an optimal compact support principle is given for a class of elliptic differential inequalities.

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Keywords: Compact support principle; Non-Lipschitz continuity; Singular; Sub-supersolution

1. Introduction

Consider the model problem

\[-\Delta u = K(x)(-u^q + \lambda u^p), \quad u \geq 0 \quad \text{in} \quad \Omega,\]

✩ This research is supported by the 973 program of China (Grant No. 2002CB312100).
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where $\Omega$ is a bounded domain of $R^N(N \geq 2)$ with $C^2$ boundary, $0 \leq K(x) \in C^\alpha(\Omega)$, $0 < q < p < 1$ and $\lambda > 0$ is a real parameter. Such problems arise in the contexts of population dynamics, some chemical reaction and plasma physics models, see [1–5,7,8] and the references therein.

It is well known by Brezis and Oswald’s result [5] that (1) has a unique positive solution when $K(x) \leq 0$ and $K(x)$ is bounded in $\Omega$. When $K(x) \geq 0$ or $K(x)$ changes sign, since the non-Lipschitz character of the nonlinear term, the strong maximum principle cannot be applied in the set where $K(x) > 0$ and the solutions may vanish identically on some regions. The existence of non-negative solutions of such problems has been studied by Alama [1], Cortázar et al. [7] and Franchi et al. [8]. On the other hand, Pucci et al. [11, 12] studied the maximum principle and the compact support principle for a wide class of singular inequalities involving quasilinear divergence structure elliptic operators. See also [3,14]. In these papers, the authors mainly considered the case that $K(x)$ is a bounded function, and little is known for the existence of positive solution.

The main purpose of this paper is to study the effect of the singularity of $K(x)$ to the existence of positive and compact support solutions. By using sub-supersolution method, we find that it is related to the blow-up rate of $K(x)$ near $\partial \Omega$. Roughly speaking, if $K(x) \sim d(x)^{-k}$ as $d(x) = \text{dist}\{x, \partial \Omega\} \rightarrow 0$, (1) has at least one positive solution for $\lambda > 0$ large when $k < 1 + q$; but (1) has only compact support solution when $k \geq 1 + q$. More generally, an optimal compact support principle is given for a class of elliptic differential inequalities. We now state precisely our main results.

**Theorem 1.** Suppose that $K(x) > 0$ in $\Omega$ and $K(x) \sim d(x)^{-k}$ as $x \rightarrow \partial \Omega$, $k < 1 + q$. Then there exists a constant $\Lambda > 0$ such that

(i) For $\lambda > \Lambda$, (1) has a maximal positive solution $\bar{u}_\lambda$ which is increasing with respect to $\lambda$.

(ii) $\bar{u}_\lambda(x) \leq Cd(x)$ in $\Omega$ for some $C > 0$ and $\bar{u}_\lambda(x) \in C^2(\Omega) \cap C^{1,1+q-k}$. (\bar{\Omega})$.

(iii) For $\lambda < \Lambda$, (1) has no positive solution.

**Theorem 2.** Assume that $\Omega$ is a $C^2$ bounded domain of $R^N(N \geq 2)$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies

\[
\Delta u \geq K(x) f(u), \quad u \geq 0 \quad \text{for} \quad d(x) \text{ small},
\]

\[
u = 0 \quad \text{on} \quad \partial \Omega,
\]

where $K(x) \geq 0$ in $\Omega$, $f(0) = 0$, $f' > 0$, $f' > 0$ on some interval $[0, a_0)$ and

\[
\int_0^{a_0} F(s)^{-1/2} ds < +\infty,
\]

where $F(s) = \int_0^s f(t) dt$. Furthermore, assume that

\[
K(x) f(d(x)) \geq C_0 d(x)^{-1}
\]
for some $C_0 > 0$ and $d(x)$ small, and as $\varepsilon \to 0^+$,

$$\frac{f(\varepsilon t)}{\varepsilon f(t)} \to +\infty \text{ uniformly for } t > 0 \text{ small.}$$

(5)

Then $u(x) \equiv 0$ for $d(x)$ small.

