Estimates of Christoffel Functions of Generalized Freud-Type Weights*

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DEDICATED TO THE MEMORY OF GÉZA FREUD

Upper and lower bounds are found for the generalized Christoffel functions $\lambda_{n,p}(d\mu, x) (0 < p < \infty)$ of Freud-type weights. These weights have the form

$$w(x) = |x|^r \exp(-Q(x)) \quad (x \in \mathbb{R}, r > -1)$$

with a singularity at the origin and non-compact support. The proof requires an inequality reducing weighted integrals of polynomials over $\mathbb{R}$ to integrals over compact intervals.

I. INTRODUCTION

Géza Freud initiated investigations into the polynomials orthogonal with respect to $W(x) = \exp\{-Q(x)\}$ with $Q(x)$ chosen as $x^{2k}/2k$ [2, 4-7]. Nevai [15, 17] and Sheen [19, 20] have successfully handled the cases $k = 2$ and $k = 3$, respectively, where, as in much of Freud’s work, estimates of the Christoffel functions gave crucial information needed in bounding the orthogonal polynomials. Freud also used the bounds to find weighted Markov-Bernstein-type inequalities [3] when $Q$ is a Freud exponent (see (2.1)). Recently Lubinsky [9], Mhaskar and Saff [14], and Zalik [22] have investigated similar weighted inequalities; further, Lubinski [10] and Mhaskar and Saff [13] have bounded the generalized Christoffel functions for a wider class of smooth weights. Both the bounds of the Christoffel functions and the weighted inequalities are used in Magnus’ proof [11, 12] of the Freud conjecture [4].

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In this paper we will investigate the Christoffel functions of Freud-type weights that have a singularity at the origin, that is, weights of the form

\[ w_r(x) = |x|^r \exp(-Q(x)) \quad (-\infty < x < +\infty, r > -1), \]

with \( Q(x) \) being a Freud exponent. We intend to use the estimates given below to find the asymptotics of orthogonal polynomials associated with these generalized Freud-type weights.

The organization of the paper is as follows: In Section II we define our notation; Section III contains the statements of the main results; Section IV is the proof of the integral inequality; Section V contains the derivation of the bounds; and, lastly, Section VI relates \( q_n \) (see (2.3)) to the largest zero and to the ratios of leading coefficients of the orthogonal polynomials associated with these weights.

II. NOTATION

The following notations will be observed throughout. \( Q(x) \) will be called a "Freud exponent" when \( Q \) is an even function and satisfies:

\begin{align}
(i) \quad & Q'(t) > 0, \quad Q''(t) > 0 \quad \text{for} \quad t \in (0, \infty), \\
(ii) \quad & Q''(t) \text{ is continuous on } \mathbb{R}, \\
(iii) \quad & \frac{Q'(2t)}{Q'(t)} > c_0 > 1 \quad \text{for} \quad t \in (0, \infty), \\
(iv) \quad & \frac{tQ''(t)}{Q'(t)} < c \quad \text{for} \quad t \in (0, \infty).
\end{align}

The weight function, \( w_r(x) \), will then be \( w_r(x) = |x|^r \exp(-Q(x)) \). The polynomials orthonormal with respect to \( w_r \) are \( p_n(w_r; x) = y_n x^n + \cdots \), denote the greatest zero of \( p_n(x) \) by \( x_{n,\gamma}(w_r) \) and let

\[ a_n(w_r) = \frac{y_{n-1}(w_r)}{y_n(w_r)}. \]

For \( n \) suitably large let \( q_n \) be defined by the equation

\[ q_n Q'(q_n) = n \quad (n > n_0). \]

By \( \mathcal{P}_n \), denote the set of all polynomials with real coefficients of degree at most \( n \). The generalized Christoffel functions of the distribution \( d\mu \) are (see Nevai [16], where they were first introduced)

\[ \lambda_{n,p}(d\mu; x) = \inf_{\pi \in \mathcal{P}_{n-1}} \left[ \int_{\mathbb{R}} |\pi(t)|^p d\mu(t) / |\pi(x)|^p \right]. \]
We note that, for the special case $p=2$, the following identity is well known (e.g., Freud [8, Theorem 1.4.1]):

$$\lambda_{n,2}(d\mu; x) = \left[ \sum_{k=0}^{n-1} p_k^2(d\mu; x) \right]^{-1}.$$ 

Denote by $c_1, c_2, \ldots$ positive constants independent of $x$ or $n$.