In particular, if $f(u) = u^q$ with $q \in (0, 1)$ and $K(x) \geq C d(x)^{-1+q}$ near $\partial \Omega$, the result of Theorem 2 holds. Thus (1) has no positive solution for any $\lambda$ in this case and condition (4) is optimal by Theorem 1. Condition (3) is referred to [11,12] and is known to be necessary for the validity of the compact support principle. Example 1 in Section 3 shows that condition (5) is really necessary when $K(x)f(d(x)) = O(d(x))$ as $d(x) \to 0$.

The proof of Theorem 1 relies on finding a positive subsolution and the iteration scheme following the same strategy as in [13]. The proof of Theorem 2 in the critical case $k = 1 + q$ is delicate. It uses a priori estimate and the construction of a supersolution in a neighborhood of the boundary, which serves to control the behavior of the solutions near boundary.

Our last result is an application of Theorem 2 concerning with the existence of compact support solution. It is easy to see that more general results would yield by combining variational method and the result of Theorem 2. We do not give them because they are the same nature.

**Theorem 3.** Suppose that $K(x) \geq 0$ in $\Omega$ and $K(x) \sim d(x)^{-k}$ as $x \to \partial \Omega$, $k \in [1+q, 2)$.

Then there exists a constant $\Lambda_0 > 0$ such that

(i) For $\lambda > \Lambda_0$, (1) has a maximal compact support solution $\bar{u}_\lambda$ which is increasing with respect to $\lambda$.

(ii) For $\lambda < \Lambda_0$, (1) has no non-trivial solution.

Throughout this paper, we use the following notations:

- $d(x) = \text{dist}\{x, \partial \Omega\}$ for $x \in \Omega$;
- $f(x) \sim g(x)$ means that there exist positive constants $C_1$ and $C_2$ such that $C_1 f(x) \leq g(x) \leq C_2 f(x)$ as $d(x) \to 0$;
- $\Omega_r = \{x \in \Omega \mid d(x) < r\}$ and $\Gamma_r = \{x \in \Omega \mid d(x) = r\}$ for $r > 0$;
- $\phi_1$ denotes an eigenfunction corresponding to the smallest eigenvalue $\lambda_1$ of the Laplacian with zero Dirichlet boundary condition;
- $C, C_1, C_2, \ldots$ denote positive constants possibly different from line to line.

2. Positive solution

In this section, the proof of Theorem 1 is given. The following two lemmas are required.
Lemma 1. Let $\Omega$ be a $C^2$ bounded domain in $\mathbb{R}^N$ and $f \in C^\mu(\Omega)$ such that $\sup_\Omega d(x)^\gamma \times |f(x)| < +\infty$ for some $1 < \gamma < 2$. Then the problem $\Delta u = f$ in $\Omega$ with $u = 0$ on $\partial\Omega$ has a unique solution $u \in C^{2,\mu}(\Omega) \cap C(\overline{\Omega})$, and
\[
\sup_\Omega d(x)^\gamma |u(x)| \leq C \sup_\Omega d(x)^\gamma |f(x)|.
\]
where $C$ is a positive constant depending only on $\Omega$ and $\gamma$.

Proof. See Lemma 4.9 and Problem 4.6 in [9] or Lemma 4 in [6].

Lemma 2. Suppose that $f : \Omega \times [0, +\infty) \to \mathbb{R}$ is a continuous function such that $f(x, s)/s$ is strictly decreasing for $s > 0$ at each $x \in \Omega$. Let $w, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy
(a) $\Delta w + f(x, w) \leq 0 \leq \Delta v + f(x, v)$ in $\Omega$,
(b) $w > 0, v \geq 0$ in $\Omega$ and $w \geq v$ on $\partial\Omega$,
(c) $\Delta v \in L^1(\Omega)$.

Then $w \geq v$ in $\overline{\Omega}$.

Proof. See Lemma 3 in [13].

Now, we prove Theorem 1 by splitting it into several lemmas.

Lemma 3. Under conditions of Theorem 1, there exists $\lambda^* > 0$ such that (1) has at least one positive solution $u_\lambda \in C^2(\Omega) \cap C(\overline{\Omega})$ for $\lambda > \lambda^*$.

Proof. Put
\[
u_\lambda = M\phi_1^\tau,
\]
where $\tau = (2 - k)/(1 - q)$, $M$ is a positive constant. Then
\[
-\Delta u_\lambda - K(x)\left(-u_\lambda^q + \lambda u_\lambda^p\right) \leq \lambda_1 M \tau \phi_1^\tau + M \tau (1 - \tau)|\nabla \phi_1|^2 \phi_1^{\tau - 2} + C_1 M^q \phi_1^{\tau q - k} - \lambda C_2 M^p \phi_1^{\tau p - k} \quad \text{in } \Omega. \tag{6}
\]

Note that $\tau > 1$ and $\tau - 2 = \tau q - k$. By Hopf’s maximum principle, there exist $\delta_0 > 0$ and $d_1 > 0$ such that
\[
|\nabla \phi_1| \geq \delta_0 \quad \text{in } \Omega_{d_1}.
\]
Then
\[
C_1 M^q \leq \frac{1}{2} M \tau (1 - 1)|\nabla \phi_1|^2 \quad \text{in } \Omega_{d_1}
\]
provided $M = (2C_1/\tau (\tau - 1)\delta_0^2)^{1/q}$, and
\[
\lambda_1 \tau \phi_1^\tau < \frac{1}{2} \tau (\tau - 1)|\nabla \phi_1|^2 \phi_1^{\tau - 2} \quad \text{in } \Omega_{d_2}.
for some small $d_2 \in (0, d_1)$. It follows from (6) that

$$-\Delta u_{\lambda} - K(x) \left(-u_{\lambda}^q + \lambda u_{\lambda}^p\right) \leq 0 \quad \text{in } \Omega_{d_2}.$$ 

On $\Omega \setminus \Omega_{d_2}$, $\varphi_1 \geq \delta_2$ for some $\delta_2 > 0$. Then there exists $\lambda^* > 0$ sufficiently large such that for $\lambda > \lambda^*$,

$$-\Delta u_{\lambda} - K(x) \left(-u_{\lambda}^q + \lambda u_{\lambda}^p\right) \leq 0 \quad \text{on } \Omega \setminus \Omega_0.$$ 

Thus $u_{\lambda}$ is a subsolution of (1) for $\lambda > \lambda^*$.

Let $\bar{w}_{\lambda} = \bar{M} U$, where $\bar{M}$ is a positive constant and $U$ is the unique positive solution of

$$-\Delta U = K(x) \quad \text{in } \Omega,$$

$$U = 0 \quad \text{on } \partial \Omega.$$ 

By Lemma 1, $U(x) \leq Cd(x)$ in $\Omega$ for some $C > 0$ and $u(x) \in C^{1,1+q-k}(\tilde{\Omega})$.

Proof. Since $K(x) \geq 0$ in $\Omega$ and $0 < q < p < 1$, we have

$$-\Delta u \leq 0 \leq -\Delta (C_0 \varphi_1) \quad \text{on } \Omega_{d_0}$$

for some small $d_0$ and $C_0 > 0$ be chosen to satisfy

$$u(x) \leq C_0 \varphi_1(x) \quad \text{on } \Gamma_{d_0}.$$ 

By maximum principle, $u(x) \leq C_0 \varphi_1(x)$ on $\Omega_{d_0}$. As is well known, $\varphi_1(x) \sim d(x)$ as $x \to \partial \Omega$. Thus $u(x) \leq Cd(x)$ in $\Omega$ for some $C > 0$. Moreover, by using Green’s formula, we have

$$\left|\nabla u(x_1) - \nabla u(x_2)\right| \leq \int_{\Omega} K(y) \left|G_x(x_1, y) - G_x(x_2, y)\right| \cdot \left|-u^q(y) + \lambda u^p(y)\right| dy \leq C \int_{\Omega} d^{-k+q}(y) \left|G_x(x_1, y) - G_x(x_2, y)\right| dy,$$