### III. The Main Results

The first result is the main tool with which the bounds were obtained.

**Theorem 3.1.** Let $Q(x)$ be a Freud exponent and $q_n$ be as defined in (2.3), then for a fixed $\theta > 0$, and $p, r$ such that $\theta < p < \infty$ and $pr > -1$, there exist constants $\rho = \rho(\theta) \in (0, 1)$, $c = c(\theta, r)$, and $B > 0$ so that for all $n > n_0$,

$$\|\pi(x) w_r(x)\|_{L_p(R)} \leq (1 + c\rho^n)^{1/\theta} \|\pi(x) w_r(x)\|_{L_p(-Bq_n + Bq_n)}$$

where $\pi(x) \in \mathbb{P}_n$.

**Remark.** The above inequality can be significantly sharpened using the techniques of Potential Theory (e.g., see Mhaskar and Saff [14]). We have chosen the methods used for simplicity of exposition since they do produce results sharp enough for the purposes of the following theorems. We also note that using $q_n \sim q_{n_0}$ for $n < n_0$ and standard compactness arguments we can extend the inequality to $n = 1, 2, \ldots$.

With this "Infinite to Finite Range" inequality in hand we can proceed to the main results, upper and lower bounds of the generalized Christoffel functions; Nevai [18] was the first to use the method of reducing weights over the real line to compact intervals in order to estimate the Christoffel functions.

**Theorem 3.2.** Let $Q(x)$ be a Freud exponent with $q_n$ as defined in (2.3), let $0 < p < \infty$ and $pr > -1$, then, for $w_r(x) = |x|^{p} \exp(-Q(x))$, for every $\epsilon$, $0 < \epsilon < 1$, there is a constant $A = A(\epsilon)$, independent of $x$ and $n$, such that

$$w_r^{-p}(x) \lambda_{n,p}(w_r^p; x) \geq A(q_n/n)(1 + (q_n/n)/|x|)^{pr} \quad (|x| \leq \epsilon Bq_n)$$

where $B$ is the constant of Theorem 3.1.

**Theorem 3.3.** Let $Q(x)$ be a Freud exponent with $q_n$ as defined in (2.3),
let $0 < p < \infty$ and $pr > -1$, then, for $w_r(x) = |x|^r \exp(-Q(x))$, there is a $\delta > 0$ and constant $A'$, independent of $x$ and $n$, such that

$$w_r^p(x) \lambda_{n,p}(w_r^n; x) \leq A'(q_n/n)(1 + (q_n/n)|x|)^p \quad (|x| \leq \delta q_n).$$

We immediately obtain the following

**COROLLARY 3.4.** Under the conditions of Theorems 3.2 and 3.3

$$w_r^p(x) \lambda_{n,p}(w_r^n; x) \sim (q_n/n)(1 + (q_n/n)|x|)^p \quad (|x| \leq \delta q_n).$$

**Remark.** We note that from the definition of Freud exponent, (ii) $Q''$ continuous is used for the lower bound but not for the upper bound while (iii) $Q'(2t)/Q'(t) > c_0$ is used for the upper bound and not the lower.