where $x_1, x_2 \in \Omega$ and $G$ is the Green function. Then

$$\left|\nabla u(x_1) - \nabla u(x_2)\right| \leq Cd^{1+q-k}(x_1, x_2),$$

by the proof of Theorem 1 in [10] (see also [13]). Therefore $u(x) \in C^{1,1+q-k}(\tilde{\Omega})$. The proof of Lemma 4 is completed. □
Lemma 5. Suppose that $K(x) \geq 0$ in $\Omega$ and $K(x) \sim d(x)^{-k}$ as $x \to \partial\Omega$, $k < 2$. Then there exists $\lambda_* > 0$ such that (1) has no non-trivial solution for $\lambda \leq \lambda_*$.

Proof. Let

$$
\lambda_{1,K} = \inf_{u \in H^1_0(\Omega)} \frac{\int_{\Omega} |\nabla u(x)|^2 \, dx}{\int_{\Omega} K(x)u^2(x) \, dx}.
$$

Note that $\lambda_{1,K} > 0$ by Hardy inequality. Let $u \not\equiv 0$ be a solution of (1). Then $\text{supp} \, u \cap \text{supp} \, K \neq \emptyset$ by the maximum principle, and

$$
\lambda_{1,K} \int_{\Omega} K(x)u^2(x) \, dx \leq \int_{\Omega} |\nabla u(x)|^2 \, dx = \int_{\Omega} K(x)(-u^{q+1}(x) + \lambda u^{p+1}(x)) \, dx.
$$

This is impossible for $\lambda \leq \lambda_* := \min\{1, \lambda_{1,K}\}$. The proof of Lemma 5 is completed.

Proof of Theorem 1 completed. (i) Let $\Lambda = \inf\{\mu > 0 : (1) \text{ has a positive solution for } \lambda = \mu\}$.

By Lemmas 3 and 5, $0 < \lambda_* \leq \Lambda \leq \lambda_*^* < +\infty$. For any $\lambda \in (\Lambda, +\infty)$, there exists $\bar{\lambda} \in (\Lambda, \lambda)$ such that (1) with $\lambda = \bar{\lambda}$ has a positive solution $u_{\bar{\lambda}}$ which is a subsolution of (1). Recall that $\bar{w}_{\bar{\lambda}} = M U$ defined in Lemma 3 is a supersolution of (1). Moreover, by using Lemmas 2 and 4, we have $\Delta u_{\bar{\lambda}} \in L^1(\Omega)$ and $u_{\bar{\lambda}} \leq \bar{w}_{\bar{\lambda}}$ in $\Omega$. Thus (1) has at least one positive solution for $\lambda > \Lambda$. To show the existence of the maximal positive solution, we consider the sequence $\{w_n\}$: $w_0 = \bar{w}_{\lambda}$ and

$$
-\Delta w_n + K(x)w_n^q = \lambda K(x)w_n^{p+1} \quad \text{in } \Omega,
$$

$$
w_n = 0 \quad \text{on } \partial\Omega,
$$

for $n = 1, 2, 3, \ldots$. The existence of $w_n$ $(n = 1, 2, 3, \ldots)$ follows from the fact that for any positive solution $u_{\lambda}$ of (1), $u_{\lambda}$ and $\bar{w}_{\lambda}$ is still a pair of sub and supersolutions for problem (7). By using the maximum principle, it is not hard to show that $u_{\lambda} \leq w_{n+1} \leq w_n \leq \bar{w}_{\lambda}$ in $\Omega$.

Let $\bar{u}_{\lambda}(x) = \lim_{n \to \infty} w_n(x)$, $x \in \Omega$. By using a standard regularity argument (see [9]), we conclude that $\bar{u}_{\lambda}$ is a solution of (1) satisfying $\bar{u}_{\lambda} \geq u_{\lambda}$ in $\Omega$, i.e., $\bar{u}_{\lambda}$ is the maximal positive solution of (1) for $\lambda > \Lambda$. Moreover, replacing $u_{\lambda}$ by $\bar{u}_{\lambda}$ and repeating the above arguments, we get that $\bar{u}_{\lambda}$ is increasing with respect to $\lambda$. (ii) and (iii) follow from Lemma 4 and the definition of $\Lambda$. The proof of Theorem 1 is completed.