The relation of $q_n$ to the polynomials $p_n(w_r; x)$ (see Freud [5]) is seen in

**THEOREM 3.5.** Let $Q(x)$ be a Freud exponent with $q_n$ as defined in (2.3) and let $r > -1$; define $w_r(x) = |x|^r \exp\{-Q(x)\}$. Let $x_{1n}(w_r)$ be the greatest zero of $p_n(w_r; x)$ and let $a_n(w_r)$ be defined by (2.2). Then we have

$$x_{1n}(w_r) \sim q_n \quad \text{and} \quad a_n(w_r) \sim q_n.$$

**IV. PROOF OF THE "INFINITE TO FINITE RANGE" INEQUALITY**

Following the method of Lubinsky [10] we use Cartan's Lemma for the

**Proof.** (Theorem 3.1). If $\pi(x) \equiv 0$ the inequality is trivial. Let $\pi \in \mathcal{P}_n$, $n > n_0$, we can express

$$\pi(x) = c \prod_{i=1}^m (x - x_i); \quad c \neq 0, 0 \leq m \leq n; \quad x_1, ..., x_m \in \mathbb{C} \quad \text{with} \quad |x_1| \leq \cdots \leq |x_m|.$$

Let $q_n$ be defined by (2.3). Determine $j \geq 0$ such that for $1 \leq i \leq j$, $|x_i| \leq 3q_{2n}/2$ and for $j < i \leq m$, $|x_i| > 3q_{2n}/2$. If $|x| > Bq_{2n}$, $|u| \leq q_{2n}$, and $j < i \leq m$, then

$$|x - x_i|/|u - x_i| \leq (1 + |x|/|x_i|)/(1 - |u|/|x_i|) \leq 3(1 + (2/3)|x|/q_{2n}).$$

i.e.,

$$|x - x_i|/|u - x_i| \leq 5(|x|/q_{2n}). \quad (4.1)$$

If $|x| > Bq_{2n}$, $|u| \leq q_{2n}$, and $1 < i \leq j$, then

$$|x - x_i|/|u - x_i| \leq (|x| + (3/2)q_{2n})/|u - x_i| \leq 2|x|/|u - x_i|. \quad (4.2)$$
Putting (4.1) and (4.2) together yields

\[ |\pi(x)/\pi(u)| \leq \prod_{i=1}^{j} \left( 2|x|/|u - x_i| \right) \prod_{i=j+1}^{m} \left( 5|x|/q_{2n} \right) \]

\[ = 2^{j} 5^{m-j} (|x|^m/(q_{2n}^{m-j})) \left[ \prod_{i=1}^{j} |u - x_i| \right]^{-1}. \]

We shall now apply Cartan's lemma (see, e.g., Baker [1, p. 174]) to \( \{\prod_{i=1}^{j} |u - x_i| \} \) to obtain

\[ |\pi(x)/\pi(u)| \leq 5^m [48|x|/q_{2n}]^m \]

for \( |x| \geq Bq_{2n}, |u| \leq q_{2n}, \) and \( u \notin \mathcal{S} \subset \mathbb{R}, \) where \( \mathcal{S} \) is a set which can be covered by intervals, the sum of whose lengths is at most \( q_{2n}/8. \) Let \( \mathcal{M} = (-q_{2n}, +q_{2n}) \setminus \mathcal{S}, \) then \( \mathcal{M} \) has Lebesgue measure at least \( (15/8) q_{2n}. \) So for \( u \in \mathcal{M}, |x| \geq Bq_{2n}, \)

\[ |\pi(x) w_r(x)/|\pi(u) w_r(u)| \leq 5^m [48|x|/q_{2n}]^m w_r(x)/w_r(u). \]

Let \( c_1 = \min\{1, (3/8)^r\} \) and \( u \in \mathcal{M}^* = \mathcal{M} \setminus (-q_{2n}, +q_{2n}), \) then

\[ |\pi(x) w_r(x)/|\pi(u) w_r(u)| \leq 5^m [48|x|/q_{2n}]^m w_r(x)/[w_0(q_{2n}) c_1 q_{2n}^{r}] \]

\[ \leq [2^{8n/c_1} [q_{2n}/|x|]^{n-r} [q_{2n}^{-n} w_0(q_{2n})/(q_{2n}^{n} w_0(q_{2n}))]]. \]

But, by the maximality of \( q_{2n}^{-n} w_0(q_{2n}), \) we have

\[ |\pi(x) w_r(x)/|\pi(u) w_r(u)| \leq [2^{8n/c_1} [q_{2n}/|x|]^{n-r}, \]

i.e., for \( |x| \geq Bq_{2n}, \) and \( u \in \mathcal{M}^*, \)

\[ |\pi(x) w_r(x)| \leq [2^{8n/c_1} [q_{2n}/|x|]^{n-r} |\pi(u) w_r(u)|. \]