3. Compact support principle and compact support solution

In this section, we mainly give the proofs of Theorems 2 and 3. At first, a prior estimate is given by comparison, which plays a critical role for the proof of Theorem 2.

Lemma 6. Under conditions of Theorem 2, there exist $\alpha \in (1, 2)$ and $M > 0$ such that $u(x) \leq M \varphi_1(x)^\alpha$ for $d(x)$ small.
Proof. Recall first that $C_1 d(x) \leq \varphi_1(x) \leq C_2 d(x)$ for some $C_1, C_2 > 0$. Let $\tilde{u} = M \varphi_1^\alpha$, where $M > 0$ and $\alpha \in (1, 2)$. Then

$$\Delta \tilde{u} - K(x) f(\tilde{u}) = -M \lambda_1 \alpha \varphi_1^\alpha + M \alpha (\alpha - 1) |\nabla \varphi_1|^2 \varphi_1^{\alpha - 2} - K(x) f(M \varphi_1^\alpha) \leq C' M \alpha (\alpha - 1) d(x)^{\alpha - 2} - K(x) f(M C_1^\alpha d(x)^\alpha)$$

(8)

for some $C' > 0$ and $d(x)$ small. By condition (5), for any $G > 0$, there exists $\varepsilon_0 > 0$ such that as $\varepsilon \in (0, \varepsilon_0)$,

$$f(\varepsilon t) \geq G \varepsilon f(t) \text{ for } t > 0\text{ small.} \quad (9)$$

Using this fact and condition (4), we have

$$K(x) f(M C_1^\alpha d(x)^\alpha) \geq K(x) f((\alpha - 1) M C_1^\alpha d(x)^\alpha) \geq G (\alpha - 1) M C_1^\alpha C_0 d(x)^{\alpha - 2} \text{ in } \Omega_{d_\alpha}$$

(10)

for sufficiently large $M$ depending on $\alpha$ and $d_{\alpha} = (\frac{\varepsilon_0}{MC_1^\alpha (\alpha - 1)})^{\frac{1}{\alpha - 1}}$. Combining (8) and (10), we have

$$\Delta \tilde{u} - K(x) f(\tilde{u}) \leq 0 \text{ in } \Omega_{d_\alpha}$$

provided $G \geq \frac{C C_0}{C_1}$ (note that $G$ may be chosen independent of $\alpha$). Furthermore, by the proof of Lemma 4, $u(x) \leq C'' \varphi_1(x)$ for some $C'' > 0$ and $d(x)$ small. Thus

$$u(x) \leq \tilde{u} \text{ on } \Gamma_{d_\alpha}$$

if $d_\alpha \geq (\frac{C'}{MC_1^\alpha})^{\frac{1}{\alpha - 1}}$. It is easy to see that this condition holds if $\alpha - 1$ is sufficiently small.

Therefore we conclude that $u(x) \leq \tilde{u}$ in $\Omega_{d_\alpha}$ for $\alpha - 1$ small and $M$ large. In fact, suppose by contradiction that $\{x \in \Omega_{d_\alpha} | u(x) > \tilde{u}(x)\} \neq \emptyset$, then there exists $x_0 \in \Omega_{d_\alpha}$ such that

$$\tilde{u}(x_0) - u(x_0) = \inf_{\Omega_{d_\alpha}} (\tilde{u} - u) < 0.$$ 

Note that $u(x_0) \leq a_0$ and by (4), $K(x_0) > 0$ for large $M$. So

$$0 \leq \Delta (\tilde{u} - u)|_{x = x_0} \leq K(x_0) (f(\tilde{u}(x_0)) - f(u(x_0))) < 0.$$ 