Therefore

\[ |\pi(x) w_r(x)|^p \leq [2^{8n/c_1} [q_{2n}/|x|]^{(n-r)p} \min_{u \in \mathcal{M}^*} |\pi(u) w_r(u)|^p, \]

or

\[ |\pi(x) w_r(x)|^p \leq [2^{8n/c_1} [q_{2n}/|x|]^{(n-r)p} (1/q_{2n}) \int_{\mathcal{M}^*} |\pi(u) w_r(u)|^p du \]

\[ \leq [2^{8n/c_1} [q_{2n}/|x|]^{(n-r)p} (1/q_{2n}) \int_{q_{2n}}^{1/q_{2n}} |\pi(u) w_r(u)|^p du. \]
Thus for $B$ suitably large and $n > n_0$

$$\int_{|x| > Bq_{2n}} |\pi(x) w_r(x)|^p dx \leq A \left[ \frac{\rho^n}{\rho_1} \right]^p \left[ \frac{pn}{\rho_0} \right]^{-1} \int_{-q_{2n}}^{+q_{2n}} |\pi(u) w_r(u)|^p du.$$ 

Now

$$\int_{\mathbb{R}} |\pi(x) w_r(x)|^p dx = \left[ \int_{|x| \leq Bq_{2n}} + \int_{|x| > Bq_{2n}} \right] |\pi(x) w_r(x)|^p dx$$

thus

$$\int_{\mathbb{R}} |\pi(x) w_r(x)|^p dx \leq \left[ 1 + \left( \frac{c_1}{(pn)} \right)^{\rho^n} \right] \int_{|x| \leq Bq_{2n}} |\pi(x) w_r(x)|^p dx.$$ 

So we have

$$\|\pi(x) w_r(x)\|_{L_p(\mathbb{R})} \leq \left[ 1 + \left( \frac{c_1}{(pn)} \right)^{\rho^n} \right]^{1/p} \|\pi(x) w_r(x)\|_{L_p(-Bq_{2n} + Bq_{2n})}$$

choosing $B$ possibly larger, since $q_{2n} < 2q_n$ (Freud [3, p. 223]). Fix $\theta > 0$ then for $0 < \theta \leq p < \infty$

$$\|\pi(x) w_r(x)\|_{L_p(\mathbb{R})} \leq \left[ 1 + \left( \frac{c_1}{(\theta n)} \right)^{\rho^n} \right]^{1/\theta} \|\pi(x) w_r(x)\|_{L_p(-Bq_{2n} + Bq_{2n})}.$$ 

By the continuity of $\|\cdot\|_{L_p}$ norms and the independence of the constants upon $p$, the limit as $p \to \infty$ may be taken and the inequality holds for $0 < \theta \leq p \leq \infty$. 

V. PROOFS OF THE UPPER AND LOWER BOUNDS
OF THE CHRISTOFFEL FUNCTIONS

First, we shall require a technical lemma

**Lemma 5.1.** Let $R_n(x) = \sum_{k=0}^{n-1} x^k/k!$ then

$$3/4 \exp(x) \leq R_n(x) \leq (5/4) \exp(x) \quad (|x| \leq n/5, n \geq 12).$$
Proof. From Taylor's theorem, we have, for $|x| \leq cn$,

$$|\exp(x) - R_n(x)| \leq (n!) \max_{|x| \leq cn} \{\exp(x)|x|^n\} \leq (n!) \exp(cn)(cn)^n.$$  

Applying Stirling's approximation gives

$$|\exp(x) - R_n(x)| \leq \exp((c + 1)n) c^n,$$

in particular, for $c = 1/5$,

$$|1 - \exp(-x) R_n(x)| \leq (8/9)^n.$$  

We shall now construct the polynomials that will be used to approximate $w_0(x)$ (as in Freud [3]).

**Lemma 5.2.** Let $Q(x)$ be a Freud exponent, $q_n$ be defined by (2.3), and fix $x \in \mathbb{R}$. There exists a polynomial $S_n(x; t)$ such that

1. $S_n(t) \in \mathbb{Z}_{2q_n}(t)$ for each fixed $x$ and some integer $k = k(Q, B)$,
2. $S_n(x; x) = w_0(x)$,
3. $0 < S_n(t) \leq (5/4) w_0(t)$ for $|t| \leq Bq_n$,

where $B$ is the constant of Theorem 3.1.

**Proof.** Let $V_n(t) = Q'(x)(t - x) + \left[c_0 n/2q_n^2\right](t - x)^2$ for $t \in \mathbb{R}$. Define

$$S_n(t) = w_0(x) R_{kn}(-V_n(t)) \quad (|t| \leq Bq_n),$$

then (i) and (ii) follow directly. Now to prove (iii): For $|t| \leq Bq_n$

$$|V_n(t)| \leq |Q'(x)| 2Bq_n + \left[c_0 n/2q_n^2\right] 4B^2 q_n^2 \leq c_1 |Q'(q_n)| 2Bq_n + 2B^2 c_0 n \leq 2B[c_1 + Bc_0] n.$$

Therefore, if $k$ is a large enough positive integer, so that $k/5 \geq 2B[c_1 + c_0 B]$, then, by Lemma 5.1,

$$R_{kn}(-V_n(t)) \sim \exp(-V_n(t)) \quad (|t| \leq Bq_n);$$

so that

$$S_n(t) = w_0(x) R_{kn}(-V_n(t)) \sim w_0(x) \exp(-V_n(t)),$$

and hence

$$S_n(t) w_0^{-1}(t) \sim \exp\{Q(t) - Q(x) - Q'(x)(t - x) - \left[c_0 n/2q_n^2\right](t - x)^2\}.$$
Since $Q''$ is continuous, $Q(t) = Q(x) + Q'(x)(t-x) + Q''(\xi)(t-x)^2/2$ for some $\xi$ between $t$ and $x$, but, since $Q$ is a Freud exponent, $|Q''(\xi)| \leq c_0 n/q_n^2$, and thus (iii) holds.

We are now in a position to determine the lower bound.

**Proof.** (Theorem 3.2). Let $p > 0$, fix $r$ such that $pr > -1$, and let $n > 12$. Then

$$
\lambda_{n,p}(w_r^n; x) = \inf_{\pi \in P_{n-1}} \int_{-B_{q_n}}^{B_{q_n}} |\pi(t)|^p w_r^n(t) \frac{dt}{\pi(x)^p} \\
\geq \inf_{\pi \in P_{n-1}} \int_{-B_{q_n}}^{B_{q_n}} |\pi(t)|^p w_r^n(t) \frac{dt}{\pi(x)^p} \\
\geq c_1 w_0^n(x) \inf_{\pi \in P_{n-1}} \int_{-B_{q_n}}^{B_{q_n}} |\pi(t) S_{2k_n}(t)|^p |t|^{pr} dt/\pi(x)^p \\
\geq c_2 w_0^n(x) q_n^{pr+1} \inf_{R \in \mathbb{P}_{k_n-1}} \int_{-1}^{+1} |R(t B_{q_n})|^p |t|^{pr} dt/\pi(x)^p \\
\geq c_2 w_0^n(x) q_n^{pr+1} \inf_{R \in \mathbb{P}_{k_n-1}} \int_{-1}^{+1} |R(t B_{q_n})|^p |t|^{pr} dt/\pi(x)^p,
$$

so that

$$
\lambda_{n,p}(w_r^n; x) \geq c_2 w_0^n(x) q_n^{pr+1} \lambda_{k_n,p}(|t|^{pr} \chi_{[-1, +1]}(t) dt; x/B_{q_n}).
$$

Using Nevai [16, Theorem 6.3.25] we have, for $|x| \leq \varepsilon B_{q_n}$ ($0 < \varepsilon < 1$),

$$
\lambda_{n,p}(w_r^n; x) \geq A w_r^n(x) [q_n/n] [1 + B(q_n/n)(1/|x|)]^{pr}.
$$

Now we shall construct the polynomials to estimate $w_0(x)$ for the upper bound.