A contradiction is obtained. The proof of Lemma 6 is completed.  \Box

Proof of Theorem 2. By Lemma 6, for sufficiently small $\delta > 0$,

$$u(x) \leq MC_2^\alpha d(x)^\alpha < \delta \text{ in } \Omega_{d_\delta},$$

where $d_\delta = (\delta MC_2^\alpha)^{\frac{1}{\alpha}}$ with $\alpha \in (1, 2)$ and $M > 0$ fixed. Let $w(x) \geq 0$ be defined implicitly by

$$\int_{w(x)}^{\delta} F(s)^{-\frac{1}{2}} ds = g(x) \text{ in } \Omega_{d_\delta},$$

where $F(s) = \int_0^s f(t) dt$. This completes the proof of Theorem 2.
where
\[ g(x) = \min_{\Gamma_{d\delta}} \varphi_1 \int_{m\varphi_1(x)} (\frac{s}{2} f(\frac{s}{2}))^{-\frac{1}{2}} \, ds, \quad m > 0. \]

By computation, we have \( w \in C^2(\Omega) \) and satisfies
\[ \Delta w + \Delta g F(w)^{\frac{1}{2}} - \frac{|\nabla g|^2}{2} f(w) = 0 \quad \text{in} \ \Omega_{d\delta} \]
with
\[ \Delta g = \lambda_1 (2m\varphi_1)^{\frac{1}{2}} f(\frac{1}{2} m\varphi_1)^{-\frac{1}{2}} + |\nabla \varphi_1|^2 \left( \frac{m}{2} \right)^{-\frac{1}{2}} \left( \varphi_1 f\left( \frac{1}{2} m\varphi_1 \right) \right)^{-\frac{1}{2}} \times \left( \frac{1}{2} m f\left( \frac{1}{2} m\varphi_1 \right) + \frac{1}{4} m^2 \varphi_1 f'\left( \frac{1}{2} m\varphi_1 \right) \right) \geq 0, \]
and by (4) and (5),
\[ \frac{1}{2} |\nabla g|^2 = m |\nabla \varphi_1|^2 \frac{m C'}{\varphi_1 f\left( \frac{1}{2} m\varphi_1 \right)} \leq \frac{m C'}{C_1 d(x) f\left( \frac{1}{2} m C_1 d(x) \right)} \leq \frac{2C'}{G C_1^2 (d(x) f(d(x))}
\leq \frac{2C'}{G C_1^2 C_0} K(x) \leq K(x) \quad \text{in} \ \Omega_{d\delta} \quad (11) \]
if \( G \geq \frac{2C'}{C_0 C_1^2} \) and \( \delta, m \) is small enough. So \( \Delta w \leq K(x) f(w) \) in \( \Omega_{d\delta} \). Moreover, since \( g \leq 0 \) on \( \Gamma_{d\delta} \), we have \( u(x) \leq \delta \leq w(x) \) on \( \Gamma_{d\delta} \). Therefore \( u(x) \leq w(x) \) in \( \Omega_{d\delta} \) by the same comparison argument as in Lemma 6. It remains to show that \( w(x) \equiv 0 \) as \( d(x) \) small.

In fact, for \( s \in (0, a_0) \), \( f(s)s \geq F(s) \geq \int_{\frac{s}{2}}^{s} f(t) \, dt \geq \frac{s}{2} f\left( \frac{s}{2} \right) \). Then for \( \delta \in (0, a_0) \),
\[ \delta \int_{0}^{\delta} F(s)^{-\frac{1}{2}} \, ds \leq \int_{0}^{\delta} \left( \frac{s}{2} f\left( \frac{s}{2} \right) \right)^{-\frac{1}{2}} \, ds. \]
So as \( x \to \partial \Omega \),
\[ g(x) \to \int_{0}^{\delta} \left( \frac{s}{2} f\left( \frac{s}{2} \right) \right)^{-\frac{1}{2}} \, ds \geq \int_{0}^{\delta} \left( \frac{s}{2} f\left( \frac{s}{2} \right) \right)^{-\frac{1}{2}} \, ds > \int_{0}^{\delta} F(s)^{-\frac{1}{2}} \, ds \quad (12) \]
for \( \delta > 0 \) small. This fact and the definition of \( w(x) \) imply the result. The proof of Theorem 2 is completed. \( \Box \)