**Lemma 5.3.** Let $x \in \mathbb{R}$ be fixed and let $n > 12$. Then there exists a polynomial $S_n(x; t)$ and $\delta > 0$ such that for $|x| \leq \delta q_n$ and $|t| \leq B_{q_n}$,

(i) $S_n(t) \in \mathbb{P}_{[n/2]}(t)$,

(ii) $S_n(x; x) = w_0^{-1}(x)$,

(iii) $0 \leq S_n(t) w_0(t) \leq 5/4$.

where $B$ is the constant of Theorem 3.1 and $q_n$ is defined by (2.3).

**Proof.** Define $S_n(x; t) = w_0^{-1}(x) R_m(Q(x)(t-x))$ where $m = \lceil n/2 \rceil$ and $R_m$ is defined in Lemma 5.1, then (i) and (ii) follow immediately. For
$|x| \leq \delta q_n$ and $|t| \leq Bq_n$, we have $|t-x| \leq (B+\delta) q_n$; now, since $Q'$ is increasing

$$|Q'(x)(t-x)| \leq Q'((\delta q_n)(B+\delta) q_n = \left[ Q'(\delta q_n)/Q'(q_n) \right] q_n Q'(q_n)(B+\delta).$$

Since $Q$ is a Freud exponent

$$\left[ Q'(\delta q_n)/Q'(q_n) \right] \leq \left[ Q'(q_n^{2^{-k}})/Q'(q_n) \right] \leq c_0^{-k}. \text{Thus we can take } \delta > 0 \text{ so small that}$$

$$|Q'(x)(t-x)| \leq c_0^{-k} n(B+\delta) \leq n/20 \leq m/5,$$

therefore, by Lemma 5.1 and the convexity of $Q$,

$$S_n(t) \leq Cw_0^{-1}(x) \exp \{ (Q'(x)|t-x|) \} \leq c \exp \{ Q(t) \} = Cw_0^{-1}(t). \quad \blacksquare$$

Let us proceed to the

Proof (Theorem 3.3). As before let $p > 0$, fix $r$ such that $pr > -1$, and let $n > 12$. Then

$$\lambda_{n,p}(w_p^r; x) = \inf_{n \in \mathbb{N}} \int_{t \in [a,Bq_n]} |\tau(t)|^p w_p^r(t) \, dt/|\tau(x)|^p$$

$$\leq C_1 \inf_{n \in \mathbb{N}} \int_{t \in [-Bq_n,Bq_n]} |\tau(t)|^p w_p^r(t) \, dt/|\tau(x)|^p,$$

which, applying Lemma 5.3, is

$$C_1 \inf_{R \in \mathbb{P}_{n/2}} \left[ |R(t)| S_n(t)|w_0(t)|^p |t|^{pr} \, dt/\left[ |R(x)| S_n(x) \right]^p \right]$$

$$\leq C_2 w_0^p(x) \inf_{R \in \mathbb{P}_{n/2}} \left[ |R(t)|^p |t|^{pr} \, dt/\left[ |R(x)| \right]^p \right].$$

We apply the same change of variables as in the derivation of the lower bound to obtain

$$\leq C_3 w_0^p(x) q_n^{pr+1} \inf_{R \in \mathbb{P}_{n/2}} \left[ \int_{-1}^{+1} \left| R^*(u) \right|^p |u|^{pr} du/\left[ |R^*(x/Bq_n)| \right]^p \right]$$

so that

$$\lambda_{n,p}(w_p^r; x) \leq C_3 w_0^p(x) q_n^{pr+1} \lambda_{n/2,p}(|t|^{pr} \lambda_{[-1,+1]}(t) \, dt, x/Bq_n).$$

Once more using Nevai [16, Theorem 6.3.25] we have, for $|x| \leq \delta Bq_n$,

$$\lambda_{n,p}(w_p^r; x) \leq A w_p^r(x) [q_n/n] [1 + B(q_n/n)(1/|x|)]^{pr}. \quad \blacksquare$$
VI. CONNECTIONS TO THE ORTHONORMAL POLYNOMIALS $p_n(w_r; x)$

While Freud originally used the property that $q_{2n}^2 Q'(q_{2n})$ maximized $xQ'(x)$, there are other significant relations concerning $q_n$.

**Lemma 6.1.** Let $x_{1n}(w_r)$ denote the greatest zero of the orthonormal polynomial, $p_n(w_r; x)$, and $q_n$ be defined by (2.3); then

$$\limsup_{n \to \infty} x_{1n}(w_r)/q_n \leq \text{const.}$$

**Proof.** From a well-known result of Chebyshev (see, e.g., Szegő [21, p. 187]) we have

$$x_{1n}(w_r) = \max_{n \in \mathbb{N}} \left[ \int_{\mathbb{R}} x \pi^2(x) w_r(x) \, dx / \int_{\mathbb{R}} \pi^2(x) w_r(x) \, dx \right].$$

According to Theorem 3.1

$$\int_{\mathbb{R}} |x| \pi^2(x) w_r(x) \, dx \leq \left[ 1 + \sigma r^{2n+1} \right] \int_{-B_q n}^{+B_q n} |x| \pi^2(x) w_r(x) \, dx$$

or

$$\leq 2 \left[ 1 + \sigma r^{2n+1} \right] B_q n \int_{-B_q n}^{+B_q n} \pi^2(x) \, w_r(x) \, dx,$$

and the result is seen to hold.

**Lemma 6.2.** Let $r > -1$. Then

$$\gamma_n(w_r)/\gamma_{n-1}(w_r) = (n + r \Delta_n)^{-1} \int_{-\infty}^{\infty} p_n(x) p_{n-1}(x) Q'(x) w_r(x) \, dx;$$

$$\Delta_n = \sin^2 (n\pi/2).$$

**Remark.** For $Q(x) = |x|^\beta$ Lemma 6.2 was proven for $r \geq 0$ and $\beta > 0$ in Freud [6] and for $r > -1$ and $\beta > 1$ in Nevai [18].

**Proof.** First integrate directly

$$\int_{-\infty}^{\infty} p_n(x) p_{n-1}(x) w_r(x) \, dx = \int_{-\infty}^{\infty} (n \gamma_n x^{n-1} + \cdots) p_{n-1}(x) w_r(x) \, dx$$

$$= \int_{-\infty}^{\infty} (n(\gamma_n/\gamma_{n-1}) p_{n-1}(x) + \pi_{n-2}(x)) p_{n-1}(x) w_r(x) \, dx = n(\gamma_n/\gamma_{n-1})$$

(6.1)

where $\pi_{n-2}(x) \in \mathbb{P}_{n-2}$, the last equality holding by virtue of orthogonality.
Now integrate by parts

\[
\int_{-\infty}^{\infty} p'_n(x) p_{n-1}(x) w_r(x) \, dx = -\int_{-\infty}^{\infty} p_n(x)(p_{n-1}(x) w_r(x))' \, dx
\]

\[
= \int_{-\infty}^{\infty} p_n(x) p_{n-1}(x) Q'(x) w_r(x) \, dx
\]

\[
- r \int_{-\infty}^{\infty} p_n(x) p_{n-1}(x) x^{-1} w_r(x) \, dx. \tag{6.2}
\]

Since \( w_r(x) \) is an even weight, \( p_n \) is an even/odd polynomial as \( n \) is even/odd, resp., therefore

\[
\int_{-\infty}^{\infty} p_n(x) p_{n-1}(x) x^{-1} w_r(x) \, dx = (\gamma_n/\gamma_{n-1}) A_n.
\]

Combining (6.1) and (6.2), the result follows. 

**Lemma 6.3.** Let \( r > -1, n > n_0 \), and \( a_n(w_r) = \gamma_{n-1}(w_r)/\gamma_n(w_r) \), then

\[
A q_n \leq a_n(w_r),
\]

where \( A \) is an absolute constant.