Remark 1. In case \( K(x) f(d(x))d(x) \to +\infty \) as \( d(x) \to 0 \), condition (5) is not needed for Theorem 2. In fact, Lemma 6 is not needed for the proof of Theorem 2 in this case because (11) and (12) with \( \alpha = 1 \) hold for large \( m \) and \( d(x) \) small. But the following example shows that condition (5) is really necessary when \( K(x) f(d(x))d(x) = O(d(x)) \) as \( d(x) \to 0 \).
Example 1. Let
\[ f(u) = \begin{cases} 0, & u = 0, \\ u \ln^{2m} u, & u > 0, \end{cases} \]
where \( m \geq 2 \) is an integer. The function \( u(x) = \varphi_1(x)^\beta \) with \( \beta > 1 \) satisfies
\[ \Delta u = K(x)f(u) \quad \text{for } d(x) \text{ small,} \]
where
\[ K(x) = \left(-\lambda_1 \beta + \beta(\beta - 1)|\nabla \varphi_1|^2 \varphi_1^{-2}\right) \left(\ln \varphi_1^\beta\right)^{-2m} \geq 0 \]
and
\[ K(x)f(d(x)) \geq C d(x)^{-1} \]
for some \( C > 0 \). It is easy to verify that
\[ f'(u) = (2m + \ln u) \ln^{2m-1} u \quad \text{for } u > 0 \text{ small,} \]
\[ \int_{a_0}^0 F(s)^{-\frac{1}{2}} ds < +\infty \quad \text{for } a_0 > 0 \text{ small,} \]
and as \( \varepsilon \to 0^+ \),
\[ \frac{f(\varepsilon t)}{\varepsilon f(t)} \to +\infty \]
for each \( t > 0 \), but not uniformly for \( t > 0 \) small.
Conclusion: condition (5) is indeed necessary for Theorem 2.

Proof of Theorem 3. Consider the functional
\[ I_\lambda(u) = \frac{1}{2} \|u\|^2 + \int_\Omega K(x) \left(\frac{1}{q+1}|u|^{q+1} \Big/ \frac{\lambda}{p+1}|u|^{p+1}\right) dx, \quad u \in H_0^1(\Omega) \]
with norm \( \|u\| = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}} \).

Claim. \( I_\lambda : H_0^1(\Omega) \to R^1 \cup \{ +\infty \} \) is coercive and bounded from below on \( H_0^1(\Omega) \).

In fact, by using Sobolev, Hölder and Hardy inequalities, we have
\[
\int_\Omega K(x)|u|^{p+1} dx = \int_{\Omega \setminus \Omega_\varepsilon} K(x)|u|^{p+1} dx + \int_{\Omega_\varepsilon} K(x)|u|^{p+1} dx \\
\leq C_1 \|u\|^{p+1} + \left( \int_{\Omega_\varepsilon} K(x)|u|^{q+1} dx \right)^{\tau} \left( \int_{\Omega_\varepsilon} K(x)|u|^2 dx \right)^{1-\tau} \\
\leq C_1 \|u\|^{p+1} + C_2 \varepsilon^{(1-\tau)(2-k)} \left( \int_{\Omega_\varepsilon} K(x)|u|^{q+1} dx \right)^{\tau}
\]
× \left( \int_{\Omega} u^2 \, d(x) \right)^{1-\tau} \leq C_1 \|u\|^{p+1} + C_2 \varepsilon^{(1-\tau)(2-k)} \\
\times \left( \tau \int_{\Omega} K(x)|u|^{q+1} \, dx + (1-\tau)C_3\|u\|^2 \right), \tag{13}

where \( \tau = (1-p)/(1-q) \), \( \varepsilon > 0 \) and \( C_2, C_3 \) is independent of \( \varepsilon \). So