**Proof.** From Lemma 6.2 we have

\[
\gamma_n(w_r)/\gamma_{n-1}(w_r) = (n + r A_n)^{-1} \int_{-\infty}^{\infty} p_n(x) p_{n-1}(x) Q'(x) w_r(x) \, dx.
\]

Since \( Q \) is a Freud exponent, for \( x > 0 \)

\[
Q'(x) = Q'(q_n) \exp \left\{ \log(Q'(x)) - \log(Q'(q_n)) \right\}
\]

\[
Q'(q_n) \exp \left\{ \int_{q_n}^{x} (Q''(t)/Q'(t)) \, dt \right\}
\]

\[
\leq Q'(q_n) \exp \left\{ \int_{q_n}^{x} (c/t) \, dt \right\} = Q'(q_n) |x/q_n|^c
\]

with \( c \) being the constant of (2.1)(iv), whereupon

\[
\gamma_n(w_r)/\gamma_{n-1}(w_r) \leq c_1 (n + r A_n)^{-1} Q'(q_n) \int_{-\infty}^{+\infty} |p_n(x) p_{n-1}(x)| |x/q_n|^c w_r(x) \, dx.
\]
We now apply Theorem 3.1 to obtain
\[ \leq c_2(n + rA_n)^{-1}Q'(q_n) \int_{-B_{2q_n}}^{B_{2q_n}} |p_n(x) p_{n-1}(x)| |x/q_n| w_1(x) \, dx, \]
so that
\[ \gamma_n w_r/(\gamma_{n-1} w_r) \leq c_3 n^{-1} Q'(q_n) \int_{-B_{2q_n}}^{B_{2q_n}} |p_n(x) p_{n-1}(x)| w_r(x) \, dx, \]
i.e.,
\[ \leq c_3 n^{-1} Q'(q_n) = c_3/q_n. \]
The last equality follows from the definition of \( q_n \).

**Proof (Theorem 3.5).** The inequality
\[ \text{const} \, q_{n-1} \leq a_{n-1} \leq \max_{1 < j < n-1} a_j \leq x_1 \leq 2 \max_{1 < j < n-1} a_j \leq 2x_{1n} \leq \text{const} \, q_n \]
follows from (Freud [6, Theorem 1])
\[ \max_{1 < j < n-1} a_j \leq x_1 \leq 2 \max_{1 < j < n-1} a_j \]
and Lemmas 6.1 and 6.3. Since \( q_n \) is an increasing function of \( n \) and \( 1 < q_{2n}/q_n < 2 \) (Freud [3, p. 222]) the Theorem holds.

Remark. When \( Q \) is an even polynomial of degree \( 2m \) then \( q_n \sim n^{1/2m} \) and given that \( A_n(w_r) = a_n(w_r)/(n^{1/2m}) \) has a limit, it is an easy calculation to find the value. Following the method of Freud [4] we integrate \( \int p'(x) p_{n-1}(x) w_r(x) \, dx \) in two ways (as in Lemma 6.2) and we arrive at the recurrence relation for \( a_n(w_r) \)
\[ n + r \sin^2((m/2)^2) = 2a_n \sum_{k=1}^{m} k d_{2k} \int_{-\infty}^{+\infty} x^{2k-1} p_n(x) p_{n-1}(x) w_r(x) \, dx \]
where \( Q(x) = \sum d_{2k} x^{2k} \); now, noting that the “order” of each of the integrals is \( \sim C_{2k-1,m} a_n^{-1} \) (\( C_{i,j} \) being the binomial coefficient), we find
\[ \lim_{n \to \infty} a_n(w_r)/(n^{1/2m}) = (2m d_{2m} C_{2m-1,m})^{-1/2m} \]
which is consistent with the Freud conjecture [4] (recently proven by A. Magnus [11] for the case \( Q(x) = x^{2m} \), also see Magnus [12], where Freud’s conjecture was discussed for \( Q(x) = |x|^r \), \( r > 1 \).
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