\[
I_{\lambda}(u) \geq \left( \frac{1}{2} - \frac{\lambda C_2 C_3 (1-\tau)\varepsilon^{(1-\tau)(2-k)}}{p+1} \right) \|u\|^2 - \frac{\lambda C_1}{p+1} \|u\|^{p+1} \\
+ \left( \frac{1}{q+1} - \frac{\lambda C_2 \varepsilon^{(1-\tau)(2-k)}}{p+1} \right) \int_{\Omega} K(x)|u|^{q+1} \, dx \\
\geq \frac{1}{4} \|u\|^2 - \frac{\lambda C_1}{p+1} \|u\|^{p+1} + \frac{1}{2(q+1)} \int_{\Omega} K(x)|u|^{q+1} \, dx, \tag{14}
\]

if \( \varepsilon > 0 \) is sufficiently small. The claim follows because of \( 0 < q < p < 1 \).

Let

\[ c_\lambda = \inf_{u \in H_0^1(\Omega)} I_{\lambda}(u). \]

We show that \( c_\lambda \) is attainable. Suppose that \( \{u_m\} \) is a minimizing sequence for \( c_\lambda \), i.e., \( I_{\lambda}(u_m) \to c_\lambda \) as \( m \to \infty \). The coerciveness of \( I_{\lambda} \) implies that \( \{u_m\} \) is bounded in \( H_0^1(\Omega) \). Then there exist a subsequence of \( \{u_m\} \) (still denoted by \( \{u_m\} \)) and \( u_\lambda \in H_0^1(\Omega) \) such that \( u_m \rightharpoonup u_\lambda \) weakly in \( H_0^1(\Omega) \), strongly in \( L^t(\Omega)(1 < t < \frac{2N}{N-2}) \) and a.e. in \( \Omega \). Moreover, by using Fatou’s lemma and inequality (14), we have

\[
\limsup_{m \to \infty} \int_{\Omega} K(x)|u_m|^{q+1} \, dx < +\infty \quad \text{and} \quad \int_{\Omega} K(x)|u_\lambda|^{q+1} \, dx < +\infty.
\]

So, similar to (13), we have

\[
\limsup_{m \to \infty} \int_{\Omega} K(x)|u_m|^{p+1} \, dx \leq C_4 \varepsilon^{(1-\tau)(2-k)} \quad \text{and} \quad \int_{\Omega} K(x)|u_\lambda|^{p+1} \, dx < +\infty.
\]

Thus

\[
c_\lambda = \lim_{m \to \infty} I_{\lambda}(u_m) \geq \frac{1}{2} \|u_\lambda\|^2 + \frac{1}{q+1} \int_{\Omega} K(x)|u_\lambda|^{q+1} \, dx - \frac{\lambda}{p+1} \int_{\Omega} K(x)|u_\lambda|^{p+1} \, dx \\
- \frac{C_4 \lambda}{p+1} \varepsilon^{(1-\tau)(2-k)} \\
\geq I_{\lambda}(u_\lambda) - \frac{C_4 \lambda}{p+1} \varepsilon^{(1-\tau)(2-k)}
\]
\[ \geq c_\lambda - \frac{C_4\lambda}{p + 1} \varepsilon^{(1-\tau)(2-k)}. \]

Let \( \varepsilon \to 0 \) and then \( I(u_\lambda) = c_\lambda. \) Since \( I_\lambda(u) = I_\lambda(|u|), \) we may assume that \( u_\lambda \geq 0. \) Furthermore, it is not hard to see that there exists some \( \lambda^* > 0 \) such that \( c_\lambda < 0 \) for \( \lambda > \lambda^*. \) So \( u_\lambda \) is a nontrivial weak solution of (1) for \( \lambda > \lambda^*. \) By using the Maximum principle, we have \( u_\lambda \leq Cd(x). \) Therefore standard regularity theory shows that \( u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega}) \) and \( u_\lambda \) has a compact support by Theorem 2. Let

\[ \Lambda_0 = \inf\{\lambda > 0: \text{ (1) has a compact support solution}\}. \]

\[ 0 < \lambda_* < \Lambda_0 < \lambda^* < +\infty \]

by Lemma 5. The other part of the proof of this theorem remains the same as Theorem 1. The proof of Theorem 3 is completed. \( \square \)

References